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RESEARCH ARTICLE

Classification of the pairs of matrices of fixed Jordan types and representations of bundles of semichains

V. M. Bondarenko and O. V. Zubaruk

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Abstract. We study the problem of classifying the pairs of matrices over an algebraically closed field with some restrictions on their Jordan canonical forms using earlier results of the first author. All tame and wild, polynomial and non-polynomial growth cases are described.

1. Introduction

In this paper, we study the problem of classifying up to simultaneous similarity pairs of matrices over fields with some restrictions on their canonical forms, using results of the first author on representations of pairs of categories [1] and bundles of semichains [2, 3]. When considering various matrix problems, we use the concepts of tame and wild ones (see general definitions and theorems in [4]); these and other types of matrix problems (including polynomial and non-polynomial growth), in concrete cases closed to the considered here, are given and analyzed in [1]. Formally, cases of finite type (when the number of indecomposable objects is finite up to the corresponding equivalence) are not excluded by us from tame ones however as rule are considered separately.

Throughout the paper, we fix an algebraically closed field k and as rule omit it in the notations. All matrices are considered over k. The

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Jordan block of size m with eigenvalue a is denoted by $J_m(a)$. For a matrix $A \in M_n(k)$, we define $J_0(A)$ as the set all $(i,a) \in \mathbb{N} \times k$ such that the Jordan canonical form A^0 of A contains $J_i(a)$. The identity matrix of size m is denoted by I_m.

Let F be a finite subset of $\mathbb{N} \times k$. We say that a matrix A is of *Jordan type* F if $J_0(A) \subseteq F$. Obviously, if A and A' are of Jordan type F then the same Jordan type has also $A \oplus A'$.

The problem of classifying up to (simultaneous) similarity the pairs of matrices (A,B) with $A, B \in M_n(k)$ (n runs through $\mathbb{N}$), respectively, of Jordan type F, G will be called by us as the problem $\mathcal{PJ}(F,G)$.

Our aim is to prove the following theorems.

Theorem 1. *Let F, G be finite subsets of $\mathbb{N} \times k$. Then the problem $\mathcal{PJ}(F,G)$ is of finite type if and only if F or G consists of a single element of the form $(1,a)$.*

Theorem 2. *Let a problem $\mathcal{PJ}(F,G)$ be of infinite type. Then it is*

(1) *tame of polynomial growth if and only if $F = \{(1,a),(1,b)\}$, $G = \{(1,c),(1,d)\}$;*

(2) *tame of non-polynomial growth if and only if up to the permutation of F and G one of the following conditions holds:*

(2.1) $F = \{(1,a),(1,b)\}$, $G = \{(2,c)\}$;

(2.2) $F = \{(1,a),(1,b)\}$, $G = \{(1,c),(2,c)\}$;

(2.3) $F = \{(2,a)\}$, $G = \{(2,c)\}$;

(2.4) $F = \{(1,a),(2,a)\}$, $G = \{(2,c)\}$;

(2.5) $F = \{(1,a),(2,a)\}$, $G = \{(1,c),(2,c)\}$.

Otherwise, the problem $\mathcal{PJ}(F,G)$ is wild.

Let k_f denotes the set of roots of the polynomial $f(x)$ and m_a, where $a \in k_f$, the multiplicity of a.

From Theorem 2 follows the next theorem.

Theorem 3. *Let a problem $\mathcal{PJ}(F,G)$ be of infinite type and $\mathcal{PJ}(F',G')$ be wild for any inclusions $F \subseteq F'$, $G \subseteq G'$ at least one of which is strict. Then the problem $\mathcal{PJ}(F,G)$*

(a) *is tame if and only if, for some polynomials $f(x)$ and $g(x)$ of the second degree, $F = \{(i,a) \mid a \in k_f, 1 \le i \le m_a\}$ and $G = \{(j,b) \mid b \in k_g, 1 \le j \le m_b\}$;*

(b) *is of polynomial growth if and only if F and G are as in (a) relative to $f(x)$ and $g(x)$ without multiple roots.*

2. Representations of pairs of completed posets

Throughout, all posets are finite. When considering a partitioned matrix $U = (U_{xy})$ with blocks $U_{xy}, x \in I_1, y \in I_2$, we denote by d_x the number of its rows in the horizontal band or columns in the vertical band indexed by the element x, and puts $d(U) = (d_1, d_2)$, where $d_1 = (d_x)$ and $d_2 = (d_y)$ are the vectors with $x \in I_1$ and $y \in I_2$; in the case when $I_1 = I_2$ and $d_1 = d_2$, we identified (d_1, d_2) with d_1.

By definition, a completed poset S consists of a poset A and an equivalence relation $\sim$ on $A^{\leq} = \{(x,y) \in A \times A |\, x \leq y\}$. These data are subjected to the condition that $x \leq z \leq y$ and $(x,y) \sim (x',y')$ imply the existence of a unique z' satisfying $x' \leq z' \leq y', (x,z) \sim (x',z')$ and $(z,y) \sim (z',y')$. It is easy to see that $(x,y) \sim (x',x')$ implies $x = y$, and $(x,y) \sim (x',y')$ implies $x \sim x'$ and $y \sim y'$. The last property for $x = y$, $x' = y'$ determines an equivalence relation on A. A special case of such posets are posets with equivalence when the relation $\sim$ on $A^{<}$ is trivial (special cases of which in turn are usual posets and sets with equivalence when, respectively, $\sim$ and $\leq$ is trivial).

For a pair (P,Q) of disjoint sets with equivalence $P = (A_0, \sim_1), Q = (B_0, \sim_2)$ and vectors $d_1 = (d_x)$ with $x \in A_0$ and $d_2 = (d_y)$ with $y \in B_0$, where $d_x, d_y \in \mathbb{N} \cup 0$, we denote by $M_{d_1,d_2}(P,Q)$ the set of all partitioned matrices $U = (U_{xy})$ with blocks U_{xy}, $x \in A_0, y \in B_0$, such that

(1) U_{xy} is a $d_x \times d_y$ matrix;

(2) $d_z = d_{z'}$ if $z, z' \in A_0$ and $z \sim_1 z'$ or $z, z' \in B_0$ and $z \sim_2 z'$.

Note that $U \in M_{d_1,d_2}(P,Q)$ is of size $d_{01} \times d_{02}$, where $d_{01} = \sum_{x \in A_0} d_x$, $d_{02} = \sum_{y \in B_0} d_y$.

Let now $S = (A, \sim)$ be a completed poset and $S_0 = (A_0, \sim)$ the corresponding set with equivalence. By an S-matrix we mean a partitioned matrix $U = (U_{xy}), x, y \in A_0$, from $M_{d_1,d_2}(S_0, S_0)$ for some d_1, d_2, such that

(3) $U_{xy} = U_{x'y'}$ if $x \sim x'$ and $y \sim y'$;

(4) $U_{xy} = 0$ if $x \not\leq y$.

The set of all such matrices is denoted by $M_{d_1,d_2}(S)$ or also by $M_{d_1}(S)$ if $d_1 = d_2$.

Let $S = (A, \sim_1)$ and $T = (B, \sim_2)$ be completed posets and $S_0 = (A_0, \sim_1)$, $T_0 = (B_0, \sim_2)$ the corresponding sets with equivalence. We call *representation of* (S,T) *of dimension* $d = (d_1, d_2)$ a partitioned matrix R from $M_{d_1,d_2}(A_0, B_0)$. The representation R is called exact if all coordinates of d_1 and d_2 differ from 0. Two representations R and R' is called equivalent if there exist invertible matrices $X \in M_{d_1}(S)$ and $Y \in M_{d_2}(T)$

such that $XR = R'Y$.

We call a pair (S, T) of completed posets of finite type, tame (respectively, of inv-finite type, inv-tame) and so on if so is the problem of classification up to equivalence all (respectively, all invertible) its representations.

Theorem 4. *Let $S = (A, \sim_1)$ and $T = (B, \sim_2)$ be completed posets that are not linearly ordered with trivial equivalence relations.*

Then it is of infinite type and the following conditions are equivalent:

I. *The pair (S, T) is tame.*

II. *The pair (S, T) is inv-tame.*

III. *For each of the completed posets*

(a) *the relation on $\{(x, y) \mid x < y\}$ is trivial;*

(b) *the order of any class of equivalent elements is less than 3;*

(c) *an element that does not belong to a two-element equivalence class is incomparable with no more than one element, otherwise it is comparable with all.*

Proof. Note that a completed poset that satisfies $\mathrm{III} - a)$ and $\mathrm{III} - b)$ can be considered as a poset with involution ($x^* = y$ if either x is not equal to and is equivalent to y, or x is equal to y and is not equivalent to $z \neq x$). Then (by definition) $\mathrm{III} - c)$ means that the completed poset is a $*$-semichain. Therefore II $\Leftrightarrow$ III follows from Theorem 6 [1], and III $\Rightarrow$ I from the main classification theorem of [2] and [3, §1]. The implication I $\Leftrightarrow$ II is trivial. $\square$

From the last two mentioned papers it also follows the next theorem.

Theorem 5. *Let completed posets $S = (A, \sim_1)$ and $T = (B, \sim_2)$ be as in Theorem 4, and the condition* III *holds. Then the pair (S, T) is of polynomial growth if and if both the equivalence relations are trivial and each of the posets has the only pair of incomparable elements.*

3. Block Jordan canonical form

Let $A \in M_n(k)$ be a matrix with a fixed set $J_0(A)$ (see Introduction) and $E(A)$ the set of its eigenvalues. Denote by $r(i, a)$, where $(i, a) \in J_0(A)$, the number of Jordan block $J_i(a)$ in the Jordan canonical form A^0 of A and put $N(a) = \{i \in \mathbb{N} \mid (i, a) \in J_0(A)\}$. Then A^0 can be writen up to renumbering the rows and columns as the following partitioned matrix:

$$A^{\square} := \oplus_{a \in E(A)} \oplus_{i \in N(a)} J_i(a) \otimes I_{r(i,a)}.$$

We call $A^{\square}$ the *block Jordan canonical form of the matrix* A.

Large blocks $A^{\square}_{(i,a)(i',a')}$ of $A^{\square}$, where $(i,a),(i',a') \in J_0(A^{\square})$, are of sizes $ir(i,a) \times i'r(i',a')$ and in turn consist of the blocks $A^{\square}_{(i,a,j)(i',a',j')} := (A^{\square}{}_{(i,a)(i',a')})_{j_1 j_2}$ of sizes $r(i,a) \times r(i',a')$, where $1 \le j \le i$, $1 \le j' \le i'$. So as the result we have that $A^{\square}$ consist of the blocks $A^{\square}{}_{xy}$ with x,y running through the set $I(A^{\square}) := \{(i,a,j) \,|\, (i,a) \in J_0(A^{\square}), 1 \le j \le i\}$. By $d(A^{\square})$ we denote the vector (d_x) with x running through $I(A^{\square})$, where d_x is equal to the size of the (square) block $A^{\square}_{xx}$.

Now we determine the structure of matrices that commute with a block Jordan canonical matrix.

Let $A = A^{\square}$ be a block Jordan canonical matrix with $d := d(A)$ and $M_d(k)$ the algebra of all partitioned matrices M with $d(M) = d$. Let further $St(A)$ be the stabilizer of A by which we mean the subalgebra of all matrices $X \in M_d(k)$ such that $AX = XA$. Define the completed poset $CP_A = (P_A, \sim_A)$ as follows. The poset P_A consists of the elements from $I(A)$ and for elements $x = (i,a,j)$, $y = (i',a',j')$,

(a) $x \le y$ if and only if $a = a'$ and $i \ge i'$, $j \le j'$, or $i < i'$, $j \le j' + i - i'$;

(b) $x \sim_A y$ if and only if $(i,a) = (i',a')$.

Elements (u,v) and (u',v') of $P_A^{<} = \{(x,y) \in P_A \times P_A | \, x < y\}$ with $u = (i,a,j)$, $v = (i',a,j')$, $u' = (s,b,t)$, $v' = (s',b,t')$ are in the equivalence relation $\sim_A$ if and only if $u \sim_A u'$, $v \sim_A v'$ and $j - t = j' - t'$.

Theorem 6. *The algebra $St(A)$ coincides in $M_d(k)$ with the algebra $M_d(CP_A)$ of all CP_A-matrices.*

Proof. Let us introduce some notations for matrices over the field k. Denote by $\Delta^{=}(p)$, where $p \in \mathbb{N}$, the set of all upper triangular $p \times p$ matrices $A = (a_{ij})$ such that $a_{ij} = a_{i'j'}$ holds whenever $j - i = j' - i' \ge 0$. For $p,q \in \mathbb{N}$ denote by $\Delta^{=}(p,q)$ the set $\Delta^{=}(p)$ if $p = q$, the set of all $p \times q$ matrices of the form $\begin{pmatrix} 0 & A \end{pmatrix}$ with $A \in \Delta^{=}(p)$ if $p < q$, and of the form $\begin{pmatrix} A \\ 0 \end{pmatrix}$ with $A \in \Delta^{=}(q)$ if $p > q$. $\square$

Lemma 1 (see e.g. [5], Ch. VIII)**.** *For Jordan blocks $J_p(a)$, $J_q(b)$ and a $p \times q$ matrix X, the equality $J_p(a)X = XJ_q(b)$ holds if and only if $a = b$, $X \in \Delta^{=}(p,q)$ or $a \ne b$, $X = 0$.*

Now we replace $J_p(a)$, $J_q(b)$ on their direct sums $J_p^m(a) (= J_p(a) \oplus \ldots \oplus J_p(a)$, m summands) and $J_q^n(b)$, where $m,n \in \mathbb{N}$, and renumbering the rows and columns (in the same way) each of them, we write these matrices, reprectively, in the form $J_p(a) \otimes I_m$ and $J_q(a) \otimes I_n$ (in other words, in the above terms they are block Jordan matrices). Denote by $\Delta^{=}_{mn}(p)$ the set of upper block-triangular $pm \times pn$ matrices with blocks A_{ij} of size $m \times n$ satisfying the same equalities as a_{ij} in $\Delta^{=}(p)$. The sets $\Delta^{=}_{mn}(p,q)$ are defined similarly to $\Delta^{=}(p,q)$, however in the case $p < q$ (respectively, $p > q$) the matrix $\Delta^{=}_{mn}(p)$ should already be supplemented by $q-p$ vertical (respectively, $p-q$ horizontal) zero bands.

From Lemma 1 we have the following one.

Lemma 2. *For a $pm \times qn$ partitioned matrix X with blocks of size $m \times n$, the equality $J_p(a) \otimes I_m X = X J_q(b) \otimes I_n$ holds if and only if $a = b$, $X \in \Delta^{=}_{mn}(p,q)$ or $a \neq b$, $X = 0$.*

Let now $X \in St(A)$. We consider A and X as matrices with large block, i.e. $A = (A_{(i,a)(i',a')})$ and $X = (X_{(i,a)(i',a')})$, where $(i,a),(i',a') \in J_0(A)$. Since the matrix A is block diagonal with $A_{(i,a)(i,a)} = J_i(a) \otimes I_{r(i,a)}$, the equality $AX = XA$ is equivalent to the equalities of the form $(J_p(a) \otimes I_m)X_{(p,a)(q,b)} = X_{(p,a)(q,b)}(J_q(b) \otimes I_n)$ with (p,a), (q,b) running through $J_0(A)$ and therefore by Lemma 2 $X_{(p,a)(q,b)}$ belongs to $\Delta^{=}_{mn}(p,q)$ if $a = b$ and ie equal to 0 if $a \neq b$. And this, as is easy to see, means that $X \in M_d(CP_A)$ and moreover that $St(A)$ with $d = d(A)$ and $M_d(CP_A)$ coincide in $M_d(k)$.

4. Proof of Theorems 1 and 2

Let us associate a finite subset of $\mathbb{N} \times k$ the completed poset $CP_F = (P_F, \sim_F)$ in the same way as for a block canonical matrix $A^{\square}$, replacing $J_0(A^{\square})$ by F. Note that if a matrix A is of Jordan type $F \neq J_0(A)$, it means that the horizontal and vertical bands of $A^{\square}$ corresponding $x \in F \setminus J_0(A)$ are empty.

The following theorem will allow us to use Theorems 4 and 5 (and ultimately the results of papers [2, 3]).

Theorem 7. *Let F and G be finite subsets in $\mathbb{N} \times k$. Then the problem $\mathcal{PJ}(F,G)$ is equivalent to the problem of classifying up to equivalence the invertible representations of the pair of completed posets (CP_F, CP_G).*

Proof. Let $A, B \in M_n(k)$ (with n any fixed) be matrices of Jordan type F and G, respectively. Then the pair (A,B) is similar to the pair $T_{AB}(R) = (A^{\square}, RB^{\square}R^{-1})$, where R is an invertible $n \times n$ matrix. Since the block Jordan canonical form of a matrix is uniquely determined by the matrix itself (like the usual one), when studying pairs of matrices with the property under consideration, it is sufficient to limit ourselves to pairs of the form $T_{AB}(R)$ with R running through the group of invertible $n \times n$ matrices.

So let we have pairs $T_{AB}(R) = (A^{\Box}, RB^{\Box}R^{-1})$ and $T_{AB}(R') = (A^{\Box}, R'B^{\Box}(R')^{-1})$ and let they are similar, i.e., for some invertible $n \times n$ matrix X, hold the equations $XA^{\Box}X^{-1} = A^{\Box}$ and $XRB^{\Box}R^{-1}X^{-1} = R'B^{\Box}(R')^{-1}$ Then from the first equality it follows that $X \in St(A^{\Box})$ and from the second one that $Y := (R')^{-1}XR \in St(B^{\Box})$. Therefore $XR = R'Y$, where all the matrices are invertible and $X \in St(A^{\Box}), Y \in St(B^{\Box})$, and it is natural to consider X and Y as block matrices the divisions of which into horizontal and vertical bands are consistent with the divisions of matrices $A^{\Box}$ and $B^{\Box}$. By Theorem 6 $St(A^{\Box})$ coincides with the algebra of all CP_F-matrices and $St(B^{\Box})$ of all CP_G-matrices, and hence the equation $XR = R'Y$ means that R and R' are equivalent as invertible representations of the pair of completed posets (CP_F, CP_G). The opposite statement – i.e., that the equivalence of $T_{AB}(R)$ and $T_{AB}(R')$ follows from the equivalence of R and R' as representations of (CP_F, CP_G) – is obvious. $\Box$

Note that the construction given in the proof preserves types and growth of the both problems what can be proved by the standard method of the representation theory.

Now Theorems 1 and 2 follow from the two theorems of Section 1 and elementary considerations concerning connections between properties of completed posets CP_F and sets F.

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Contact information

Vitaliy M. Bondarenko Institute of Mathematics, Tereshchenkivska str., 3, 01024, Kyiv, Ukraine
E-Mail: vitalij.bond@gmail.com
URL:

Olesya V. Zubaruk Kyiv National Taras Shevchenko Univ., Volodymyrska str., 64, 01033 Kyiv, Ukraine
E-Mail: Sambrinka@ukr.net
URL:

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