© Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **35** (2023). Number 2, pp. 168–179
DOI:10.12958/adm2139

Ideally finite Leibniz algebras

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Abstract. The aim of this paper is to consider Leibniz algebras, whose principal ideals are finite dimensional. We prove that the derived ideal of L has finite dimension if every principal ideal of a Leibniz algebra L has dimension at most b, where b is a fixed positive integer.

Let L be an algebra over a field F with the binary operations $+$ and $[-,-]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]]$$

for all $a, b, c \in L$. We will also use another form of this identity:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

Leibniz algebras first appeared in the paper of A. Bloh [3], but the term "Leibniz algebra" appears in the book of J. L. Loday [11] and in the article of J. L. Loday [12]. In [13], J. Loday and T. Pirashvili began the in-depth study of the properties of Leibniz algebras. The theory of Leibniz algebras has developed significantly in many different directions. Some of the results of this theory were presented in the book [1]. Note that Lie algebras are a partial case of Leibniz algebras. Conversely, if L is a Leibniz algebra in which $[a,a] = 0$ for every element $a \in L$, then it is a Lie algebra. Thus, Lie algebras can be characterized as anticommutative Leibniz algebras. As for Lie algebras, the theory of finite-dimensional Leibniz algebras is far more developed than the theory of infinite-dimensional Leibniz algebras. However, even when considering Leibniz algebras of small dimensions,

2020 MSC: 17A32, 17A31.

Key words and phrases: Leibniz algebra, ideally finite algebras, Lie algebra, breadth of elements.

significant differences in of Lie algebras become noticeable. Therefore, a natural step is to consider Leibniz algebras that are, in one sense or another, close to finite-dimensional ones. The corresponding topic in the theory of Lie algebras turned out to be very fruitful and rich in interesting results (see, for example, the book [2]). One of these algebras, which retained quite a few properties of finite-dimensional algebras, turned out to be the ideally finite Lie algebras. The book of I. Stewart [14] was devoted to these algebras. In this paper, we begin the study of the ideally finite Leibniz algebras.

Let L be a Leibniz algebra over a field F. If M is a non-empty subset of L, then $\langle M\rangle$ denotes the subalgebra of L generated by M, and $\langle M\rangle^L$ denotes the ideal generated by M. As usual, an ideal generated by one single element is called a *principal ideal*.

If A, B are subspaces of L, then denote by $[A, B]$ a subspace generated by all elements $[a, b]$, where $a \in A, b \in B$.

The Leibniz algebra L is called *ideally finite* if its every principal ideal has finite dimension.

The Leibniz algebra L is called *boundedly-ideally finite* if there exists a positive integer b such that $\dim_F(\langle a\rangle^L) \leqslant b$ for every element $a \in L$. In this case, we will also say that L is *b-ideally finite*.

The first main result of this paper gives the description of the boundedly-ideally finite Leibniz algebras.

Theorem A. *Let L be a b-ideally finite Leibniz algebra over a field F. Then the derived ideal of L has a finite dimension at most $(1/6)b(b+1)(2b^2+b+3)$.*

Let L be a Leibniz algebra over a field F, M be non-empty subset of L, and H be a subalgebra of L. Put

$$\mathrm{Ann}_H^{\mathrm{left}}(M) = \{a \in H | [a, M] = 0\}, \mathrm{Ann}_H^{\mathrm{right}}(M) = \{a \in H | [M, a] = 0\}.$$

The subset $\mathrm{Ann}_H^{\mathrm{left}}(M)$ is called the *left annihilator* or the *left centralizer* of M in subalgebra H. The subset $\mathrm{Ann}_H^{\mathrm{right}}(M)$ is called the *right annihilator* or the *right centralizer* of M in subalgebra H. The intersection

$$\mathrm{Ann}_H(M) = \mathrm{Ann}_H^{\mathrm{left}}(M) \cap \mathrm{Ann}_H^{\mathrm{right}}(M) = \{a \in H | [a, M] = \langle 0\rangle = [M, a]\}$$

is called the *annihilator* or the *centralizer* of M in subalgebra H.

It is not hard to see that all of these subsets are subalgebras of L. Moreover, if M is a left ideal of L, then $\mathrm{Ann}_L^{\mathrm{left}}(M)$ is an ideal of L. If M is an ideal of L, then $\mathrm{Ann}_L(M)$ is an ideal of L.

Let L be a Leibniz algebra and a be an element of L. Then the number $\dim_F(A/\mathrm{Ann}_L(a)) = \mathrm{br}(a)$ is called a *breadth of an element a in algebra L.*

If a Leibniz algebra L is ideally finite, then the breadth of its every element is finite. In fact, if a is an arbitrary element of L, then an ideal A generated by a has a finite dimension k. Using Proposition 3.2 of paper [9], we obtain that factor-algebra $L/\mathrm{Ann}_L(A)$ is isomorphic to some subalgebra of the algebra of derivations of A. Since $\dim_F(A) = k$, $\dim_F(L/\mathrm{Ann}_L(A)) \leqslant k^2$.

The obvious inclusion $\mathrm{Ann}_L(A) \leqslant \mathrm{Ann}_L(a)$ shows that an element a has a breadth at most k^2.

The Leibniz algebra L is called an *FB-algebra* if its every element has a finite breadth.

A converse question naturally appears: Is a Leibniz algebra L, in which every element has finite breadth, ideally finite?

The second main result of this paper is as follows.

Theorem B. *Let L be a Leibniz algebra over a field $F, a_1, \ldots, a_n$ be the elements of L such that $br(a_j) = k_j$ is finite, $1 \leqslant j \leqslant n$. Let A be an ideal of L, generated by the elements $a_1, \ldots, a_n$, $C = \mathrm{Ann}_L\ (\{a_1, \ldots, a_n\})$, and let $e_1 + C, \ldots, e_t + C$ be the basis of L/C. Suppose that $br(e_m) = s_m$ is finite, $1 \leqslant m \leqslant t$. Then $\dim_F(A)$ is finite, and moreover,*

$$\dim_F(A) \leqslant (k_1 + \ldots + k_n + s_1 + \ldots + s_t)^2 + n + t.$$

We obtain the following corollary:

Corollary. *Let L be a Leibniz algebra over a field F. Then L is ideally finite if and only if every element of L has a finite breadth.*

Note that this topic is not specific to only Lie algebras and Leibniz algebras. It is inherent in other algebraic structures as well. It originated in group theory, where a large array of articles is devoted to it (see, for example, the instead of a survey [7]); this topic was also developed in the theory of modules over the group rings [5], [6], [8].

1. Structure of Leibniz algebras whose principal ideals have bounded finite dimensions

A Leibniz algebra L has one specific ideal. Denote by $\mathrm{Leib}(L)$ the subspace generated by the elements $[a, a], a \in L$.

It is possible to show that $\mathrm{Leib}(L)$ is an ideal of L, and if H is an ideal of L such that L/H is a Lie algebra, then $\mathrm{Leib}(L) \in H$.

The ideal $\mathrm{Leib}(L)$ is called the *Leibniz kernel* of algebra L.

We also note the following important property of a Leibniz kernel:

$$[[a, a], x] = 0 \text{ for arbitrary elements } a, x \in L.$$

The *left* (respectively, *right*) *center* $\zeta^{\mathrm{left}}(L)$ (respectively, $\zeta^{\mathrm{right}}\ (L)$) of a Leibniz algebra L is defined by the rule

$$\zeta^{\mathrm{left}}(L) = \{x \in L | [x, y] = 0 \text{ for each element } y \in L\}.$$

(respectively,

$$\zeta^{\mathrm{right}}(L) = \{x \in L | [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal. Moreover, $\mathrm{Leib}(L) \leqslant \zeta^{\mathrm{left}}(L)$ and L/ζ^{left} are Lie algebras. In general, the left and the right centers are different. Moreover, the left center is an ideal, but such is not true for the right center (one can find a corresponding example in [9]).

The *center* $\zeta(L)$ of L is defined by the rule

$$\zeta(L) = \{x \in L | [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L. We can say the same about the factor-algebra $L/\zeta(L)$, in particular.

We note the following elementary properties:

Lemma 1.1. *Let L be a b-ideally finite Leibniz algebra over a field F.*

(i) *If K is a subalgebra of L, then K is a k-ideally finite Leibniz algebra for some positive integer $k \leqslant b$.*

(ii) *If K is an ideal of L, then L/K is a k-ideally finite Leibniz algebra for some positive integer $k \leqslant b$.*

(iii) *If K is a subalgebra of L and A is an ideal of K, then the section K/A is a k-ideally finite Leibniz algebra for some positive integer $k \leqslant b$.*

The following result will play an essential role in describing boundedly-ideally finite Leibniz algebras.

Proposition 1.2. *Let L be a b-ideally finite Leibniz algebra over a field F. Let d be an element of L such that $\dim_F(\langle d \rangle^L) = b > 1$. If D is an ideal of L, generated by an element d, and $K = \mathrm{Ann}_L(D)$, then $\dim_F(\langle a + D \rangle^{(K+D)/D}) \leqslant b - 1$.*

Proof. Suppose the contrary. Let c be an element of L such that the ideal $\langle c+D\rangle^{(K+D)/D}$, generated by the coset $c+D$ in the subalgebra $(K+D)/D$, has dimension b. Since

$$\dim_F(\langle c+D\rangle^{(K+D)/D}) \leqslant \dim_F(\langle c+D\rangle^{L/D}) \text{ and } \dim_F(\langle c+D\rangle^{L/D}) \leqslant b,$$

by Lemma 1.1, we have $\dim_F(\langle c+D\rangle^{(K+D)/D}) = \dim_F(\langle c+D\rangle^{L/D}) = b$. Denote by C the ideal generated in L by element c. Then clearly, $(C+D)/D = \langle c+D\rangle^{L/D}$, so that $\dim_F((C+D)/D) = b$. It is obvious that $\dim_F((C+D)/D) \leqslant \dim_F(C) \leqslant b$. It follows that $\dim_F(C) = b$. We have $\dim_F(C+D) \leqslant \dim_F(C)+\dim_F(D) = 2b$. Suppose that $\dim_F(C+D) < 2b$. Then $\dim_F((C+D)/D) = \dim_F(C+D) - \dim_F(D) < 2b-b$, and we obtain a contradiction. This contradiction shows that $\dim_F(C+D) = 2b$. On the other hand,

$$\dim_F(C+D) = \dim_F(C) + \dim_F(D) - \dim_F(C\cap D),$$

so that $\dim_F(C\cap D) = 0$, and hence $C\cap D = \langle 0\rangle$.

Let $x+D \in \mathrm{Ann}_{L/D}(c+D)$. Then

$$D = [x+D, c+D] = [x,c]+D, \text{ and similarly, } [c,x]+D = D.$$

It follows that $[x,c],[x,c] \in D$. On the other hand, c belongs to the ideal C of L, and therefore $[x,c],[c,x] \in C$. Thus, we obtain that $[x,c],[c,x] \in D\cap C = \langle 0\rangle$ (i.e., $[x,c] = 0 = [c,x]$). Now, taking into account the obvious inclusion $\mathrm{Ann}_L(c) \leqslant \mathrm{Ann}_L(c+D)$, we obtain the equation $\mathrm{Ann}_L(c) = \mathrm{Ann}_L(c+D)$.

Let $z \in \mathrm{Ann}_L(c+d)$. Then

$$[z+D, c+D] = [z+D, c+d+D] = [z, c+d]+D = D,$$

and similarly,

$$[c+D, z+D] = D.$$

It follows that $z \in \mathrm{Ann}_L(c+D)$. From what has been proven above, it follows that $z \in \mathrm{Ann}_L(c)$ (i.e., $\mathrm{Ann}_L(c+d) \leqslant \mathrm{Ann}_L(c)$). Furthermore, $0 = [z, c+d] = [z,c]+[z,d] = [z,d]$, and similarly, $0 = [d,z]$. Thus, we obtain that $z \in \mathrm{Ann}_L(c) \cap \mathrm{Ann}_L(d)$. In other words, $\mathrm{Ann}_L(c+d) \leqslant \mathrm{Ann}_L(c) \cap \mathrm{Ann}_L(d)$.

Put $e = c+d$ and denote by E the ideal generated by an element e in L. We have:

$$\begin{aligned}(E+C)/C = \langle e+C\rangle^{L/C} = \langle c+d+C\rangle^{L/C} = \langle d+C\rangle^{L/D}\\ = (\langle d\rangle^{L}+C)/C = (D+C)/C.\end{aligned}$$

As we have seen above, $D \cap C = \langle 0 \rangle$. It follows that

$$(E + C)/C = (D + C)/C \cong D/(D \cap C) \cong D.$$

Now, we obtain that $\dim_F((E + C)/C) = \dim_F(D) = b$, and therefore

$$\dim_F(E) \geqslant \dim_F((E + C)/C) = b.$$

On the other hand, $\dim_F(E) = \dim_F(\langle e \rangle^L) \leqslant b$. It follows that $\dim_F(E) = b$. It is only possible if $E \cap C = \langle 0 \rangle$. Using the arguments above, we obtain the equation $\mathrm{Ann}_L(e) = \mathrm{Ann}_L(e + C)$. However,

$$e + C = c + d + C = d + C.$$

The equality $D \cap C = \langle 0 \rangle$ implies that $\mathrm{Ann}_L(d) = \mathrm{Ann}_L(d + C)$. All these equalities imply that $\mathrm{Ann}_L(e) = \mathrm{Ann}_L(d+c) = \mathrm{Ann}_L(d)$. As we have seen above, $\mathrm{Ann}_L(c+d) \leqslant \mathrm{Ann}_L(c) \cap \mathrm{Ann}_L(d)$, hence $\mathrm{Ann}_L(d) \leqslant \mathrm{Ann}_L(c)$. An obvious inclusion $\mathrm{Ann}_L(D) \leqslant \mathrm{Ann}_L(d)$ implies that $\mathrm{Ann}_L(D) \leqslant \mathrm{Ann}_L(c)$. In other words, $\langle c \rangle^K = Fc$, and therefore $\langle c + D \rangle^{(K+D)/D} = \langle c \rangle^K + D = \{c + D\}$. This contradicts that a subalgebra $\dim_F(\langle c + D \rangle^{(K+D)/D}) = b$. This contradiction proves the result. $\square$

A Leibniz algebra is called *strong extraspecial* if $[E, E] = \zeta(E)$ is a subalgebra of dimension 1 and $[x, x] \neq 0$ for each element $x \notin \zeta(E)$.

Leibniz algebras whose subalgebras are ideals have been described in the paper [10]. Such algebra L has the following structure: $L = E \oplus Z$, where Z is a subalgebra of the center of L and E is a strong extraspecial algebra.

Proof of Theorem A

If $b = 1$, then every subspace of L is an ideal. Such Leibniz algebras have been described in the paper [10]. As we have noted above, such algebras are either abelian or strong extraspecial. In any case, $\dim_F(L) \leqslant 1$.

Now, suppose that $b > 1$. In the Leibniz algebra L, choose the element d_1 such that the ideal D_1 generated in L by an element d_1 has dimension b. Put $K_1 = \mathrm{Ann}_L(D_1)$. Using Proposition 3.2 from [9], we obtain that factor-algebra L/K_1 is isomorphic to some subalgebra of the algebra of derivations of D_1. Since $\dim_F(D_1) = b$, $\dim_F(L/K_1) \leqslant b^2$. Proposition 1.2 shows that an ideal generated by an arbitrary element of $(K_1 + D_1)/D_1$ has dimension at most $b - 1$.

In the section $(K_1 + D_1)/D_1$, choose the element $d_2 + D_1$ such that the ideal D_2/D_1, generated by $d_2 + D_1$ in $(K_1 + D_1)/D_1$, has the greatest dimension k. Then, as we remarked above, $k \leqslant b-1$. Put $K_2/D_1 = \mathrm{Ann}_{(K_1+D_1)/D_1}(D_2/D_1)$. Using Proposition 3.2 from [9] again, we obtain that the section $((K_1+D_1)/D_1)/(K_2/D_1)$ has dimension at most $k^2 \leqslant (b-1)^2$. It follows that a subalgebra K_2+D_2 has, in K_1+D_1, codimension at most $(b-1)^2$. Hence, subalgebra K_2+D_2 has codimension at most $b^2+(b-1)^2$ in L. Proposition 1.2 shows that the ideal generated by every element of $(K_2+D_2)/D_2$ has dimension at most $k-1 \leqslant b-2$.

Repeating these arguments, we finally construct the subalgebras A, S such that S is an ideal of $A, \dim_F(S) \leqslant b+(b-1)+\ldots+2$, every subalgebra of A/S is an ideal, and A has codimension t in L where $t \leqslant b^2+(b-1)^2+\ldots+2^2$. As we have seen above, the derived subalgebra E/S of A/S has dimension at most 1, so that $\dim_F(E) \leqslant b+(b-1)+\ldots+2+1$.

Let $\{a_1+A, \ldots, a_t+A\}$ be a basis of the factor-space L/A. Denote by A_j the ideal generated by an element a_j in L. Then $\dim_F(A_j) \leqslant b, 1 \leqslant j \leqslant t$. Put $A_1+\ldots+A_t = B$. Then $\dim_F(B) \leqslant bt$ and $L = B+A$. It follows that the factor-algebra L/B is isomorphic to some section of A. As we have noted above, the derived ideal of A has dimension at most $b+(b-1)+\ldots+2+1$, and therefore the derived ideal of L has dimension at most

$$
\begin{aligned}
&bt+b+(b-1)+\ldots+2+1 \\
&\quad \leqslant b(b^2+(b-1)^2+\ldots+2^2+1)+\frac{1}{2}b(b+1) \\
&\quad = (1/6)b^2(b+1)(2b+1)+\frac{1}{2}b(b+1) = (1/6)b(b+1)(2b^2+b+3).
\end{aligned}
$$

2. Structure of Leibniz algebras whose elements have finite breadth

Lemma 2.1. *Let L be a Leibniz algebra over a field F, a be an element of L, and A be a subalgebra generated by a. Then* $\mathrm{Ann}^{\mathrm{left}}_L(A) = \mathrm{Ann}^{\mathrm{left}}_L(a), \mathrm{Ann}^{\mathrm{right}}_L(A) = \mathrm{Ann}^{\mathrm{right}}_L(a)$, *and therefore* $\mathrm{Ann}_L(A) = \mathrm{Ann}_L(a)$.

Proof. Put $a_1 = [a,a], a_2 = [a,a_1], a_n+1 = [a,a_n], n \in \mathbb{N}$. Then the subalgebra A is a subspace of L generated by elements $a, a_n, n \in \mathbb{N}$ [[4], Corollary 2.2]. Since $a_n \in \mathrm{Leib}(L) \leqslant \zeta^{\mathrm{left}}(L), \mathrm{Ann}^{\mathrm{right}}_L(A) = \mathrm{Ann}^{\mathrm{right}}_L(a)$.

Let $z \in \mathrm{Ann}^{\mathrm{left}}_L(a)$. We have:

$$[z,a_1] = [z,[a,a]] = [[z,a],a]+[a,[z,a]] = 0.$$

Suppose we have already proven that $[z, a_2] = \ldots = [x, a_n] = 0$, and consider the element $[z, a_{n+1}]$. We have:

$$[z, a_{n+1}] = [z, [a, a_n]] = [[z, a], a_n] + [a, [z, a_n]] = 0.$$

It follows that $z \in \mathrm{Ann}_L^{\mathrm{left}}(A)$, so that $\mathrm{Ann}_L^{\mathrm{left}}(a) \leqslant \mathrm{Ann}_L^{\mathrm{left}}(A)$. The converse inclusion is obvious. □

Lemma 2.2. *Let L be a Leibniz algebra over a field $F, a_1, \ldots, a_n$ be elements of L, and A be a subalgebra generated by these elements. Then* $\mathrm{Ann}_L^{\mathrm{left}}(A) = \mathrm{Ann}_L^{\mathrm{left}}(\{a_1, \ldots, a_n\})$, $\mathrm{Ann}_L^{\mathrm{right}}(A) = \mathrm{Ann}_L^{\mathrm{right}}(\{a_1, \ldots, a_n\})$, *and therefore* $\mathrm{Ann}_L(A) = \mathrm{Ann}_L(\{a_1, \ldots, a_n\})$.

Proof. Let

$$z \in \mathrm{Ann}_L^{\mathrm{left}}(\{a_1, \ldots, a_n\}) \quad (\text{respectively, } z \in \mathrm{Ann}_L^{\mathrm{right}}(\{a_1, \ldots, a_n\})).$$

For the proof, we will use induction by the weight of commutators of elements $a_1, \ldots, a_n$. We have:

$$[z, [a_j, a_k]] = [[z, a_j], a_k]] + [a_j, [z, a_k]] = 0$$
$$(\text{respectively, } [[a_j, a_k], z] = [a_j, [a_k, z]] - [a_k, [a_j, z]] = 0).$$

Suppose that we have already proven that the element z left (respectively, right) annihilates all commutators whose weight is at most $m, m \geqslant 2$. Let b be a commutator of some elements of $a_1, \ldots, a_n$ having weight $m+1$. Then either $b = [c, d]$ where c, d are commutators whose weights are at most $m-1$, or $b = [a_j, u]$ where u is a commutator having weight m, or $b = [v, a_k]$ where v is a commutator having a weight $m, 1 \leqslant j, k \leqslant n$. We have:

$$[z, [c, d]] = [[z, c], d]] + [c, [z, d]] = 0$$
$$(\text{respectively, } [[c, d], z] = [c, [d, z]] - [d, [c, z]] = 0),$$
$$[z, [a_j, u]] = [[z, a_j], u] + [a_j, [z, u]] = 0$$
$$(\text{respectively, } [[a_j, u], z] = [a_j, [u, z]] - [u, [a_j, z]] = 0),$$
$$[z, [v, a_k]] = [[z, v], a_k] + [v, [z, a_k]] = 0$$
$$(\text{respectively, } [[v, a_k], z] = [v, [a_k, z]] - [a_k, [v, z]] = 0).$$

Since subalgebra A as a vector space is generated by all commutators of elements $a_1, \ldots, a_n$, we obtain that $\mathrm{Ann}_L^{\mathrm{left}}(A) = \mathrm{Ann}_L^{\mathrm{left}}(a_1, \ldots, a_n)$ (respectively, $\mathrm{Ann}_L^{\mathrm{right}}(A) = \mathrm{Ann}_L^{\mathrm{right}}(a_1, \ldots, a_n)$. □

Lemma 2.3. *Let L be a Leibniz algebra over a field $F, a_1, \ldots, a_n$ be elements of L, and A be a subalgebra generated by these elements. Suppose that an element a_j has a breadth $k_j, 1 \leqslant j \leqslant n$. Then subalgebra A has finite dimension at most $(k_1 + \ldots + k_n)^2 + n$.*

Proof. The obvious inclusion

$$\mathrm{Ann}_L(a_1) \cap \ldots \cap \mathrm{Ann}_L(a_n) \leqslant \mathrm{Ann}_L(\{a_1, \ldots, a_n\})$$

implies that $\mathrm{codim}_F(\mathrm{Ann}_L(\{a_1, \ldots, a_n\})) \leqslant k_1 + \ldots + k_n$. Lemma 2.2 shows that $\mathrm{Ann}_L(\{a_1, \ldots, a_n\}) = \mathrm{Ann}_L(A)$. Clearly, $A \cap \mathrm{Ann}_L(A) \leqslant \zeta(A)$. It follows that $A/\zeta(A)$ has dimension at most $k_1 + \ldots + k_n$. Then the derived ideal $[A, A]$ of a subalgebra A has dimension at most $(k_1 + \ldots + k_n)^2$ [[9]; Corollary B1]. The factor-algebra $A/[A, A]$ is abelian and generated by cosets $a_1 + [A, A], \ldots, a_n + [A, A]$. Then $\dim_F(A/[A, A]) \leqslant n$. Hence, $\dim_F(A) \leqslant n + (k_1 + \ldots + k_n)2$. □

Let L be a Leibniz algebra, A be a subalgebra of L, and S be a non-empty subset of L. We say that A is *S-invariant* if $[a, s], [s, a] \in A$ for every $a \in A, s \in S$.

Clearly, the zero subalgebra and the algebra L are S-invariant for every non-empty subset S of L. Also, it is clear that the intersection of S-invariant subalgebras is also S-invariant. It follows that for each subalgebra K of L, there exists the least S-invariant subalgebra including K. We will denote this subalgebra by K^S. If $S = L$, then every S-invariant subalgebra is an ideal of L and K^L is an ideal generated by K in L (the *ideal closure of K in L*).

Proposition 2.4. *Let L be a Leibniz algebra over a field F, A be a subalgebra of L and S be a non-empty subset of L. Let $A_0 = A, A_1$ be a subalgebra generated by $[S, A_0], [A_0, S], A_{n+1}$ be a subalgebra generated by $[S, A_n]$, and $[A_n, S], n \in \mathbb{N}$. Then $A^S = \sum_{n \in \mathbb{N}} A_n$.*

Proof. Clearly, $A \leqslant A^S$ and $A_n \leqslant A^S$ for every $n \in \mathbb{N}$, so that $\sum_{n \in \mathbb{N}} A_n \leqslant A^S$. Conversely, let $x \in \sum_{n \in \mathbb{N}} A_n$. Then $x = u_0 + u_1 + \ldots + u_k$ for some positive integer k where $u_j \in A_j, 0 \leqslant j \leqslant k$. For every element $y \in S$, we have $[y, u_j], [u_j, y] \in A_{j+1}, 0 \leqslant j \leqslant k$. Hence, $[x, y], [y, x] \in \sum_{n \in \mathbb{N}} A_n$. This means that subalgebra $\sum_{n \in \mathbb{N}} A_n$ is S-invariant. Since it includes $A, A^S \leqslant \sum_{n \in \mathbb{N}}$. □

Proof of Theorem B

Let D be a subalgebra of L generated by elements $a_1, \ldots, a_n, D_0 = D, D_1$ be a subalgebra generated by $[L, D_0], [D_0, L], A_{n+1}$ be a subalgebra generated by $[L, D_n]$, and $[D_n, L], n \in \mathbb{N}$. Then Proposition 2.4 implies that $A = D^L = \sum_{n \in \mathbb{N}} D_n$.

By Lemma 2.2, $\mathrm{Ann}_L(D) = \mathrm{Ann}_L(\{a_1, \ldots, a_n\}) = C$. If x is an arbitrary element of L, then $x = \alpha_1 e_1 + \ldots + \alpha_t e_t + z$ for some elements $\alpha_1, \ldots, \alpha_t \in F$ and $z \in C$. For every element $d \in D$, we have

$$\begin{aligned}[d, x] &= [d, \alpha_1 e_1 + \ldots + \alpha_t e_t + z] = [d, \alpha_1 e_1] + \ldots + [d, \alpha_t e_t] + [d, z] \\ &= \alpha_1 [d, e_1] + \alpha_t [d, e_t]\end{aligned}$$

and

$$[x, d] = \alpha_1 [e_1, d] + \ldots + \alpha_t [e_t, d].$$

Denote by E a subalgebra generated by the elements $e_1, \ldots, e_t$. Then we can see that $D_1 \leqslant \langle D, E \rangle$.

Now, consider the element $[[d, e_m], x], 1 \leqslant m \leqslant t$. We have:

$$\begin{aligned}[[d, e_j], x] &= [[d, e_j], \alpha_1 e_1 + \ldots + \alpha_t e_t + z] \\ &= \alpha_1 [[d, e_j], e_1] + \ldots + \alpha_t [[d, e_j], e_t] + [[d, e_j], z].\end{aligned}$$

Further, $[[d, e_j], z] = [d, [e_j, z]] - [e_j, [d, z] = [d, [e_j, z]]$. For the element $[e_j, z]$, we have the following presentation: $[e_j, z] = \beta_1 e_1 + \ldots + \beta_t e_t + z_1$ for some elements $\beta_1, \ldots, \beta_t \in F$ and $z_1 \in C$. Then

$$[d, [e_j, z]] = [d, \beta_1 e - 1 + \ldots + \beta_t e_t + z_1] = \beta_1 [d, e_1] + \ldots + \beta_t [d, e_t] \in \langle D, E \rangle.$$

Hence, $[[d, e_m], x] \in \langle D, E \rangle, 1 \leqslant m \leqslant t$. It follows that $[D_1, L] \leqslant \langle D, E \rangle$. Similarly, $[L, D_1] \leqslant \langle D, E \rangle$, so that $D_2 \leqslant \langle D, E \rangle$.

Using similar arguments and ordinary induction, we obtain that $D_n \leqslant \langle D, E \rangle$ for each positive integer n. Thus, $A \leqslant \langle D, E \rangle$, and therefore $\dim_F(A) \leqslant \dim_F(\langle D, E \rangle$. The subalgebra $\langle D, E \rangle$ is generated by the elements $a_1, \ldots, a_n, e_1, \ldots, e_t$. Now, using Lemma 2.3, we obtain that $\dim_F(\langle D, E \rangle \leqslant (k_1 + \ldots + k_n + s_1 + \ldots + s_t)^2 + n + t$.

Proof of the corollary of Theorem B

If a Leibniz algebra L is ideally finite, then we have already noted that the breadth of its every element is finite. Conversely, if the breadth of its every element is finite, then Theorem 2.5 shows that every element of L generates a finite-dimensional ideal.

Acknowledgments

The first author is grateful for the support Isaac Newton Institute for Mathematical Sciences and the University of Edinburgh provided in the frame of LMS Solidarity Supplementary Grant Program. The authors are sincerely grateful to Professor A. Smoktunowicz for useful consultations.

References

[1] Ayupov SH.A., Omirov B.A., Rakhimov I.S. *Leibniz Algebras: Structure and Classification*, CRC Press, Taylor & Francis Group, (2020).

[2] Amayo R.K., Stewart I.N. *Infinite dimensional Lie algebras*, Noordhoff International Publishing Leyden, The Netherlands (1974).

[3] Bloh A.M. *On a generalization of the concept of Lie algebra.* Doklady AN USSR - 165 (1965), 471 - 473.

[4] Chupordya V.A., Kurdachenko L.A., Subbotin I.Ya. *On some minimal Leibniz algebras.* Journal of Algebra and Its Applications - 16(2017), no. 2. DOI: 10.1142/S0219498817500827

[5] Dixon M.R., Kurdachenko L.A., Otal J. *Linear groups with bounded action*, Algebra Colloquium - 18 (2011), no. 3, 487 - 498.

[6] Dixon M.R., Kurdachenko L.A., Otal J. *Linear groups with finite dimensional orbits.* Ischia Group Theory 2010, Proceedings of the conference in Group Theory, World Scientific, 2012, 131 - 145.

[7] de Giovanni F., Subbotin I.Ya. *Some topics of classical group theory: the genesis and current stage.* Advances in Group Theory and Applications-2019, Volume 8, pp 119 - 153.

[8] Kurdachenko L.A., Muños - Escolano J.M., Otal J. *Groups acting on vector spaces with a large family of invariant subspaces.* Linear and Multilinear Algebra, 2012, Volume 60, Issue 4, 487 - 498.

[9] Kurdachenko L.A., Otal J., Pypka A.A. *On some relationships between the factors of the canonical central series in Leibniz algebras.* European Journal of Mathematics June 2016, Volume 2, Issue 2, pp 565 - 577.

[10] Kurdachenko L.A., Semko N.N., Subbotin I.Ya. *The Leibniz algebras whose subalgebras are ideals.* Open Mathematics 2017, Volume15, p. 92 - 100

[11] Loday J.L. *Cyclic homology.* Grundlehren der Mathematischen Wissenschaften, Vol. 301, 2nd ed., Springer - Verlag, Berlin, 1992

[12] Loday J.L. *Une version non commutative des algebres de Lie; les algebras de Leibniz.* Enseign. Math. 39 (1993), 269 - 293.

[13] Loday J.L. and Pirashvili T. *Universal enveloping algebras of Leibniz algebras and (co)homology*, Math. Annalen 296 (1) (1993) 139- 158.

[14] Stewart I.N. *Lie algebras, generated by finite dimensional ideals.* Pitman Publishing, London (1975).

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Received by the editors: 15.06.2023.