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RESEARCH ARTICLE

On a finite state representation of $GL(n, \mathbb{Z})$

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Dedicated to the 75th birthday of Professor V. M. Bondarenko

Abstract. It is examined finite state automorphisms of regular rooted trees constructed in [6] to represent groups $GL(n, \mathbb{Z})$. The number of states of automorphisms that correspond to elementary matrices is computed. Using the representation of $GL(2, \mathbb{Z})$ over an alphabet of size 4 a finite state representation of the free group of rank 2 over binary alphabet is constructed.

1. Introduction

Representations of residually finite groups and semigroups by automorphisms and endomorphisms of regular rooted trees is an attractive and challenging research direction. It is inspired mainly by brilliant examples of infinite finitely generated periodic groups constructed as automorphism groups of rooted trees ([14, 15, 16]). Results on ubiquity of free groups and semigroups in automorphism groups of rooted trees ([7, 8]) stimulated explicit representations of groups and semigroups containing free subgroups and subsemigroups. Among them, in [6] a natural and brilliant representation of groups $GL(n, \mathbb{Z})$, $n > 1$, by finite state automorphisms of 2^n-regular rooted tree was found.

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The purpose of this note is further investigation of the construction presented in [6]. For elementary matrices in $GL(n, \mathbb{Z})$, $n > 1$, it is calculated the number of states of finite state automorphisms corresponding to them. An algorithm for constructing a finite state automorphism by a given unimodular matrix is discussed and implemented. Finally, a method from [1, 9] is applied to construct a quite surprising representation of the free group of rank 2 by finite state automorphisms of a regular rooted tree based on the representation of $GL(2, \mathbb{Z})$ (cf. with Section 7 of [17]).

The organization of the paper is follows. In Section 2, required concepts and properties related to finite state automorphisms of regular rooted trees are briefly reviewed. More details on rooted trees, their automorphisms and in particular finite state automorphisms can be found in [2, 3, 4, 5]. Section 3 contains computations of the number of states of automorphisms corresponding to elementary unimodular matrices. In Section 4, a representation of the free group of rank 2 is constructed.

2. Rooted trees and their automorphisms

Let $\mathcal{T}_n$, $n > 1$, be a rooted n-regular tree. Denote by X the set of vertices of $\mathcal{T}_n$, connected with the root. Then $|\mathsf{X}| = n$. It is convenient to treat the tree $\mathcal{T}_n$ as the (right) Cayley graph of the free monoid X^* with basis X. From this point of view each vertex of $\mathcal{T}_n$ is a finite word over X, the root is the empty word Λ. Two words u, v are connected by an edge if and only if $u = vx$ or $v = ux$ for some $x \in \mathsf{X}$.

The automorphism group $Aut\mathcal{T}_n$ of $\mathcal{T}_n$ is a permutational wreath product

$$Sym(\mathsf{X}) \wr Aut\mathcal{T}_n$$

of the symmetric group on X with $Aut\mathcal{T}_n$ itself. Each automorphism $g \in Aut\mathcal{T}_n$ can be uniquely defined by a permutation $\sigma_g \in Sym(\mathsf{X})$ and a multiset $g_x \in Aut\mathcal{T}_n, x \in \mathsf{X}$ that form the so-called wreath recursion

$$g = (g_x, x \in \mathsf{X})\sigma_g.$$

The right action of g on vertices of $\mathcal{T}_n$ can be written recursively as follows

$$(xw)^g = x^{\sigma_g} w^{g_x}, \quad x \in \mathsf{X}, w \in \mathsf{X}^*.$$

The permutation σ_g is called the rooted permutation of g. Automorphisms $g_x, x \in \mathsf{X}$ are called states of the first level of g. Using wreath recursions the product of automorphisms

$$g = (g_x, x \in \mathsf{X})\sigma_g, \quad h = (h_x, x \in \mathsf{X})\sigma_h$$

can be expressed as

$$gh = (g_x h_{x^{\sigma_g}}, x \in \mathsf{X})\sigma_g \sigma_h.$$

The identity automorphism will be denoted by e.

For an arbitrary vertex $v \in \mathsf{X}^*$ the state of g at v is a uniquely defined automorphism g_v such that

$$(vw)^g = v^g w^{g_v}, \quad w \in \mathsf{X}^*.$$

The set $Q(g) = \{g_v : v \in \mathsf{X}^*\}$ is called the set of states of g. Since $g_\Lambda = g$ the automorphism g is its state as well. If $Q(g)$ is finite then the automorphism g is called the finite state. All finite state automorphisms of $\mathcal{T}_n$ form a countable subgroup $FAut\mathcal{T}_n$ in $Aut\mathcal{T}_n$. We say that a group G has a finite state representation if it is isomorphic to a subgroup of $FAut\mathcal{T}_n$ for some n. The self-similar closure of an automorphism g is a subgroup of $Aut\mathcal{T}_n$ generated by the set $Q(g)$. The multiplication rule for automorphisms imply $Q(g^{-1}) = \{h^{-1} : h \in Q(g)\}$.

For an arbitrary automorphisms g, h and a vertex v the state of their product gh at v is the product $g_v h_{v^g}$. In particular, the set $Q(gh)$ of states of the product gh is a subset of the product $Q(g)Q(h)$ of multipliers' sets of states.

Each automorphism $g \in Aut\mathcal{T}_n$ can be defined by its Moore diagram, i.e. a directed graph with $Q(g)$ as the vertex set. The vertex g is marked. For arbitrary state $h \in Q(g)$ the Moore diagram of g has exactly n labelled arrows starting from h. For each $x \in \mathsf{X}$ exactly one arrow starts in h and terminates in h_x. It has a label of the form $x|x^{\sigma_h}$.

3. Finite state representation of $GL(n,\mathbb{Z})$

Let $n > 1$. In [6], the authors constructively proved that the group $GL(n,\mathbb{Z})$ is isomorphic to a subgroup of $FAut\mathcal{T}_{2^n}$. Let us recall this embedding. We will identify the vertex set of $\mathcal{T}_{2^n}$ with the set of all finite words over the vector space $\mathbb{Z}_2^n$ of dimension n over the binary field $\mathbb{Z}_2$. Define the following permutations $\tau, \sigma, \pi_{i,j}$, $1 \le i < j \le n$:

$$(x_1, x_2, \ldots, x_n)^\tau = (x_1 + x_2, x_2, \ldots, x_n),$$

$$(x_1, x_2, \ldots, x_n)^{\sigma} = (x_1 + 1, x_2, \ldots, x_n),$$

$$(x_1, \ldots, x_i, \ldots, x_j, \ldots x_n)^{\pi_{ij}} = (x_1, \ldots, x_j, \ldots, x_i, \ldots x_n),$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_2^n$. Then all these permutations and the product $\tau\sigma$ are involutions. Consider automorphisms t_1, t_2, $s_{i,j}$,$1 \leq i < j \leq n$, of $Aut\mathcal{T}_{2^n}$, defined by the following wreath recursions:

$$t_1 = (t_{1(x_1,x_2,\ldots,x_n)}, (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_2^n)\tau,$$

$$t_2 = (t_{2(x_1,x_2,\ldots,x_n)}, (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_2^n)\tau\sigma$$

$$s_{ij} = (s_{ij(x_1,x_2,\ldots,x_n)}, (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_2^n)\pi_{ij},$$

where

$$t_{1(x_1,x_2,\ldots,x_n)} = \begin{cases} t_2, & \text{if } x_1 = x_2 = 1 \\ t_1, & \text{otherwise} \end{cases},$$

$$t_{2(x_1,x_2,\ldots,x_n)} = \begin{cases} t_1, & \text{if } x_1 = 1, x_2 = 0 \\ t_2, & \text{otherwise} \end{cases},$$

$s_{ij(x_1,x_2,\ldots,x_n)} = s_{ij}$. Then $Q(t_1) = Q(t_2) = \{t_1, t_2\}$, $Q(s_{ij}) = \{s_{ij}\}$.

It is shown in [6], that the subgroup of $FAut\mathcal{T}_{2^n}$ generated by the set $\{t_1, s_{ij}, 1 \leq i < j \leq n\}$ is isomorphic to $GL(n, \mathbb{Z})$. More precisely, the required isomorphism is defined as follows. Denote by $T_{ij}(k)$ the elementary $n \times n$ matrix obtained from the identity matrix by adding the ith column multiplied by k to the jth column, $1 \leq i, j \leq n$, $i \neq j$, $k \in \mathbb{Z}$, $k \neq 0$, and by E_i the elementary matrix obtained from the identity by multiplying its ith column by -1, $1 \leq i \leq n$. Denote by E_{ij} the permutation matrix that correspond to the transposition (ij), $1 \leq i < j \leq n$. Then the mapping φ_n that sends elementary matrix $T_{21}(1)$ to t_1 and permutation matrix E_{ij} to s_{ij}, $1 \leq i < j \leq n$, defines the required isomorphic embedding.

This construction gives rise to the following natural algorithm of constructing a finite state automorphism $\varphi_n(A)$ corresponding to a given matrix $A \in GL(n, \mathbb{Z})$:

1. factorize A as a product of elementary matrices $F_1 \ldots F_m$;

2. compute finite state automorphisms $\varphi(F_i)$, $1 \leq i \leq m$;

3. compute $\varphi(A)$ as the product $\varphi(F_1) \ldots (F_m)$.

We implemented this algorithm using `GAP` ([10]) and `AutomGrp`[11] package.

In order to estimate the number of states of $\varphi_n(A)$ we examine the subgroup generated by automorphisms t_1, t_2.

Lemma 1. *Automorphisms t_1 and t_2 commute.*

Proof. Denote by g and h the products t_1t_2 and t_2t_1 correspondingly. Then their rooted permutations are $\tau\tau\sigma$ and $\tau\sigma\tau$. Each of them equals σ. For arbitrary $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_2^n$ states of the first level of g and h at $(x_1, x_2, \ldots, x_n)$ have the form

$$g_{(x_1,x_2,\ldots,x_n)} = \begin{cases} t_1^2, & \text{if } x_1 = 1, x_2 = 0 \\ t_2^2, & \text{if } x_1 = 1, x_2 = 1 \\ t_1t_2, & \text{otherwise} \end{cases},$$

$$h_{(x_1,x_2,\ldots,x_n)} = \begin{cases} t_1^2, & \text{if } x_1 = 1, x_2 = 0 \\ t_2^2, & \text{if } x_1 = 1, x_2 = 1 \\ t_2t_1, & \text{otherwise} \end{cases}.$$

Since rooted permutations of g and h are equal we obtain by induction the equality $g = h$. The proof is complete. □

Lemma 2. *Let $n \geq 1$ and $Q = \{t_1^n, t_1^{n-1}t_2, \ldots, t_1t_2^{n-1}, t_2^n\}$. Then automorphisms from Q are pairwise different and for arbitrary $g \in Q$ the set of states of g is Q. In particular, automorphisms t_1^n and t_2^n have exactly $n+1$ states.*

Proof. Let $g = t_1^{2k_1+\varepsilon_1}t_2^{2k_2+\varepsilon_2}$ for non-negative integers k_1, k_2 and $\varepsilon_1, \varepsilon_1 \in \{0, 1\}$. Using induction on n and Lemma 1 one can directly verify that for arbitrary $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_2^n$ the states of the first level of g at $(x_1, x_2, \ldots, x_n)$ has the form

$$g_{(x_1,x_2,\ldots,x_n)} = \begin{cases} t_1^{2k_1+\varepsilon_1+k_2}t_2^{k_2+\varepsilon_2}, & \text{if } x_1 = x_2 = 0 \\ t_1^{2k_1+\varepsilon_1+k_2+\varepsilon_2}t_2^{k_2}, & \text{if } x_1 = 1, x_2 = 0 \\ t_1^{k_1+\varepsilon_1}t_2^{k_1+2k_2+\varepsilon_2}, & \text{if } x_1 = 0, x_2 = 1 \\ t_1^{k_1}t_2^{k_1+\varepsilon_1+2k_2+\varepsilon_2}, & \text{if } x_1 = x_2 = 1 \end{cases}. \quad (1)$$

Since $t_1 \neq t_2$ by induction on k using (1) one obtains inequality $t_1^k \neq t_2^k$, $k \geq 1$. Hence, the automorphisms from Q are pairwise different and $|Q| = n + 1$.

Since states of the first level of g belong to Q the inclusion $Q(g) \subseteq Q$ holds. To prove the equality it is sufficient to show that the Moore diagram of g is a strongly connected graph. Direct computations show that this statement holds for $n \leq 6$. Then we assume that $n > 6$.

Using states at $(1, 0, \ldots, 0)$ one constructs a directed path from g to $t_1^{n-1}t_2$ and t_1^n. Now it is sufficient to show that for an arbitrary l, $0 \leq l \leq n$, the state $t_1^l t_2^{n-l}$ is accessible from t_1^n. It follows from (1) that the accessibility of $t_1^l t_2^{n-l}$ implies accessibility of $t_1^{n-l}t_2^l$. This property will be called the symmetricity.

Assume on the contrary that n is the least positive integer such that there exists not accessible states from Q. Let k be an integer such that the state $t_1^k t_1^{n-k}$ is not accessible. Then $0 < k < n$. Since the statement about accessibility holds for $n-1$ the state $t_1^k t_1^{n-k-1}$ is accessible from t_1^{n-1}. As soon as each state of the product is a product of states of the multipliers it means that in the product $t_1^{n-1} \cdot t_1$ the state $t_1^k t_1^{n-k-1}$ is multiplied by t_1 only. Hence, the state $t_1^{k+1} t_1^{n-k-1}$ is accessible. This property will be called the inconsistency.

Assume now that l is the least number such that $t_1^l t_1^{n-l}$ is not accessible. Then the symmetricity implies $l < n/2$. Since $t_1 t_2^{n-1}$ is accessible at least one of the states $t_1^2 t_2^{n-2}$ and $t_1^3 t_2^{n-3}$ is accessible. In both cases $t_1^2 t_2^{n-2}$ is accessible. Hence $2 \leq l$. Applying (1) for cases $x_2 = 0$ one obtains that the states

$$t_1^{2l+1} t_2^{n-2l-1}, \quad t_1^{2l+2} t_2^{n-2l-2}$$

are not accessible. This contradicts with the inconsistency. The proof is complete. □

Proposition 1. *The self-similar closure of each of automorphisms t_1 and t_2 is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

Proof. The statement immediately follows from Lemma 1 and Lemma 2. □

Theorem 1. *Let $A \in GL(n, \mathbb{Z})$. Denote by $m(A)$ the number of states of the automorphism $\varphi_n(A) \in FAut\mathcal{T}_{2^n}$.*

1. *If A is a permutation matrix then $m(A) = 1$.*
2. *If $A = E_i$, $1 \leq i \leq n$, then $m(A) = 8$.*
3. *If $A = T_{ij}(k)$, $1 \leq i, j \leq n$, $i \neq j$, $k \in \mathbb{Z}$, $k \neq 0$, then $m(A) = |k| + 1$.*

Proof. 1. Definition of φ_n implies $m(E_{ij}) = 1$ for a permutation matrix E_{ij}, $1 \le i < j \le n$. Since the product of automorphisms with exactly 1 state has 1 state the required equality holds for arbitrary permutation matrix.

2. Since the automorphism t_1 has 2 states its inverse

$$t_1^{-1} = \varphi_n(T_{21}(-1))$$

has 2 states as well. Then the equality

$$E_1 = T_{21}(1) \cdot E_{12} \cdot T_{21}(-1) \cdot E_{12} \cdot T_{21}(1) \cdot E_{12}.$$

implies $m(E_1) \le 8$. Direct verification shows that the equality holds. Since $E_i = E_{1i} \cdot E_1 \cdot E_{1i}$ one obtains $m(E_i) = 8$, $2 \le i \le n$.

3. Since $\varphi_n(T_{21}(1)) = t_1$ Lemma 2 implies $m(T_{21}(k)) = k + 1$, $k > 0$. The inverse automorphism t_1^{-k} equals $\varphi_n(T_{21}(-k))$ and has $k + 1$ states as well. Hence $m(T_{21}(-k)) = k + 1$, $k > 0$.

Then from the equalies

$$T_{12}(k) = E_{12} \cdot T_{21}(k) \cdot E_{12},$$

$$T_{i1}(k) = E_{2i} \cdot T_{21}(k) \cdot E_{2i}, \quad 3 \le i \le n,$$

$$T_{2j}(k) = E_{1j} \cdot T_{21}(k) \cdot E_{1j}, \quad 3 \le j \le n,$$

$$T_{ij}(1) = E_{2i} \cdot E_{1j} \cdot T_{21}(1) \cdot E_{1j} \cdot E_{2i}, \quad 3 \le i, j \le n, i \ne j,$$

it follows $m(T_{ij}(k)) = |k| + 1$, $1 \le i, j \le n$, $i \ne j$, $k \in \mathbb{Z}$, $k \ne 0$. □

This theorem together with a factorization of a matrix $A \in GL(n,\mathbb{Z})$ in a product of elementary matrices give an upper estimation on the size of $Q(\varphi_n(A))$. In particular, from the third statement of the theorem we immediately have

Corollary 1. *Let $A \in SL(n,\mathbb{Z})$ be a triangular matrix, $A = (a_{ij})_{i,j=1}^n$. Then*

$$|Q(\varphi_n(A))| \le \prod_{i \ne j} (1 + |a_{ij}|).$$

In general, to obtain such an estimation a factorization is required. Moreover, such a factorization strongly depends on an algorithm applied. For instance, it is shown in [13] that for $n \ge 3$ each matrix $A \in SL(n,\mathbb{Z})$ is a product of at most $(3n^2 - n)/2 + 36$ elementary matrices. However, elementary multipliers of the form $T_{ij}(k)$ may contain enormous k.

4. Finite state representation of a free group

Let $n = 2$. We will show how the isomorphic embedding φ_2 of $GL(2, \mathbb{Z})$ in $FAut\mathcal{T}_4$ gives rise to an isomorphic embedding of the free group of rank 2 in $FAut\mathcal{T}_2$. Let $\mathsf{X} = \{0, 1\}$ be the set of vertices of the first level of $\mathcal{T}_2$. Consider finite state automorphisms $a, d \in FAut\mathcal{T}_2$ defined by their Moore diagrams, see Figure 1 and Figure 2 correspondingly.

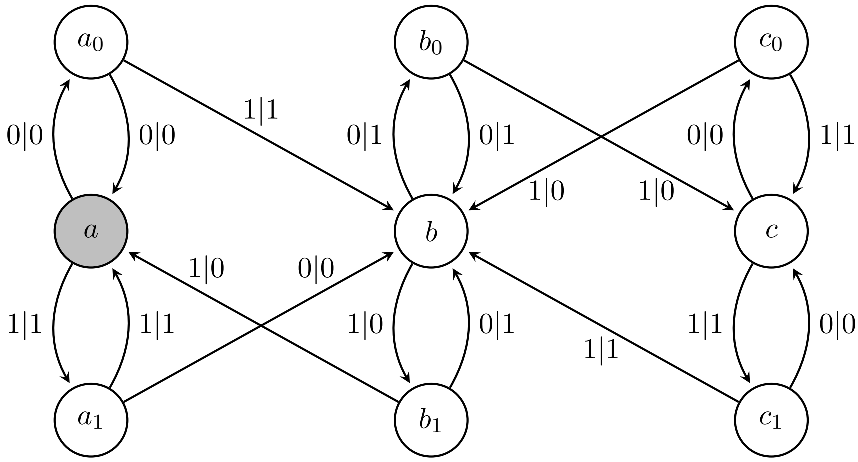

Figure 1: Moore diagram $\mathcal{A}_1$ that defines generator a of the free group

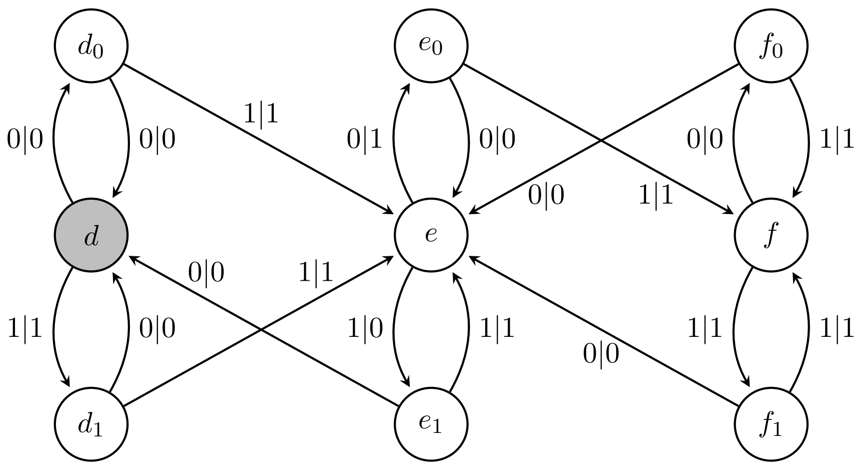

Figure 2: Moore diagram $\mathcal{A}_2$ that defines generator d of the free group

Theorem 2. *The group generated by finite state automorphisms a and d is free of rank 2.*

Proof. Let $\mathsf{X} = \{1,2,3,4\}$. To simplify notation we use the following bijection from $\mathbb{Z}_2^2$ to X:

$$(0,0) \mapsto 1, (1,0) \mapsto 2, (0,1) \mapsto 3, (1,1) \mapsto 11.$$

Then the isomorphism φ_2 maps the matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL(2,\mathbb{Z})$$

to the finite state automorphisms $t_1, s_1 \in FAut\mathcal{T}_4$ such that

$$t_1 = (t_1,t_1,t_1,t_2)(34), \quad t_2 = (t_2,t_1,t_2,t_2)(12),$$

$$s_1 = (s_1,s_1,s_1,s_2)(24), \quad s_2 = (s_2,s_2,s_1,s_2)(13).$$

Here $s_1 = s_{12} \cdot t_1 \cdot s_{12}$. Since the matrices

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

generate a free subgroup of rank 2 in $GL(2,\mathbb{Z})$ (see [12]) then their images under φ_2, finite state automorphisms t_1^2 and s_1^2, generate a free subgroup of rank 2 in $FAut\mathcal{T}_4$. Direct computations show that

$$t_1^2 = (t_1^2, t_1^2, t_1t_2, t_2t_1), \quad t_2^2 = (t_2t_1, t_1t_2, t_2^2, t_2^2),$$

$$t_1t_2 = t_2t_1 = (t_1t_2, t_1^2, t_1t_2, t_2^2)(12)(34),$$

and

$$s_1^2 = (s_1^2, s_1s_2, s_1^2, s_2s_1), \quad s_2^2 = (s_2s_1, s_2^2, s_1s_2, s_2^2),$$

$$s_1s_2 = s_2s_1 = (s_1s_2, s_1s_2, s_1^2, s_2^2)(13)(24).$$

Consider the bijection between the set of vertices of the first level of $\mathcal{T}_4$ and the set of vertices of the second level of $\mathcal{T}_2$ defined by the rule

$$1 \mapsto 00, 2 \mapsto 11, 3 \mapsto 10, 4 \mapsto 01.$$

It defines an injection f from the vertex set of $\mathcal{T}_4$ to $\mathcal{T}_2$. Then one directly verifies that for an arbitrary vertex v of $\mathcal{T}_4$ the following equalities hold

$$f(v^{t_1^2}) = (f(v))^a, \quad f(v^{t_1t_2}) = (f(v))^b, \quad f(v^{t_2^2}) = (f(v))^c,$$

$$f(v^{s_1^2}) = (f(v))^d, \quad f(v^{s_1 s_2}) = (f(v))^d, \quad f(v^{s_2^2}) = (f(v))^f.$$

It means that the groups generated by the sets $\{t_1, s_1\}$ and $\{a, d\}$ are isomorphic as permutation groups. In particular, the latter group is free of rank 2. The proof is complete. □

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