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# Combinatorial properties of non-serial posets with positive Tits quadratic form

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Abstract. We study properties of partially ordered sets with the positive Tits quadratics forms, associated with the calculation of their transitivity coefficients. Such posets, which were classified by the authors earlier, are analogs of Dynkin diagrams in the set of all finite posets.

## Introduction

The Tits quadratic forms play an important role in the modern representation theory and its applications. They were first introduced by P. Gabriel [1] for finite quivers (directed graphs) in connection with the introduction of the concept of their representations. He also proved that a quiver is of finite representation type over a field if and only if its Tits quadratic form is positive. On the other hand, simple calculations show that the positivity condition for a connected quiver holds if and only if its underlying undirected graph is a (simply faced) Dynkin diagram.

In the case of posets, the representations of which were introduce by L. A. Nazarova and A. V. Roiter [2], we have a different situation. The set of all finite posets of finite representation type form a "boundless set", which on a combinatorial language can only be characterized with the help of critical posets (i.e. those of infinite type with all proper full subposets to be of finite type). The critical posets were described by

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M. M. Kleiner in [3]; they are self-dual and their number is 5 (up to isomorphism). The Tits quadratic forms for posets first appeared in the paper [4] in which Yu. A. Drozd proved that a poset is of finite representation type if and only if its Tits quadratic form is weakly positive (i.e. positive for all nonzero vectors with non-negative coordinates). And so the Kleiner's posets are also critical with respect to weakly positivity of the Tits form.

In contrast to the quivers, the sets of posets with weakly positive and with positive Tits quadratic forms do not coincide. Therefore the investigations (from different perspectives) of posets with positive Tits form as analogs of the Dynkin diagrams are natural. They were studied by the authors in many papers (see e.g. [5] – [9]). In particular, the classification of posets with positive Tits quadratic form was first obtained in [5] (for width 2 in [6]); see also below Section 4. If one talks about critical posets with respect to positivity of the Tits quadratic form (which were first classified in [5]), their number is much more than those with respect to weakly positivity, namely 75 up to isomorphism and duality; see also Tables 1 and 2 in [7].

This paper is devoted to study of combinatorial properties of posets with positive Tits quadratic forms (which clarify and generalize some calculations in a partial case [10]).

## 1. Main results

We consider only finite posets (without elements denoted as 0). For a poset $S = (A, \prec)$ we will not write the set $A$ and keep to the following conventions: by a subset $S'$ of $S$ we mean a subset $A'$ of $A$ together with the induced order relation (also denoted by the symbol $\prec$), and we write $x \in S$ instead of $x \in A$, etc. By $S^{\mathrm{op}}$ we denote the poset dual to $S$ (i.e. $S^{\mathrm{op}} = S$ as usual sets and $x \prec y$ in $S^{\mathrm{op}}$ if and only if $x \succ y$ in $S$).

Linear ordered sets of order $n$ are also called *chains of length* $n$, and the maximum length $h(S)$ of a chain of a poset $S$ is called its *height*. Posets of order $n$ with pairwise incomparable elements are called *antichains of length* $n$, and the maximum length $w(S)$ of an antichain of $S$ is called its *width*. By Dilworth's theorem, a poset of width $w$ has a partition into $w$ chains.

We call a poset $S$ *positive* if so is its Tits quadratic form

$$q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j,i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i.$$

A positive $S$ is called *serial* positive if there is an infinite increasing sequence $S \subset S^{(1)} \subset S^{(2)} \subset \ldots$ with positive terms, and *non-serial* if otherwise. A minimal poset $S$ with non-positive $q_S(z)$ is called *$P$-critical*. Obviously, $S$ and $S^{\mathrm{op}}$ simultaneously satisfy or do not satisfy the specified properties (defined by the authors in [5].

Let $S$ be a poset and $S^2_{\prec} := \{(x, y) \mid x, y \in S, x \prec y\}$. Elements $x$ and $y$ are called *neighboring* if $(x, y) \in S^2_{\prec}$ and there no $z$ satisfying $x \prec z \prec y$. Denote by $n_w = n_w(S)$ the order of the set $S^2_{\prec}$ and by $n_e = n_e(S)$ the number of pairs $(x, y)$ of neighboring elements of $S$. On the language of the Hasse diagram $H(S)$ of $S$ (that represents $S$ in the plane), $n_e$ is equal to the number of all its edges and $n_w$ to the number of all its ways, up to parallelity, going bottom-up (two ways are called *parallel* if they start and terminate at the same vertices). The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and $n_w$, which is the probability that comparable elements of $S$ are not neighboring, is called the *coefficient of transitivity of $S$*; for $n_w = 0$, one assumes that $k_t = 0$ [11]. Obviously, dual posets have the same coefficient of transitivity.

**Theorem 1.** *Let $S$ and $T$ be non-serial positive posets. Then*

(1) $k_t(T) > k_t(S)$ *if* $h(T) > h(S) + 1$;

(2) $k_t(T) > k_t(S) - \frac{1}{20}$ *if* $h(T) = h(S) + 1$.

**Theorem 2.** *Let $S$ and $T$ be non-serial positive posets. Then*

(3) $k_t(T) \geq k_t(S)$ *if* $w(T) = w(S) = 3$ *and* $h(T) > h(S)$;

(4) $k_t(T) \geq k_t(S)$ *if* $w(T) = w(S) = 2$, $h(T) > h(S)$ *and the Hasse diagram of $T$ is not a cycle.*

An analog of Theorem 1 for $P$-critical posets (with 1/10 instead of 1/20) is formulated and proved in [12]. We prove our theorems here using the same scheme. In particular, we calculate the transitivity coefficients of all non-serial positive posets, what is interesting in itself.

## 2. Classification of non-critical positive posets

For subsets $X, Y$ of a poset $S$, we denote by $X \sqcup Y$ their direct sum (i.e. such union that elements from the different subposets are incomparable). From Dilworth's theorem it follows that any poset can be represented in the form $\sqcup_{i=1}^{m} X_i$ with $X_i$ being chains and additional relations $y < z$ for $y$

and $z$ belonging to different components. $A_s, B_s, C_s$ denote, respectively, the chains $a_1 < \ldots < a_s$, $b_1 < \ldots < b_s$, $c_1 < \ldots < c_s$.

The positive non-serial posets were classified by the authors in [5] in the terms of their Hasse diagrams. We will indicate these diagrams below in Section 4, sorting them by their order (which can be only equal to 5, 6 or 7). The next three theorems are a set-theoretic reformulation of our classification ($m$ in parentheses means the corresponding number from [5] and $m^{op}$ means that one must takes the poset dual to that with number $m$).

**Theorem 3.** *The non-serial posets of order 5 are exhausted, up to isomorphism and duality, by the following 10 posets:*

$NSP5.1(3)$ $A_2 \sqcup B_3,\ a_1 \prec b_2;$
$NSP5.2(4)$ $A_2 \sqcup B_3,\ a_2 \prec b_3;$
$NSP5.3(5)$ $A_2 \sqcup B_3,\ a_1 \prec b_2,\ a_2 \prec b_3;$
$NSP5.4(1)$ $A_1 \sqcup B_4,\ a_1 \prec b_3;$
$NSP5.5(2)$ $A_2 \sqcup B_3,\ a_1 \prec b_1,\ a_2 \prec b_3;$
$NSP5.6(46)$ $A_1 \sqcup B_2 \sqcup C_2;$
$NSP5.7(48)$ $A_1 \sqcup B_2 \sqcup C_2,\ b_1 \prec c_2;$
$NSP5.8(49)$ $A_1 \sqcup B_2 \sqcup C_2,\ a_1 \prec b_2,\ b_1 \prec c_2;$
$NSP5.9(47)$ $A_1 \sqcup B_1 \sqcup C_3,\ b_1 \prec c_3;$
$NSP5.10(50)$ $A_1 \sqcup B_3 \sqcup C_1,\ a_1 \prec b_3,\ b_1 \prec c_1;$

**Theorem 4.** *The non-serial posets of order 6 are exhausted, up to isomorphism and duality, by the following 32 posets:*

$NSP6.1(12)$ $A_3 \sqcup B_3,\ a_1 \prec b_2;$
$NSP6.2(20)$ $A_3 \sqcup B_3,\ a_1 \prec b_2,\ a_2 \prec b_3;$
$NSP6.3(10)$ $A_2 \sqcup B_4,\ a_1 \prec b_2;$
$NSP6.4(11)$ $A_2 \sqcup B_4,\ a_1 \prec b_3;$
$NSP6.5(13)$ $A_2 \sqcup B_4,\ a_2 \prec b_4;$
$NSP6.6(14)$ $A_2 \sqcup B_4,\ a_1 \prec b_2,\ a_2 \prec b_3;$
$NSP6.7(16)$ $A_2 \sqcup B_4,\ a_1 \prec b_2,\ a_2 \prec b_4;$
$NSP6.8(18)$ $A_2 \sqcup B_4,\ a_1 \prec b_3,\ a_2 \prec b_4;$
$NSP6.9(19^{op})$ $A_3 \sqcup B_3,\ a_1 \prec b_2,\ a_3 \prec b_3;$
$NSP6.10(14^{op})$ $A_3 \sqcup B_3,\ a_2 \prec b_2,\ a_3 \prec b_3;$
$NSP6.11(15)$ $A_3 \sqcup B_3,\ a_1 \prec b_1,\ a_2 \prec b_2,\ a_3 \prec b_3;$
$NSP6.12(6)$ $A_1 \sqcup B_5,\ a_1 \prec b_3;$
$NSP6.13(8)$ $A_1 \sqcup B_5,\ a_1 \prec b_4;$
$NSP6.14(7)$ $A_2 \sqcup B_4,\ a_1 \prec b_1,\ a_2 \prec b_3;$
$NSP6.15(9)$ $A_2 \sqcup B_4,\ a_1 \prec b_1,\ a_2 \prec b_4;$
$NSP6.16(58)$ $A_2 \sqcup B_2 \sqcup C_2,\ b_1 \prec c_2;$
$NSP6.17(66)$ $A_2 \sqcup B_2 \sqcup C_2,\ a_1 \prec b_2,\ b_1 \prec c_2;$
$NSP6.18(51)$ $A_1 \sqcup B_2 \sqcup C_3;$

$NSP6.19(55)$ $A_1 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_2;$
$NSP6.20(56)$ $A_1 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_3;$
$NSP6.21(57)$ $A_2 \sqcup B_1 \sqcup C_3,\ b_1 \prec c_3;$
$NSP6.22(60)$ $A_1 \sqcup B_2 \sqcup C_3,\ a_1 \prec b_2,\ b_1 \prec c_2;$
$NSP6.23(61)$ $A_1 \sqcup B_2 \sqcup C_3,\ a_1 \prec b_2,\ b_1 \prec c_3;$
$NSP6.24(62)$ $A_1 \sqcup B_3 \sqcup C_2,\ a_1 \prec b_3,\ b_1 \prec c_1;$
$NSP6.25(63)$ $A_1 \sqcup B_3 \sqcup C_2,\ a_1 \prec b_3,\ b_1 \prec c_2;$
$NSP6.26(59)$ $A_1 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_2,\ b_2 \prec c_3;$
$NSP6.27(67)$ $A_1 \sqcup B_3 \sqcup C_2,\ a_1 \prec b_3,\ b_1 \prec c_1,\ b_2 \prec c_2;$
$NSP6.28(52)$ $A_1 \sqcup B_1 \sqcup C_4,\ b_1 \prec c_3;$
$NSP6.29(54)$ $A_1 \sqcup B_1 \sqcup C_4,\ b_1 \prec c_4;$
$NSP6.30(64)$ $A_1 \sqcup B_4 \sqcup C_1,\ a_1 \prec b_3,\ b_1 \prec c_1;$
$NSP6.31(65)$ $A_1 \sqcup B_4 \sqcup C_1,\ a_1 \prec b_4,\ b_1 \prec c_1;$
$NSP6.32(53)$ $A_1 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_1,\ b_2 \prec c_3;$

**Theorem 5.** *The non-serial posets of order 7 are exhausted, up to isomorphism and duality, by the following 66 posets:*

$NSP7.1(29)$ $A_3 \sqcup B_4,\ a_1 \prec b_3;$
$NSP7.2(30)$ $A_3 \sqcup B_4,\ a_2 \prec b_4;$
$NSP7.3(42)$ $A_3 \sqcup B_4,\ a_1 \prec b_2,\ a_2 \prec b_4;$
$NSP7.4(43)$ $A_3 \sqcup B_4,\ a_1 \prec b_3,\ a_2 \prec b_4;$
$NSP7.5(44)$ $A_3 \sqcup B_4,\ a_1 \prec b_3,\ a_3 \prec b_4;$
$NSP7.6(45)$ $A_3 \sqcup B_4,\ a_1 \prec b_2,\ a_2 \prec b_3,\ a_3 \prec b_4;$
$NSP7.7(26)$ $A_2 \sqcup B_5,\ a_1 \prec b_2;$
$NSP7.8(27)$ $A_2 \sqcup B_5,\ a_1 \prec b_4;$
$NSP7.9(28)$ $A_2 \sqcup B_5,\ a_2 \prec b_5;$
$NSP7.10(31)$ $A_2 \sqcup B_5,\ a_1 \prec b_2,\ a_2 \prec b_3;$
$NSP7.11(33)$ $A_2 \sqcup B_5,\ a_1 \prec b_2,\ a_2 \prec b_4;$
$NSP7.12(36)$ $A_2 \sqcup B_5,\ a_1 \prec b_2,\ a_2 \prec b_5;$
$NSP7.13(38)$ $A_2 \sqcup B_5,\ a_1 \prec b_3,\ a_2 \prec b_5;$
$NSP7.14(40)$ $A_2 \sqcup B_5,\ a_1 \prec b_4,\ a_2 \prec b_5;$
$NSP7.15(35^{op})$ $A_3 \sqcup B_4,\ a_2 \prec b_2,\ a_3 \prec b_3;$
$NSP7.16(41^{op})$ $A_4 \sqcup B_3,\ a_1 \prec b_2,\ a_4 \prec b_3;$
$NSP7.17(39^{op})$ $A_4 \sqcup B_3,\ a_2 \prec b_2,\ a_4 \prec b_3;$
$NSP7.18(37^{op})$ $A_4 \sqcup B_3,\ a_3 \prec b_2,\ a_4 \prec b_3;$
$NSP7.19(32)$ $A_3 \sqcup B_4,\ a_1 \prec b_1,\ a_2 \prec b_2,\ a_3 \prec b_3;$
$NSP7.20(34)$ $A_3 \sqcup B_4,\ a_1 \prec b_1,\ a_2 \prec b_2,\ a_3 \prec b_4;$
$NSP7.21(21)$ $A_1 \sqcup B_6,\ a_1 \prec b_3;$
$NSP7.22(24)$ $A_1 \sqcup B_6,\ a_1 \prec b_5;$
$NSP7.23(22)$ $A_2 \sqcup B_5,\ a_1 \prec b_1,\ a_2 \prec b_3;$
$NSP7.24(25)$ $A_2 \sqcup B_5,\ a_1 \prec b_1,\ a_2 \prec b_5;$
$NSP7.25(23)$ $A_3 \sqcup B_4,\ a_2 \prec b_1,\ a_3 \prec b_3;$
$NSP7.26(75)$ $A_1 \sqcup B_3 \sqcup C_3,\ b_1 \prec c_3;$
$NSP7.27(78)$ $A_2 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_2;$

$NSP7.28(79)$ $A_3 \sqcup B_1 \sqcup C_3,\ b_1 \prec c_3;$
$NSP7.29(80)$ $A_3 \sqcup B_2 \sqcup C_2,\ b_1 \prec c_2;$
$NSP7.30(89)$ $A_1 \sqcup B_3 \sqcup C_3,\ a_1 \prec b_2,\ b_1 \prec c_3;$
$NSP7.31(91)$ $A_1 \sqcup B_3 \sqcup C_3,\ a_1 \prec b_3,\ b_1 \prec c_2;$
$NSP7.32(92)$ $A_1 \sqcup B_3 \sqcup C_3,\ a_1 \prec b_3,\ b_1 \prec c_3;$
$NSP7.33(99)$ $A_2 \sqcup B_2 \sqcup C_3,\ a_1 \prec b_2,\ b_1 \prec c_2;$
$NSP7.34(100)$ $A_2 \sqcup B_2 \sqcup C_3,\ a_1 \prec b_2,\ b_1 \prec c_3;$
$NSP7.35(101)$ $A_2 \sqcup B_3 \sqcup C_2,\ a_1 \prec b_3,\ b_1 \prec c_2;$
$NSP7.36(102)$ $A_2 \sqcup B_3 \sqcup C_2,\ a_2 \prec b_3,\ b_1 \prec c_1;$
$NSP7.37(85)$ $A_1 \sqcup B_3 \sqcup C_3,\ b_1 \prec c_2,\ b_2 \prec c_3;$
$NSP7.38(86)$ $A_2 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_2,\ b_2 \prec c_3;$
$NSP7.39(108)$ $A_2 \sqcup B_3 \sqcup C_2,\ a_2 \prec b_3,\ b_1 \prec c_1,\ b_2 \prec c_2;$
$NSP7.40(108^{op})$ $A_2 \sqcup B_3 \sqcup C_2,\ a_1 \prec b_2,\ a_2 \prec b_3,\ b_1 \prec c_2;$
$NSP7.41(68)$ $A_1 \sqcup B_2 \sqcup C_4;$
$NSP7.42(72)$ $A_1 \sqcup B_2 \sqcup C_4,\ b_1 \prec c_2;$
$NSP7.43(73)$ $A_1 \sqcup B_2 \sqcup C_4,\ b_1 \prec c_3;$
$NSP7.44(74)$ $A_1 \sqcup B_2 \sqcup C_4,\ b_1 \prec c_4;$
$NSP7.45(76)$ $A_2 \sqcup B_1 \sqcup C_4,\ b_1 \prec c_3;$
$NSP7.46(87)$ $A_1 \sqcup B_2 \sqcup C_4,\ a_1 \prec b_2,\ b_1 \prec c_2;$
$NSP7.47(88)$ $A_1 \sqcup B_2 \sqcup C_4,\ a_1 \prec b_2,\ b_1 \prec c_4;$
$NSP7.48(90)$ $A_1 \sqcup B_3 \sqcup C_3,\ a_1 \prec b_3,\ b_1 \prec c_1;$
$NSP7.49(93)$ $A_1 \sqcup B_4 \sqcup C_2,\ a_1 \prec b_3,\ b_1 \prec c_1;$
$NSP7.50(94)$ $A_1 \sqcup B_4 \sqcup C_2,\ a_1 \prec b_3,\ b_1 \prec c_2;$
$NSP7.51(95)$ $A_1 \sqcup B_4 \sqcup C_2,\ a_1 \prec b_4,\ b_1 \prec c_2;$
$NSP7.52(81)$ $A_1 \sqcup B_2 \sqcup C_4,\ b_1 \prec c_2,\ b_2 \prec c_3;$
$NSP7.53(83)$ $A_1 \sqcup B_2 \sqcup C_4,\ b_1 \prec c_2,\ b_2 \prec c_4;$
$NSP7.54(84^{op})$ $A_1 \sqcup B_3 \sqcup C_3,\ b_2 \prec c_2,\ b_3 \prec c_3;$
$NSP7.55(77)$ $A_2 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_1,\ b_2 \prec c_3;$
$NSP7.56(103)$ $A_1 \sqcup B_3 \sqcup C_3,\ a_1 \prec b_3,\ b_1 \prec c_1,\ b_2 \prec c_2;$
$NSP7.57(104)$ $A_1 \sqcup B_3 \sqcup C_3,\ a_1 \prec b_3,\ b_1 \prec c_1,\ b_2 \prec c_3;$
$NSP7.58(105)$ $A_1 \sqcup B_4 \sqcup C_2,\ a_1 \prec b_4,\ b_1 \prec c_1,\ b_2 \prec c_2;$
$NSP7.59(106)$ $A_1 \sqcup B_4 \sqcup C_2,\ a_1 \prec b_4,\ b_2 \prec c_1,\ b_3 \prec c_2;$
$NSP7.60(82)$ $A_1 \sqcup B_3 \sqcup C_3,\ b_1 \prec c_1,\ b_2 \prec c_2,\ b_3 \prec c_3;$
$NSP7.61(69)$ $A_1 \sqcup B_1 \sqcup C_5,\ b_1 \prec c_3;$
$NSP7.62(71)$ $A_1 \sqcup B_1 \sqcup C_5,\ b_1 \prec c_5;$
$NSP7.63(96)$ $A_1 \sqcup B_5 \sqcup C_1,\ a_1 \prec b_3,\ b_1 \prec c_1;$
$NSP7.64(97)$ $A_1 \sqcup B_5 \sqcup C_1,\ a_1 \prec b_4,\ b_1 \prec c_1;$
$NSP7.65(98)$ $A_1 \sqcup B_5 \sqcup C_1,\ a_1 \prec b_5,\ b_1 \prec c_1;$
$NSP7.66(70)$ $A_1 \sqcup B_2 \sqcup C_4,\ b_1 \prec c_1,\ b_2 \prec c_3.$

## 3. Calculation of the transitivity coefficients. Proof of Theorems 1 and 2

We first calculate the coefficients of transitivity $k_t$ of the non-serial positive posets, which are indicated in Theorems 3 – 5. In all three tables below the posets are ordered lexicographically with respect to their weights $w$, heights $h$, transitivity coefficients $k_t$ and numbers $N$. Horizontal lines are drawn in such a way that inside each block $w$ and $h$ are the same (except for the 1st table, the converse is not true due to the last lines of the left subtables).

The coefficients of transitivity $k_t$ are calculated up to the fifth decimal place. If the number of decimal places is less than five, then the decimal fraction is finite, and if it is five, then infinite. When two decimal fractions are equal up to five digits, then they are generally equal.

**Theorem 6.** *The following holds for the posets $NSP5.1$ – $NSP5.10$:*

| $N$ | $w$ | $h$ | $n_e$ | $n_w$ | $k_t$ | $N$ | $w$ | $h$ | $n_e$ | $n_w$ | $k_t$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5.3 | 2 | 3 | 5 | 7 | 0, 28571 | 5.6 | 3 | 2 | 2 | 2 | 0 |
| 5.1 | 2 | 3 | 4 | 6 | 0, 33333 | 5.7 | 3 | 2 | 3 | 3 | 0 |
| 5.2 | 2 | 3 | 4 | 6 | 0, 33333 | 5.8 | 3 | 2 | 4 | 4 | 0 |
| 5.5 | 2 | 4 | 5 | 8 | 0, 375 | 5.10 | 3 | 3 | 4 | 5 | 0, 2 |
| 5.4 | 2 | 4 | 4 | 8 | 0, 5 | 5.9 | 3 | 3 | 3 | 4 | 0, 25 |

**Theorem 7.** *The following holds for the posets $NSP6.1$ – $NSP6.32$:*

| $N$ | $w$ | $h$ | $n_e$ | $n_w$ | $k_t$ | $N$ | $w$ | $h$ | $n_e$ | $n_w$ | $k_t$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 6.2 | 2 | 3 | 6 | 9 | 0, 33333 | 6.17 | 3 | 2 | 5 | 5 | 0 |
| 6.1 | 2 | 3 | 5 | 8 | 0, 375 | 6.23 | 3 | 3 | 5 | 6 | 0, 16667 |
| 6.8 | 2 | 4 | 6 | 10 | 0, 4 | 6.25 | 3 | 3 | 5 | 6 | 0, 16667 |
| 6.9 | 2 | 4 | 6 | 10 | 0, 4 | 6.20 | 3 | 3 | 4 | 5 | 0, 2 |
| 6.11 | 2 | 4 | 7 | 12 | 0, 41667 | 6.21 | 3 | 3 | 4 | 5 | 0, 2 |
| 6.4 | 2 | 4 | 5 | 9 | 0, 44444 | 6.18 | 3 | 3 | 3 | 4 | 0, 25 |
| 6.5 | 2 | 4 | 5 | 9 | 0, 44444 | 6.27 | 3 | 3 | 6 | 8 | 0, 25 |
| 6.7 | 2 | 4 | 6 | 11 | 0, 45455 | 6.22 | 3 | 3 | 5 | 7 | 0, 28571 |
| 6.10 | 2 | 4 | 6 | 11 | 0, 45455 | 6.24 | 3 | 3 | 5 | 7 | 0, 28571 |
| 6.3 | 2 | 4 | 5 | 10 | 0, 5 | 6.26 | 3 | 3 | 5 | 7 | 0, 28571 |
| 6.6 | 2 | 4 | 6 | 12 | 0, 5 | 6.19 | 3 | 3 | 4 | 6 | 0, 33333 |
| 6.15 | 2 | 5 | 6 | 12 | 0, 5 | 6.31 | 3 | 4 | 5 | 8 | 0, 375 |
| 6.14 | 2 | 5 | 6 | 13 | 0, 53846 | 6.32 | 3 | 4 | 5 | 8 | 0, 375 |
| 6.13 | 2 | 5 | 5 | 12 | 0, 58333 | 6.29 | 3 | 4 | 4 | 7 | 0, 42857 |
| 6.12 | 2 | 5 | 5 | 13 | 0, 61538 | 6.30 | 3 | 4 | 5 | 9 | 0, 44444 |
| 6.16 | 3 | 2 | 4 | 4 | 0 | 6.28 | 3 | 4 | 4 | 8 | 0, 5 |

**Theorem 8.** *The following holds for the posets $NSP7.1$ – $NSP7.66$:*

| $N$ | $w$ | $h$ | $n_e$ | $n_w$ | $k_t$ | $N$ | $w$ | $h$ | $n_e$ | $n_w$ | $k_t$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 7.4 | 2 | 4 | 7 | 12 | 0,41667 | 7.27 | 3 | 3 | 5 | 7 | 0,28571 |
| 7.1 | 2 | 4 | 6 | 11 | 0,45455 | 7.28 | 3 | 3 | 5 | 7 | 0,28571 |
| 7.2 | 2 | 4 | 6 | 11 | 0,45455 | 7.39 | 3 | 3 | 7 | 10 | 0,3 |
| 7.3 | 2 | 4 | 7 | 13 | 0,46154 | 7.30 | 3 | 3 | 6 | 9 | 0,33333 |
| 7.5 | 2 | 4 | 7 | 13 | 0,46154 | 7.31 | 3 | 3 | 6 | 9 | 0,33333 |
| 7.6 | 2 | 4 | 8 | 15 | 0,46667 | 7.36 | 3 | 3 | 6 | 9 | 0,33333 |
| 7.14 | 2 | 5 | 7 | 14 | 0,5 | 7.37 | 3 | 3 | 6 | 9 | 0,33333 |
| 7.16 | 2 | 5 | 7 | 14 | 0,5 | 7.47 | 3 | 4 | 6 | 9 | 0,33333 |
| 7.20 | 2 | 5 | 8 | 17 | 0,52941 | 7.51 | 3 | 4 | 6 | 9 | 0,33333 |
| 7.13 | 2 | 5 | 7 | 15 | 0,53333 | 7.55 | 3 | 4 | 6 | 9 | 0,33333 |
| 7.17 | 2 | 5 | 7 | 15 | 0,53333 | 7.57 | 3 | 4 | 7 | 11 | 0,36364 |
| 7.8 | 2 | 5 | 6 | 13 | 0,53846 | 7.58 | 3 | 4 | 7 | 11 | 0,36364 |
| 7.9 | 2 | 5 | 6 | 13 | 0,53846 | 7.44 | 3 | 4 | 5 | 8 | 0,375 |
| 7.19 | 2 | 5 | 8 | 18 | 0,55556 | 7.48 | 3 | 4 | 6 | 10 | 0,4 |
| 7.12 | 2 | 5 | 7 | 16 | 0,5625 | 7.50 | 3 | 4 | 6 | 10 | 0,4 |
| 7.18 | 2 | 5 | 7 | 16 | 0,5625 | 7.56 | 3 | 4 | 7 | 12 | 0,41667 |
| 7.11 | 2 | 5 | 7 | 17 | 0,58824 | 7.60 | 3 | 4 | 7 | 12 | 0,41667 |
| 7.15 | 2 | 5 | 7 | 17 | 0,58824 | 7.41 | 3 | 4 | 4 | 7 | 0,42857 |
| 7.7 | 2 | 5 | 6 | 15 | 0,6 | 7.43 | 3 | 4 | 5 | 9 | 0,44444 |
| 7.10 | 2 | 5 | 7 | 18 | 0,61111 | 7.45 | 3 | 4 | 5 | 9 | 0,44444 |
| 7.24 | 2 | 6 | 7 | 17 | 0,58824 | 7.46 | 3 | 4 | 6 | 11 | 0,45455 |
| 7.23 | 2 | 6 | 7 | 19 | 0,63158 | 7.49 | 3 | 4 | 6 | 11 | 0,45455 |
| 7.25 | 2 | 6 | 7 | 19 | 0,63158 | 7.53 | 3 | 4 | 6 | 11 | 0,45455 |
| 7.22 | 2 | 6 | 6 | 17 | 0,64706 | 7.54 | 3 | 4 | 6 | 11 | 0,45455 |
| 7.21 | 2 | 6 | 6 | 19 | 0,68421 | 7.59 | 3 | 4 | 7 | 13 | 0,46154 |
| 7.34 | 3 | 3 | 6 | 7 | 0,14286 | 7.42 | 3 | 4 | 5 | 10 | 0,5 |
| 7.35 | 3 | 3 | 6 | 7 | 0,14286 | 7.52 | 3 | 4 | 6 | 12 | 0,5 |
| 7.29 | 3 | 3 | 5 | 6 | 0,16667 | 7.65 | 3 | 5 | 6 | 12 | 0,5 |
| 7.40 | 3 | 3 | 7 | 9 | 0,22222 | 7.64 | 3 | 5 | 6 | 13 | 0,53846 |
| 7.32 | 3 | 3 | 6 | 8 | 0,25 | 7.66 | 3 | 5 | 6 | 13 | 0,53846 |
| 7.33 | 3 | 3 | 6 | 8 | 0,25 | 7.62 | 3 | 5 | 5 | 11 | 0,54545 |
| 7.38 | 3 | 3 | 6 | 8 | 0,25 | 7.63 | 3 | 5 | 6 | 14 | 0,57143 |
| 7.26 | 3 | 3 | 5 | 7 | 0,28571 | 7.61 | 3 | 5 | 5 | 13 | 0,61538 |

For proving Theorem 6 – 8 we need the following lemmas from [12].

**Lemma 1.** *Let $S = S_1 \sqcup S_2$. Then*
$n_e(S) = n_e(S_1) + n_e(S_2), n_w(S) = n_w(S_1) + n_w(S_2)$.

**Lemma 2.** *Let $S = A_m$. Then*
$n_e(S) = m - 1, n_w(S) = \frac{(m-1)m}{2}$.

**Lemma 3.** *Let* $S = \{A_m \sqcup B_n, a_i < b_j\}$. *Then*
(*a*) $n_e(S) = m + n - 1$;
(*b*) $n_w(S) = \frac{(m-1)m+(n-1)n}{2} + i(n - j + 1)$.

**Lemma 4.** *Let* $S = \{A_m \sqcup B_n, a_i < b_j, a_{i'} < b_{j'}\}$, *where* $i < i', j < j'$. *Then*
(*a*) $n_e(S) = m + n$;
(*b*) $n_w(S) = \frac{(m-1)m+(n-1)n}{2} + i'(n - j' + 1) + i(j' - j)$.

**Lemma 5.** *Let* $S = \{A_m \sqcup B_n \sqcup C_s, a_i < b_j, b_{j'} < c_k$, *where* $j > j'$. *Then*
(*a*) $n_e(S) = m + n + s - 1$;
(*b*) $n_w(S) = \frac{(m-1)m+(n-1)n+s(s-1)}{2} + i(n - j + 1) + j'(s - k + 1)$.

**Lemma 6.** *Let* $S = \{A_m \sqcup B_n, a_i < b_j, a_{i+1} < b_{j+1}, a_{i+2} < b_{j+2}$. *Then*
(*a*) $n_e(S) = m + n + 1$;
(*b*) $n_w(S) = \frac{(m-1)m+(n-1)n}{2} + (i + 2)n - i(j - 1) - (2j + 1)$.

**Lemma 7.** *Let* $S = \{A_m \sqcup B_n \sqcup C_s, a_i < b_j, b_{j'} < c_k$, $b_{j'+1} < c_{k+1}$, *where* $j > j' + 1$. *Then*
(*a*) $n_e(S) = m + n + s$;
(*b*) $n_w(S) = \frac{(m-1)m+(n-1)n+s(s-1)}{2} + i(n - j + 1) + (j' + 1)(s - k) + j'$.

The data indicated in the tables of Theorems 6 – 8 for the posets from Theorems 3 – 5 are verified by direct calculations using Lemmas 1, 2 for $N = 5.6, 6.18, 7.41$, Lemma 3 for $N = 5.1, 5.2, 5.4, 6.1, 6.3 - 6.5, 6.12, 6.13, 7.1 - 7.2, 7.7 - 7.9, 7.21 - 7.22$, Lemmas 1–3 for $N = 5.7, 5.9$, $6.16$, $6.19 - 6.21$, $6.28 - 6.29$, $7.26 - 7.29, 7.42 - 7.45, 7.61 - 7.62$, Lemma 4 for $N = 5.3, 5.5, 6.2, 6.6 - 6.10, 6.14, 6.15, 7.3 - 7.5, 7.10 - 7.18, 7.23 - 7.25$, Lemmas 1, 2, 4 for $N = 6.26, 6.32, 7.37 - 7.38, 7.52 - 7.55, 7.66$, Lemma 5 for $N = 5.8, 5.10, 6.17, 6.22 - 6.25, 6.30 - 6.31, 7.30 - 7.36, 7.46 - 7.51, 7.63 - 7.65$, Lemma 6 for $N = 6.11, 7.6, 7.19 - 7.20$, Lemmas 1, 2, 6 for $N = 7.60$, Lemma 7 for $N = 6.27, 7.39 - 7.40, 7.56 - 7.59$.

Given the lexicographic notation in the tables, it is easy to check that Theorems 1 and 2 follow from them.

## 4. The List of the non-serial positive posets (in terms of Hasse diagrams)

We follow the paper [5]. Up to isomorphism and duality, there exists 108 non-serial positive posets. We sort them by their order (which can be only equal to 5, 6 or 7), and three below tables, for convenience, are broken into blocks with respect to the value of $w$ and $h$ specified at the beginning of these blocks.

Table 4.1. The non-serial positive posets of order 5 (up to isomorphism and duality):

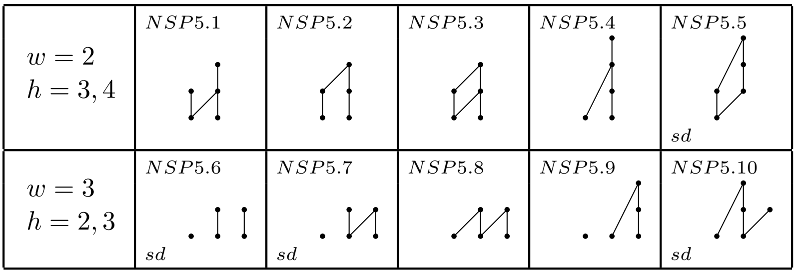



Table 4.2. The non-serial positive posets of order 6 (up to isomorphism and duality):

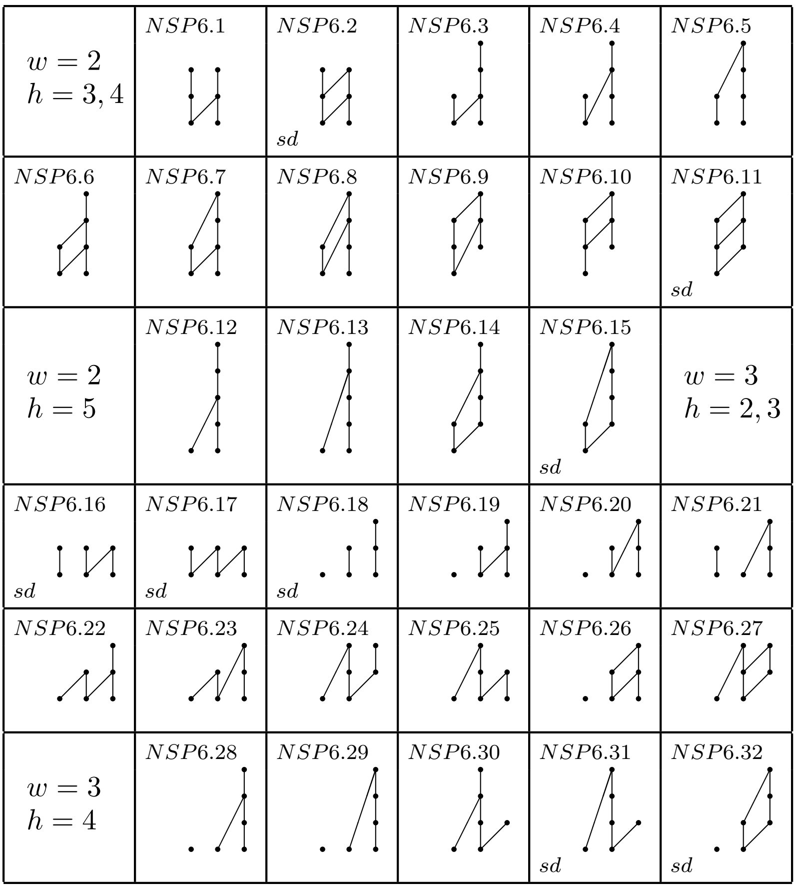
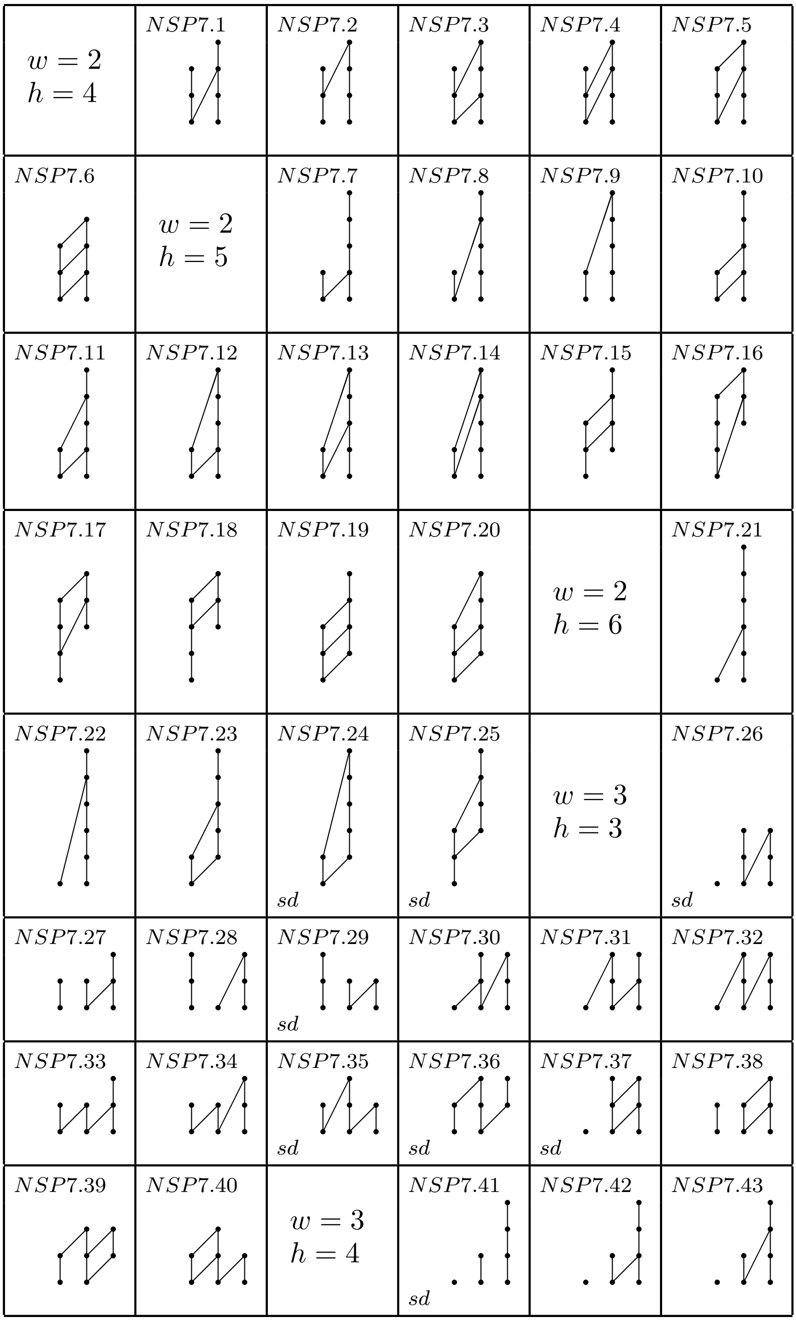
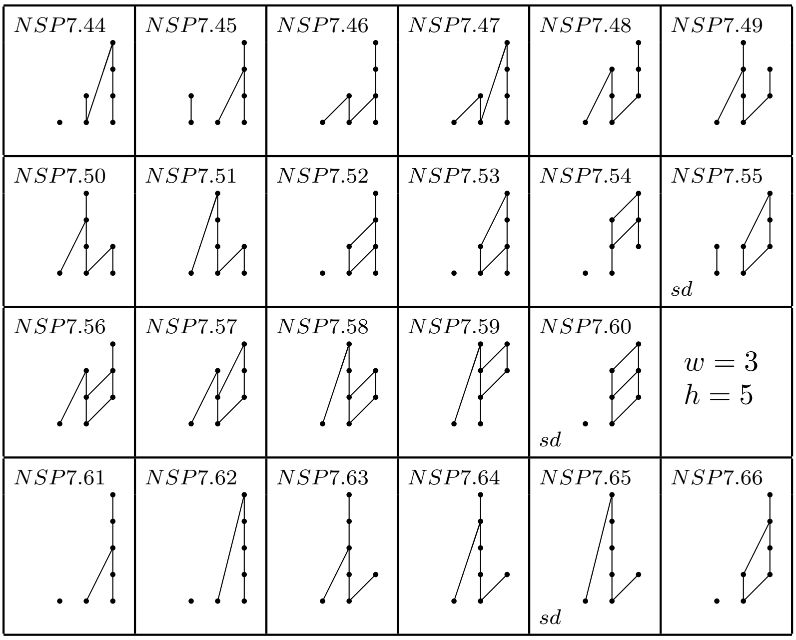


Table 4.3. The non-serial positive posets of order 7 (up to isomorphism and duality):






In conclusion, we emphasize that the self-dual posets in the tables are marked in the lower left corners by *sd*. If we add all the posets dual to unmarked ones, we obtain the classification of the non-serial posets up to isomorphism; their number is 193 (16 of order 5, 56 of order 6 and 121 of order 7).

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