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About the theory of local homology and local cohomology modules

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Abstract. We introduce local homology for the edge ideal of a graph, and study some properties of local homology modules for Artinian modules, such as the artinianness, and the characterization of Width by local homology. Moreover, we put results which involve the theory of local cohomology modules together with the edge ideal of a graph simple and finite, with no isolated vertices.

1. Introduction

Throughout this paper, R is a commutative ring with non-zero identity.

In [7] we have that Grothendieck introduced the definition of local cohomology. Let J be an ideal of R, and let M be an R-module, then

$$\mathrm{H}^i_J(M) = \varinjlim_{t\in\mathbb{N}} \mathrm{Ext}^i_R\left(R/J^t, M\right)$$

is called the i-th local cohomology module of M with respect to J. We know that there exists the theory of local homology which is dual to the theory of local cohomology of Grothendieck (see [4, 6, and 13]).

The purpose of this paper is to study some properties of local homology modules and local cohomology modules which involve the theory of graphs, together with the edge ideal of a graph.

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In the Section 2, we put some definitions and prerequisites for a better understanding of the theory and results.

We introduce preliminaries of the theory of graphs which involving the edge ideal of a graph G; associated to the graph G is a monomial ideal

$$I(G) = (v_i v_j \mid v_i v_j \text{ is an edge of } G),$$

with $v_i v_j = v_j v_i$ and with $i \neq j$, in the polynomial ring $R = K[v_1, v_2, \ldots, v_s]$ over a field K, called the **edge ideal** of G. The preliminaries of the theory of graphs were introduced in this Section 2 together with the concepts suitable for the work.

In the Section 3, we prove some properties of the local homology module, properties that involve the edge ideal of a graph G, which is a graph simple and finite, with no isolated vertices.

In the Section 4, we prove a result about the local cohomology module, and this result involve the edge ideal of a graph G, which is a graph finite simple, with no isolated vertices.

Throughout the paper, we mean by a graph G, a finite simple graph with the vertex set $V(G)$ and with no isolated vertices.

Here, we use properties of commutative algebra and homological algebra for the development of the results (see [2 and 10]).

2. Prerequisites of the graph theory

Let us present in this section the concepts of the graphs theory that we will use in the course of this work.

2.1. Edge ideal of a graph

This section is in accordance with [1 and 12].

Let $R = K[v_1, \ldots, v_s]$ be a polynomial ring over a field K, and let $Z = \{z_1, \ldots, z_q\}$ be a finite set of monomials in R. The **monomial subring** spanned by Z is the K-subalgebra,

$$K[Z] = K[z_1, \ldots, z_q] \subset R.$$

In general, it is very difficult to certify whether $K[Z]$ has a given algebraic property – e.g., Cohen-Macaulay, normal – or to obtain a measure of its numerical invariants – e.g., Hilbert function. This arises because the number q of monomials is usually large.

Thus, consider any graph G, simple and finite without isolated vertices, with vertex set $V(G) = \{v_1, \ldots, v_s\}$.

Let Z be the set of all monomials $v_iv_j = v_jv_i$, with $i \neq j$, in $R = K[v_1, \ldots, v_s]$, such that $\{v_iv_j\}$ is an edge of G, i.e., the graph finite and simple G, with no isolated vertices, is such that the squarefree monomials of degree two are defining the edges of the graph G.

Definition 2.1. A *walk* of length s in G is an alternating sequence of vertices and edges $w = \{v_1, z_1, v_2, \ldots, v_{s-1}, z_h, v_s\}$, where $z_i = \{v_{i-1}v_i\}$ is the edge joining v_{i-1} and v_i.

Definition 2.2. A walk is **closed** if $v_1 = v_s$. A walk may also be denoted by $\{v_1, \ldots, v_s\}$, the edges being evident by context. A **cycle** of length s is a closed walk, in which the points $v_1, \ldots, v_s$ are distinct.

A **path** is a walk with all the points distinct. A **tree** is a connected graph without cycles and a graph is **bipartite** if all its cycles are even. A vertex of degree one will be called an **end point**.

Definition 2.3. A subgraph $G' \subseteq G$ is called **induced** if $v_iv_j = v_jv_i$, with $i \neq j$, is an edge of G' whenever v_i and v_j are vertices of G' and v_iv_j is an edge of G.

The **complement** of a graph G, for which we write G^c, is the graph on the same vertex set in which $v_iv_j = v_jv_i$, with $j \neq i$, is an edge of G^c if and only if it is not an edge of G. Finally, let C_k denote the cycle on k vertices; a **chord** is an edge which is not in the edge set of C_k. A cycle is called **minimal** if it has no chord.

If G is a graph without isolated vertices, simple and finite, then let R denote the polynomial ring on the vertices of G over some fixed field K.

Definition 2.4 ([1])**.** According to the previous context, the **edge ideal** of a finite simple graph G, with no isolated vertices, is defined by

$$I(G) = (v_iv_j \mid v_iv_j \text{ is an edge of } G),$$

with $v_iv_j = v_jv_i$, and with $i \neq j$.

3. Local homology module of the edge ideal of a graph

We have the following definition.

Definition 3.1. Let J be an ideal of the ring R, and let M be an R-module. The i-th local homology module $\mathrm{H}_i^J(M)$ of M with respect to J is defined by

$$\mathrm{H}_i^J(M) = \varprojlim_{t \in \mathbb{N}} \mathrm{Tor}_i^R(R/J^t, M)$$

for all $i \geq 0$.

Remark 3.2. (1) Let $\Lambda_J(G) = \varprojlim_{t\in\mathbb{N}} G/J^tG$ be the J-adic completion of an R-module G, and so we have $\mathrm{H}_0^J(M) \cong \Lambda_J(M)$.

(2) Since $\mathrm{Tor}_i^R(R/J^t, M)$ has a natural structure as a module over the ring R/J^t for all $t > 0$, $\mathrm{H}_i^J(M)$ has a natural structure as a module over the ring $\Lambda_J(R) = \varprojlim_{t\in\mathbb{N}} R/J^t$.

In this section, we present some results about the local homology module which involve the theory of graphs together with the edge ideal of a graph G, which is simple and finite and with no isolated vertices.

Here, we take K a fixed field and we consider $K[v_1, v_2 \ldots, v_s]$ the ring polynomial over the field K. Since K is a field, we have that K is a Noetherian ring and then $K[v_1, \ldots, v_s]$ is also a Noetherian ring (Theorem of the Hilbert Basis).

Remark 3.3. By the previous context, $R = K[v_1, v_2 \ldots, v_s]$ is a Noetherian ring. Thus, the edge ideal $I\,(G)$ is an R-module, and thus we can get characterizations for this module under certain hypothesis.

Proposition 3.4. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Then, the local homology module $\mathrm{H}_i^{(v_1,\ldots,v_s)}(I(G))$ is $(v_1, \ldots, v_s)$-separated, for all $i \geq 0$, i.e.,*

$$\bigcap_{t>0}(v_1, \ldots, v_s)^t \mathrm{H}_i^{(v_1,\ldots,v_s)}(I(G)) = 0.$$

Proof. It should be noted that inverse limits are left exact and any two inverse limits commute. Therefore,

$$\bigcap_{n>0}(v_1, \ldots, v_s)^n \mathrm{H}_i^{(v_1,\ldots,v_s)}(I(G)) \cong \varprojlim_{n\in\mathbb{N}}(v_1, \ldots, v_s)^n \varprojlim_{t\in\mathbb{N}} \mathrm{Tor}_i^R(R/(v_1, \ldots, v_s)^t, I(G)),$$

and, as $\varprojlim_{n\in\mathbb{N}}(v_1, \ldots, v_s)^n \varprojlim_{t\in\mathbb{N}} \mathrm{Tor}_i^R(R/(v_1, \ldots, v_s)^t, I(G))$ it is contained in $\varprojlim_{n\in\mathbb{N}} \varprojlim_{t\in\mathbb{N}}(v_1, \ldots, v_s)^n \mathrm{Tor}_i^R(R/(v_1, \ldots, v_s)^t, I(G))$, which in turn is isomorphic to $\varprojlim_{t\in\mathbb{N}} \varprojlim_{n\in\mathbb{N}}(v_1, \ldots, v_s)^n \mathrm{Tor}_i^R(R/(v_1, \ldots, v_s)^t, I(G))$, it follows that

$$\bigcap_{n>0}(v_1, \ldots, v_s)^n \mathrm{H}_i^{(v_1,\ldots,v_s)}(I(G)) = 0,$$

since we have

$$(v_1, \ldots, v_s)^n \mathrm{Tor}_i^R(R/(v_1, \ldots, v_s)^t, I(G)) = 0$$

for all $s \geq t$, as required. □

Now, we have the following result.

Proposition 3.5. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is an R-module Artinian and let*

$$0 \to N^{'} \to I(G) \to N^{''} \to 0,$$

be a short exact sequence of Artinian modules. Then, we have a long exact sequence of local homology modules

$$\ldots \to \mathrm{H}_i^{(v_1,\ldots,v_s)}(N^{'}) \to \mathrm{H}_i^{(v_1,\ldots,v_s)}(I(G)) \to \mathrm{H}_i^{(v_1,\ldots,v_s)}(N^{''}) \to \ldots \to \mathrm{H}_0^{(v_1,\ldots,v_s)}(N^{'}) \to \mathrm{H}_0^{(v_1,\ldots,v_s)}(I(G)) \to \mathrm{H}_0^{(v_1,\ldots,v_s)}(N^{''}) \to 0.$$

Proof. The short exact sequence

$$0 \to N^{'} \to I(G) \to N^{''} \to 0$$

gives rise to a long exact sequence, for all $t > 0$,

$$\ldots \mathrm{Tor}_i^R(R/(v_1, \ldots, v_s)^t, N^{'}) \to \mathrm{Tor}_i^R(R/(v_1, \ldots, v_s)^t, I(G)) \to \mathrm{Tor}_i^R(R/(v_1, \ldots, v_s)^t, N^{''}) \to \ldots \to R/(v_1, \ldots, v_s)^t \otimes_R N^{'} \to R/(v_1, \ldots, v_s)^t \otimes_R I(G) \to R/(v_1, \ldots, v_s)^t \otimes_R N^{''} \to 0.$$

Since $I(G)$ is an R-module Artinian, the modules in the long exact sequence are Artinian. It should be noted that the inverse limit $\varprojlim_{t \in \mathbb{N}}$ is exact on Artinian R-modules by [8, 9.1]. Therefore, we have the long exact sequence of local homology modules, and the proof is complete. □

Let us now make a definition, which we will use in a later result.

Definition 3.6. A sequence of elements $x_1, \ldots, x_r$ in $R = K[v_1, \ldots, v_s]$ is said to be an $I(G)$-coregular sequence (see [9, 3.1]) if $\big(0 :_{I(G)} (x_1, \ldots, x_r)\big) \neq 0$ and

$$\big(0 :_{I(G)} (x_1, \ldots, x_{i-1})\big) \overset{\cdot x_i}{\to} \big(0 :_{I(G)} (x_1, \ldots, x_{i-1})\big)$$

is surjective for $i = 1, \ldots, r$. We denote by $\mathrm{Width}_{(v_1,\ldots,v_s)}(I(G))$ the length of the longest $I(G)$-coregular sequence in $(v_1, \ldots, v_s)$. In the case in that $I(G)$ is an Artinian R-module, we know that $\mathrm{Width}_{(v_1,\ldots,v_s)}(I(G)) < \infty$ (see [4, Paragraph 5]).

Theorem 3.7. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices, and such that $\mathrm{Ann}(R) \subseteq (v_1, \ldots, v_s)$. Suppose that $I(G)$ is an Artinian R-module. Then all maximal $I(G)$-coregular sequences in $(v_1, \ldots, v_s)$ have the same length. Moreover,*

$$\mathrm{Width}_{(v_1,\ldots,v_s)}(I(G)) = \inf \left\{ i \mid \mathrm{H}_i^{(v_1,\ldots,v_s)}(I(G)) \neq 0 \right\}.$$

Before doing the proof of the theorem, let's put a result that will be used in proof of the Theorem 3.7

Lemma 3.8. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices, and such that $\mathrm{Ann}(R) \subseteq (v_1, \ldots, v_s)$. Suppose that $I(G)$ is an Artinian R-module. Then $\mathrm{H}_0^{(v_1,\ldots,v_s)}(I(G)) = 0$ if and only if $xI(G) = I(G)$ for some $x \in (v_1, \ldots, v_s)$.*

Proof. If there exists $x \in (v_1, \ldots, v_s)$ such that $xI(G) = I(G)$, then

$$(v_1, \ldots, v_s)I(G) = I(G)$$

and $\Lambda_{(v_1,\ldots,v_s)}(I(G)) = 0$. Moreover,

$$R \otimes_R \Lambda_{(v_1,\ldots,v_s)}(I(G)) \cong \mathrm{H}_0^{(v_1,\ldots,v_s)}(I(G)),$$

by [6, 4.3]. It follows that $\mathrm{H}_0^{(v_1,\ldots,v_s)}(I(G)) = 0$.

We now suppose that, there exists not any element $x \in (v_1, \ldots, v_s)$ such that $xI(G) = I(G)$, then $(v_1, \ldots, v_s)I(G) \neq I(G)$ and $\Lambda_{(v_1,\ldots,v_s)}(I(G)) \neq 0$. From [14, 1.21], we get

$$\mathrm{Coass}(R \otimes_R \Lambda_{(v_1,\ldots,v_s)}(I(G))) = \mathrm{V}(\mathrm{Ann}(R)) \cap \mathrm{Coass}(\Lambda_{(v_1,\ldots,v_s)}(I(G))).$$

By [5, 3.5 and 4.2], we also have

$$\mathrm{Coass}(\Lambda_{(v_1,\ldots,v_s)}(I(G))) \subseteq \mathrm{V}\left((v_1, \ldots, v_s)\right) \subseteq \mathrm{V}(\mathrm{Ann}(R)).$$

Hence,

$$\mathrm{Coass}(\mathrm{H}_0^{(v_1,\ldots,v_s)}(I(G))) = \mathrm{Coass}(R \otimes_R \Lambda_{(v_1,\ldots,v_s)}(I(G))) = \mathrm{Coass}(\Lambda_{(v_1,\ldots,v_s)}(I(G))).$$

As $\Lambda_{(v_1,\ldots,v_s)}(I(G)) \neq 0$, we have that $\mathrm{H}_0^{(v_1,\ldots,v_s)}(I(G)) \neq 0$. $\square$

Proof of the Theorem 3.7:

Assume that $(x_1, x_2, \ldots, x_n) \subseteq (v_1, v_2, \ldots, v_s)$ is a maximal $I(G)$-coregular sequence, let us use induction on n. When $n = 0$, there does not exists any x in $(v_1, \ldots, v_s)$ such that $xI(G) = I(G)$. Thus, we have that $\mathrm{H}_0^{(v_1,\ldots,v_s)}(I(G)) \neq 0$ by Lemma 3.8. Let $n > 0$. The short exact sequence of Artinian R-modules

$$0 \to (0 :_{I(G)} x_1) \to I(G) \xrightarrow{\cdot x_1} I(G) \to 0$$

gives rise to a long exact sequence of local homology modules

$$\ldots \to \mathrm{H}_i^{(v_1,\ldots,v_s)}(0 :_{I(G)} x_1) \to \mathrm{H}_i^{(v_1,\ldots,v_s)}(I(G)) \xrightarrow{\cdot x_1} \mathrm{H}_i^{(v_1,\ldots,v_s)}(I(G)) \to \mathrm{H}_{i-1}^{(v_1,\ldots,v_s)}(0 :_{I(G)} x_1) \to \ldots .$$

By the inductive hypothesis, we have $\mathrm{H}_i^{(v_1,\ldots,v_s)}(0 :_{I(G)} x_1) = 0$ for all $i < n-1$ and $\mathrm{H}_{n-1}^{(v_1,\ldots,v_s)}(0 :_{I(G)} x_1) \neq 0$. It follows that,

$$\mathrm{H}_i^{(v_1,\ldots,v_s)}(I(G)) = x_1 \mathrm{H}_i^{(v_1,\ldots,v_s)}(I(G)) \text{ for all } i < n.$$

Hence, we have that

$$\mathrm{H}_i^{(v_1,\ldots,v_s)}(I(G)) = \bigcap_{t>0} x_1^t \mathrm{H}_i^{(v_1,\ldots,v_s)}(I(G)) = 0 \text{ for all } i < n,$$

by Proposition 3.4. We now have the exact sequence

$$\ldots \to \mathrm{H}_n^{(v_1,\ldots,v_s)}(I(G)) \xrightarrow{\cdot x_1} \mathrm{H}_n^{(v_1,\ldots,v_s)}(I(G)) \to \mathrm{H}_{n-1}^{(v_1,\ldots,v_s)}(0 :_{I(G)} x_1) \to 0.$$

As $\mathrm{H}_{n-1}^{(v_1,\ldots,v_s)}(0 :_{I(G)} x_1) \neq 0$, we get $\mathrm{H}_n^{(v_1,\ldots,v_s)}(I(G)) \neq 0$. Thus, the proof is complete.

4. Local cohomology module of the edge ideal of a graph

Definition 4.1. Let J be an ideal of the ring R, and let M be an R-module. The i-th local cohomology module $\mathrm{H}_J^i(M)$ of M with respect to J is defined by

$$\mathrm{H}_J^i(M) = \varinjlim_{t \in \mathbb{N}} \mathrm{Ext}_R^i\left(R/J^t, M\right)$$

for all $i \geq 0$.

In this section, we presented a result about the local cohomology module which involve the theory of graphs together with the edge ideal of a graph G, which is simple and finite and with no isolated vertices. Here, we continue in the same context of the previous section.

We begin by proving a lemma which will be used for the proof of the main theorem of this section.

Lemma 4.2. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is an $(v_1, \ldots, v_s)$-torsion R-module. Then $\mathrm{H}^i_{(v_1,\ldots,v_s)}(I(G))$ is a finitely generated R-module for all $i \geq 0$.*

Proof. First we observe that, using the additivity of the local cohomology functor, that $\mathrm{H}^i_{(v_1,\ldots,v_s)}(F \otimes I(G))$ is finitely generated whenever F is a finitely generated free R-module.

Since R is a finitely generated R-module, we have that R can be included in an exact sequence

$$0 \to L \to F \to R \to 0,$$

of finitely generated R-modules in which F is free. Since $I(G)$ is an $(v_1, \ldots, v_s)$-torsion R-module, by [3, 2.1.6] there exists an injective resolution $E^\bullet$ of $I(G)$ in which each term is an $(v_1, \ldots, v_s)$-torsion R-module. Since all the terms of $E^\bullet$ are injective, the above sequence induces an exact sequence

$$0 \to \mathrm{Hom}_R(R, E^\bullet) \to \mathrm{Hom}_R(F, E^\bullet) \to \mathrm{Hom}_R(L, E^\bullet) \to 0$$

of complexes.

Note that, for a finitely generated R-module S and an $(v_1, \ldots, v_s)$-torsion R-module T, we have $\Gamma_{(v_1,\ldots,v_s)}(\mathrm{Hom}_R(S, T)) = \mathrm{Hom}_R(S, T)$. Hence, in view of [3, 2.1.7(i)] we obtain the following long exact sequence of local cohomology modules, which induces from the above exact sequence of complexes by applying the functor $\Gamma_{(v_1,\ldots,v_s)}$ on it

$$\begin{gathered} 0 \to \mathrm{H}^0_{(v_1,\ldots,v_s)}(I(G)) \to \mathrm{H}^0_{(v_1,\ldots,v_s)}(F \otimes I(G)) \to \mathrm{H}^0_{(v_1,\ldots,v_s)}(L \otimes I(G)) \to \\ \mathrm{H}^1_{(v_1,\ldots,v_s)}(I(G)) \to \mathrm{H}^1_{(v_1,\ldots,v_s)}(F \otimes I(G)) \to \mathrm{H}^1_{(v_1,\ldots,v_s)}(L \otimes I(G)) \to \ldots \to \\ \mathrm{H}^i_{(v_1,\ldots,v_s)}(I(G)) \to \mathrm{H}^i_{(v_1,\ldots,v_s)}(F \otimes I(G)) \to \mathrm{H}^i_{(v_1,\ldots,v_s)}(L \otimes I(G)) \to \\ \mathrm{H}^{i+1}_{(v_1,\ldots,v_s)}(I(G)) \to \ldots. \end{gathered}$$

We have, by what was put above, that $\mathrm{H}^i_{(v_1,\ldots,v_s)}(F \otimes I(G))$ is finitely generated for all $i \geq 0$. Hence $\mathrm{H}^1_{(v_1,\ldots,v_s)}(I(G))$ is finitely generated.

But R is an finitely generated R-module, and so, $\mathrm{H}^1_{(v_1,\dots,v_s)}(L \otimes I(G))$ is finitely generated. The above long exact sequence therefore shows that $\mathrm{H}^2_{(v_1,\dots,v_s)}(I(G))$ is finitely generated. Now use induction. □

We finalize with a theorem.

Theorem 4.3. *Let $R = K[v_1, \dots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let i be a non-negative integer such that $\mathrm{H}^j_{(v_1,\dots,v_s)}(I(G))$ is finitely generated for all $j < i$. Then,*

$$\mathrm{Hom}_R\left(R/(v_1, \dots, v_s), \mathrm{H}^i_{(v_1,\dots,v_s)}(I(G))\right)$$

is a finitely generated R-module.

Proof. We proceed by induction on i. The case $i = 0$ is obvious, because

$$\mathrm{H}^0_{(v_1,\dots,v_s)}(I(G)) = \Gamma_{(v_1,\dots,v_s)}\left(\mathrm{Hom}_R(R, I(G))\right)$$

is finitely generated. So, let $i \geq 1$. Consider the exact sequence

$$0 \to \Gamma_{(v_1,\dots,v_s)}(I(G)) \to I(G) \to \frac{I(G)}{\Gamma_{(v_1,\dots,v_s)}(I(G))} \to 0,$$

to deduce the exact sequence

$$\mathrm{H}^i_{(v_1,\dots,v_s)}(\Gamma_{(v_1,\dots,v_s)}(I(G))) \xrightarrow{f} \mathrm{H}^i_{(v_1,\dots,v_s)}(I(G)) \xrightarrow{g} \mathrm{H}^i_{(v_1,\dots,v_s)}\left(\frac{I(G)}{\Gamma_{(v_1,\dots,v_s)}(I(G))}\right)$$

(note that $\left(\mathrm{H}^i_{(v_1,\dots,v_s)}(\bullet)\right)_{i\geq 0}$ is a connected right sequence of covariant functors from $\mathfrak{C}(R)$ to $\mathfrak{C}(R)$ (in the sense of [11, Paragraph 6.5])). By the Lemma 4.2, $\mathrm{H}^i_{(v_1,\dots,v_s)}(\Gamma_{(v_1,\dots,v_s)}(I(G)))$ is finitely generated and so is $\mathrm{Im}(f)$. Therefore we conclude from the exact sequences

$$0 \to \mathrm{Im}(f) \to \mathrm{H}^i_{(v_1,\dots,v_s)}(I(G)) \to \mathrm{Im}(g) \to 0, \text{ and}$$
$$0 \to \mathrm{Im}(g) \to \mathrm{H}^i_{(v_1,\dots,v_s)}\left(\frac{I(G)}{\Gamma_{(v_1,\dots,v_s)}(I(G))}\right),$$

by applying the left exact functor $\mathrm{Hom}_R(R/(v_1, \dots, v_s), \bullet)$ on them, that it is enough for us to show that

$$\mathrm{Hom}_R\left(R/(v_1, \dots, v_s), \mathrm{H}^i_{(v_1,\dots,v_s)}\left(\frac{I(G)}{\Gamma_{(v_1,\dots,v_s)}(I(G))}\right)\right)$$

is finitely generated.

Hence we can (and do) assume that $I(G)$ is an $(v_1, \ldots, v_s)$-torsion-free R-module. So $(v_1, \ldots, v_s)$ contains an element x which is a non-zerodivisor on $I(G)$. Set $I(\bar{G}) := \frac{I(G)}{xI(G)}$. Now the exact sequence

$$0 \to I(G) \xrightarrow{x} I(G) \to I(\bar{G}) \to 0$$

induces an exact sequence

$$\mathrm{H}^{i-1}_{(v_1,\ldots,v_s)}(I(G)) \xrightarrow{k} \mathrm{H}^{i-1}_{(v_1,\ldots,v_s)}(I(\bar{G})) \xrightarrow{h} \mathrm{H}^{i}_{(v_1,\ldots,v_s)}(I(G)) \xrightarrow{x} \mathrm{H}^{i}_{(v_1,\ldots,v_s)}(I(G)).$$

□

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