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RESEARCH ARTICLE

Algebraic connections between Menger algebras and Menger hyperalgebras via regularity

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ABSTRACT. Menger hyperalgebras of rank n, where n is a fixed integer, can be regarded as a natural generalization of arbitrary semihypergroups. Based on this knowledge, an interesting question arises: what a generalization of regular semihypergroups is. In the article, we establish the notion of *v-regular Menger hyperalgebras of rank n*, which can be considered as an extension of regular semihypergroups. Furthermore, we study regularity of Menger hyperalgebras of rank n which are induced by some subsets of Menger algebras of rank n. In particular, we obtain sufficient conditions so that the Menger hyperalgebras of rank n are v-regular.

1. Introduction

Based on the concept of unary functions (mappings on a nonempty set), the concept of multiplace functions (which were also called functions of many elements or functions of many variables) was developed. In 1946, K. Menger [8] investigated the algebraic property of the composition of multiplace functions, the so-called a *superassociative law*. Hence, the algebraic structure of multiplace functions was established. Nowadays, such an algebraic structure has been called as a *Menger algebras of rank n*, where n is a fixed natural number. The algebraic structure of Menger

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algebras of rank n consists of a nonempty set and an $(n+1)$-ary operation defined on the set satisfying the superassociative law. By setting the natural number n, Menger algebras can be immediately reduced to semigroups if $n = 1$. So, the concept of Menger algebras of rank n has been considered as one of the suitable generalizations of semigroups. For other generalizations of semigroups see, e.g., [10, 11].

The theories of Menger algebras of rank n and some of its applications are developed by W.A. Dudek and V.S. Trokhimenko who introduced various types of congruences on Menger algebras of rank n which can be considered as generalizations of principal right and left congruences on semigroups, see [3]. In 2012, they studied the concept of subtraction of Menger algebras of rank n, see [6]. Moreover, they also investigated some algebraic properties related to Menger algebras of rank n, see [5]. In addition, V.S. Trokhimenko [9] established the so-called v-regular Menger algebras. Furthermore, v-regular Menger algebras of rank n can be reduced to regular semigroups.

In this article, we aim to investigate algebraic connections between Menger algebras of rank n and Menger hyperalgebras of rank n via the concept of regularity. In order to achieve the main aim, we start with recalling some basic results on Menger algebras and Menger hyperalgebras. After that, we introduce the notion of v-regular Menger hyperalgebras of rank n which is an extension of regular semihypergroups. Then, we construct Menger hyperalgebras of rank n by using subsets and $(n+1)$-ary operations of Menger algebras of rank n. In particular, we also obtain some specific conditions on the Menger hyperalgebras of rank n to be v-regular Menger hyperalgebras of rank n.

2. Preliminaries

In order to obtain the main aim, we recall some basic definitions and results on Menger algebras of rank n and Menger hyperalgebras of rank n.

Let H be a nonempty set of elements and $\bullet : H^{n+1} \longrightarrow H$ be an $(n+1)$-ary operation on H. A Menger algebra of rank n is an algebraic structure of $(H, \bullet)$ such that the $(n+1)$-ary operation satisfies the *superassociative law* as follows: for each $x, y_i, z_i \in H, i = 1, \ldots, n$,

$$\begin{aligned}\bullet(\bullet(x, y_1, \ldots, y_n), &z_1, \ldots, z_n) \\ &= \bullet(x, \bullet(y_1, z_1, \ldots, z_n), \ldots, \bullet(y_n, z_1, \ldots, z_n)).\end{aligned} \tag{1}$$

According to the superassociative law defined as in (1), it is easy to see that it can be reduced to the usual associative law on semigroups by setting $n = 1$, and hence a Menger algebra $(H, \bullet)$ is immediately reduced to a semigroup.

Let $(H, \bullet)$ be a Menger algebra of rank n. For convenience, from now on, a symbol $x[\bar{y}]$ stands for $\bullet(x, y_1, \ldots, y_n)$. In case, $y_1 = \ldots = y_n = y$, we write $x[y^n]$ instead of $\bullet(x, y_1, \ldots, y_n)$. So, the superassociatibe law given in (1) can be simply written as:

$$x[\bar{y}][\bar{z}] = x[y_1[\bar{z}] \ldots y_n[\bar{z}]].$$

Here, $x[\bar{y}][\bar{z}]$ stands for $(x[\bar{y}])[\bar{z}]$. An element $(a_1, \ldots, a_n) \in H^n$ of a Menger algebra $(H, \bullet)$ of rank n is said to be a v-regular element if the following equalities:

$$a_i = a_i[x^n][\bar{a}], i = 1, \ldots, n$$

hold for some $x \in H$. Moreover, if each element $(a_1, \ldots, a_n) \in H^n$ is v-regular, then $(H, \bullet)$ is called a v-regular Menger algebra of rank n.

Example 1. (i) An $(n+1)$-ary groupoid $(H, \bullet)$ together with an $(n+1)$-ary operation $\bullet : H^n \longrightarrow H$ defined as follows:

$$x[\bar{y}] = x \quad \text{for all } x, y_i \in H, i = 1, \ldots, n,$$

forms a v-regular Menger algebra of rank n.

(ii) Let $\mathbb{R}^+$ be the set of all positive real numbers. Define an $(n+1)$-ary operation $\bullet$ on $\mathbb{R}$ by

$$x[\bar{y}] = x \times \sqrt[n]{y_1 \times \ldots \times y_n} \quad \text{for all } x, y_i \in H, i = 1, \ldots, n.$$

Then, the pair $(\mathbb{R}^+, \bullet)$ forms a Menger algebra of rank n.

Now, let H be a nonempty set and $\cdot : H^{n+1} \longrightarrow P^*(H)$ be an $(n+1)$-ary hyperoperation on H, where $P^*(H)$ is the family of all nonempty subsets of H. For any nonempty subsets $X, Y_i, i = 1, \ldots, n$ of H, we define

$$\cdot(X, Y_1, \ldots, Y_n) = \cup\{\cdot(x, y_1, \ldots, y_n) \mid x \in X, y_i \in Y_i, i = 1, \ldots, n\}.$$

Definition 1. *An $(n+1)$-ary hypergroupoid $(H, \cdot)$ is said to be a Menger hyperalgebra of rank n if its $(n+1)$-ary hyperoperation $\cdot$ satisfies the superassociative law defined as in (1).*

According to an algebraic hyperstructure of Menger algebras of rank n, we can construct a new semihypergroup such that its algebraic hyperstructure is induced by the hyperstructure of a Menger hyperalgebra of rank n as the following proposition.

Proposition 1. [7]. *Let $(H, \cdot)$ be a Menger hyperalgebra of rank n. Define a binary hyperoperation $\circ$ on H by:*

$$\circ(x, y) = x[y^n] \quad \textit{for all } x, y \in H. \tag{2}$$

Then, the binary hypergroupoid $(H, \circ)$ forms a semihypergroup.

On a Menger algebra $(H, \cdot)$ of rank n, the semihypergroup which was defined as in Proposition 1 is called a *diagonal semihypergroup* of $(H, \cdot)$.

Definition 2. [2]. *Let $(H, \cdot)$ be a Menger algebra of rank n. A nonempty subset I of H is said to be:*

(i) an s-ideal, if for every $x, y_i \in H, i = 1, \dots, n$

$$x \in I \Longrightarrow \cdot(x, y_1, \dots, y_n) \in I;$$

(ii) a v-ideal, if for every $x_i, y \in H, i = 1, \dots, n$

$$x_1, \dots, x_n \in I \Longrightarrow \cdot(y, x_1, \dots, x_n) \in I.$$

Now, we can see that s-ideals (resp. v-ideals) in a Menger algebra $(H, \cdot)$ of rank n defined as in Definition 2 and right ideals (resp. left ideals) on a semigroup are the same, if $n = 1$.

Next, we introduce some special elements of Menger algebras of rank n as follows:

Definition 3. [2]. *Let $(H, \cdot)$ be a Menger algebra of rank n. An element $e \in H$ is said to be:*

(i) a left diagonal unit if $x = \cdot(e, x^n)$ for all $x \in H$;

(ii) a right diagonal unit if $x = \cdot(x, e^n)$ for all $x \in H$;

(iii) a diagonal unit if $x = \cdot(e, x^n) = \cdot(x, e^n)$ for all $x \in H$,

where x^n is a sequence $\underbrace{x, x, \dots, x}_{n \text{ times}}$.

Let $(H, \cdot)$ be a Menger algebra of rank n containing a diagonal unit e. For $x \in H$, if there is $y \in H$ ($z \in H$) such that $\cdot(y, x^n) = e$ (resp. $\cdot(x, z^n) = e$), then the element x is said to be a left (right) invertible element of $(H, \cdot)$.

Example 2. Let p be a fixed positive integer. Define an $(n+1)$-ary hyperoperation $\cdot$ on the set $H = \{2, 4, \ldots, 2p\}$ by

$$\cdot(x, y_1, \ldots, y_n) = \{z \in H \mid z \leq \max\{x, y_1, \ldots, y_n\}\},$$

for all $x, y_i \in H, i = 1, \ldots, n$. Hence, $(H, \cdot)$ is a Menger hyperalgebra of rank n.

For more results on Menger hyperalgebras of rank n, we refer the reader to [7].

3. Regularity of Menger hyperalgebras

In this section, we first establish the so-called v-regular Menger hyperalgebras of rank n, which may be regarded as a generalization of arbitrary semihypergroups. Moreover, we investigate regularity of a Menger hyperalgebra of rank n which is induced by some nonempty subsets of the based set of a Menger algebra of rank n.

Definition 4. *Let $(H, \cdot)$ be a Menger hyperalgebra of rank n. An element $(a_1, ..., a_n) \in H^n$ is said to be:*

(i) idempotent if $a_i \in \cdot(a_i, a_1, \ldots, a_n)$ for all $i = 1, \ldots, n$;

(ii) v-regular if there is $x \in H$ such that

$$a_i \in \cdot(\cdot(a_i, x^n), a_1, \ldots, a_n) \quad \text{for all } i = 1, \ldots, n.$$

A Menger hyperalgebra of rank n is called *v-regular* if every element $(a_1, ..., a_n) \in H^n$ is v-regular.

According to Definition 4, we see that a v-regular Menger hyperalgebra of rank n is immediately reduced to a regular semihypergroup, if $n = 1$. It means that the algebraic structure of v-regular Menger hyperalgebras of rank n can be considered as a generalization of regular semihypergroups.

Example 3. (i) The Menger hyperalgebra $(H, \cdot)$ of rank n in Example 2 forms a v-regular Menger hyperalgebra of rank n.

(ii) Consider the set $\mathbb{R}$ of all real numbers under an $(n+1)$-ary hyperoperation $\cdot$ on $\mathbb{R}$ given as follows:

$$\cdot(x, y_1, \ldots, y_n) = \{z \in \mathbb{R} \mid z \leq \min\{x, y_1, \ldots, y_n\}\},$$

for all $x, y_i \in \mathbb{R}, i = 1, \ldots, n$. Then $(\mathbb{R}, \cdot)$ is a v-regular Menger hyperalgebra of rank n, because for every element $(r_1, \ldots, r_n) \in \mathbb{R}^n$, there is $x = \max\{r_1, \ldots, r_n\} \in \mathbb{R}$ such that

$$r_i \in \cdot(\cdot(r_i, x^n), r_1, \ldots, r_n) \quad \text{for all } i = 1, \ldots, n.$$

Proposition 2. *Let $(H, \cdot)$ be a v-regular Menger hyperalgebra of rank n. Then a diagonal semihypergroup $(H, \circ)$ of $(H, \cdot)$ forms a regular semihypergroup, i. e., for each $a \in H$ there exists $x \in H$ such that $a \in \circ(\circ(a, x), a)$.*

Proof. The proof is strengthforward. □

Now, let $(H, \bullet)$ be a Menger algebra of rank n. If an element $(a_1, \ldots, a_n) \in H^n$ is v-regular, then we set

$$V_{\bar{a}} = \{x \in H \mid a_i = a_i[x^n][\bar{a}] \quad \text{for all } i = 1, \ldots, n\}.$$

For every nonempty subset P of H, we denote

$$\widehat{P} = \{(p, \ldots, p) \in P^n \mid p \in P\} \text{ and } x[\widehat{P}] = \{x[p^n] \in H \mid x \in H, p \in P\}.$$

Proposition 3. *Let $(H, \bullet)$ be a Menger algebra of rank n and P a nonempty subset of H. Define an $(n+1)$-ary hyperoperation $\bullet_P$ on H as follows:*

$$\bullet_P(x, y_1, \ldots, y_n) = x[\widehat{P}][\bar{y}] \quad \textit{for all } x, y_i \in H, i = 1, \ldots, n. \tag{3}$$

That is, $\bullet_P(x, y_1, \ldots, y_n) = \{x[p^n][\bar{y}] \in H \mid x, y_i \in H, p \in P, i = 1, \ldots, n\}$. Then $(H, \bullet_P)$ forms a Menger hyperalgebra of rank n.

Proof. By using the superassociative law on the $(n+1)$-ary operation $\bullet$ on H, we have

$$\begin{aligned}
\bullet_P(\bullet_P(x, y_1, \ldots, y_n), z_1, \ldots z_n) &= x[\widehat{P}][\bar{y}][\widehat{P}][\bar{z}] \\
&= \{x[p^n][\bar{y}][q^n][\bar{z}] \mid x, y_i, z_i \in H, p, q \in P, i = 1, \ldots, n\} \\
&= \{x[p^n][y_1[q^n] \ldots y_n[q^n]][\bar{z}] \mid x, y_i, z_i \in H, p, q \in P, i = 1, \ldots, n\} \\
&= \{x[p^n][y_1[q^n][\bar{z}] \ldots y_n[q^n][\bar{z}]] \mid x, y_i, z_i \in H, p, q \in P, i = 1, \ldots, n\} \\
&= x[\widehat{P}][y_1[\widehat{P}][\bar{z}] \ldots y_n[\widehat{P}][\bar{z}]] \\
&= \bullet_P(x, \bullet_P(y_1, z_1, \ldots, z_n), \ldots, \bullet_P(y_n, z_1, \ldots, z_n)),
\end{aligned}$$

for each $x, y_i, z_i \in H, i = 1, \ldots, n$. It implies that $\bullet_P$ satisfies the superassociative law. Consequently, $(H, \bullet_P)$ is a Menger hyperalgebra of rank n. □

For every Menger algebra $(H, \bullet)$ of rank n and a nonempty subset P of H, the Menger hyperalgebra $(H, \bullet_P)$ of rank n, where an $(n+1)$-ary hyperoperation $\bullet_P$ is defined as in (3), is called a *Menger hyperalgebra of rank n induced by a subset P of a Menger algebra $(H, \bullet)$ of rank n.*

Lemma 1. *Let $(H, \bullet)$ be a Menger algebra of rank n. An element $(a_1, \ldots, a_n) \in H^n$ is v-regular in a Menger hyperalgebra $(H, \bullet_P)$ of rank n if and only if the element $(a_1, \ldots, a_n) \in H^n$ is v-regular in $(H, \bullet)$ and $V_{\bar{a}} \cap P[\widehat{H}][\widehat{P}] \neq \emptyset$.*

Proof. $(\Longrightarrow)$ Assume that an element $(a_1, \ldots, a_n) \in H^n$ is v-regular in a Menger hyperalgebra $(H, \bullet_P)$ of rank n. Then there exists $x \in H$ such that

$$a_i \in \bullet_P(\bullet_P(a_i, x^n), a_1, \ldots, a_n) \quad \text{for all } i = 1, \ldots, n.$$

It means that $a_i \in \bullet_P(\bullet_P(a_i, x^n), a_1, \ldots, a_n) = a_i[\widehat{P}][x^n][\widehat{P}][\bar{a}]$ for all $i = 1, \ldots, n$. Hence, there are $p, q \in P$ such that

$$\begin{aligned} a_i &= a_i[p^n][x^n][q^n][\bar{a}] \\ &= a_i[p[x^n] \ldots p[x^n]][q^n][\bar{a}] \\ &= a_i[p[x^n][q^n] \ldots p[x^n][q^n]][\bar{a}] \\ &= a_i[y^n][\bar{a}], \end{aligned}$$

where $y = p[x^n][q^n] \in H$. So, the element $(a_1, \ldots, a_n)$ is v-regular in $(H, \bullet)$. Moreover, $y = p[x^n][q^n] \in V_{\bar{a}} \cap P[\widehat{H}][\widehat{P}]$, and hence $V_{\bar{a}} \cap P[\widehat{H}][\widehat{P}] \neq \emptyset$.

$(\Longleftarrow)$ Assume that an element $(a_1, \ldots, a_n) \in H^n$ is v-regular in a Menger algebra $(H, \bullet)$ of rank n and $V_{\bar{a}} \cap P[\widehat{H}][\widehat{P}] \neq \emptyset$. Then there exists $x \in H$ such that $x \in V_{\bar{a}} \cap P[\widehat{H}][\widehat{P}]$. That is, $a_i = a_i[x^n][\bar{a}]$ for all $i = 1, \ldots, n$ and $x = p[y^n][q^n]$ for some $y \in H, p, q \in P$. It follows that

$$\begin{aligned} a_i &= a_i[x^n][\bar{a}] \\ &= a_i[p[y^n][q^n] \ldots p[y^n][q^n]][\bar{a}] \\ &= a_i[p[y^n] \ldots p[y^n]][q^n][\bar{a}] \\ &= a_i[p^n][y^n][q^n][\bar{a}] \\ &\in a_i[\widehat{P}][y^n][\widehat{P}][\bar{a}] \\ &= \bullet_P(\bullet_P(a_i, y^n), a_1, \ldots, a_n). \end{aligned}$$

Consequently, the element $(a_1, \ldots, a_n) \in H^n$ is v-regular in a Menger hyperalgebra $(H, \bullet_P)$ of rank n. □

Corollary 1. *Let $(H, \bullet)$ be a Menger algebra of rank n. If an element $(a_1, \ldots, a_n) \in H^n$ is v-regular in a Menger hyperalgebra $(H, \bullet_P)$ of rank n, then the element $(a_1, \ldots, a_n) \in H^n$ is v-regular in $(H, \bullet)$.*

Proof. The proof follows from Lemma 1. □

Corollary 2. *Let $(H, \bullet)$ be a Menger algebra of rank n. If a Menger hyperalgebra $(H, \bullet_P)$ of rank n is v-regular, then the Menger algebra $(H, \bullet)$ of rank n is v-regular.*

Proof. The proof follows from Corollary 1. □

The following example is given to show that the converse of Corollary 1 need not be true, i.e., there is a subset P of H such that an element $(a_1, \ldots, a_n) \in H^n$ is not v-regular in a Menger hyperalgebra $(H, \bullet_P)$ of rank n, but it is v-regular in a Menger algebra $(H, \bullet)$ of rank n.

Example 4. Let $\mathbb{N}$ be the set of all natural numbers. Define an $(n+1)$-ary operation $\bullet$ on $\mathbb{N}$ by

$$x[\bar{y}] = \max\{x, y_1, \ldots, y_n\} \quad \text{for all } x, y_i \in \mathbb{N}, i = 1, \ldots, n.$$

Then $(\mathbb{N}, \bullet)$ is a Menger algebra of rank n in which each element $(x, \ldots, x) \in \mathbb{N}^n$ is v-regular. Now, let $m \in \mathbb{N}$. Consider the set $P = \{m+1, m+2, \ldots, m+n\}$. We see that an element $(m, \ldots, m) \in \mathbb{N}^n$ is not v-regular in a Menger hyperalgebra $(\mathbb{N}, \bullet_P)$ of rank n, i.e., there is no $x \in \mathbb{N}$ such that $m \in \bullet_P(\bullet_P(m, x^n), m^n)$. Moreover, $V_{(m,\ldots,m)} = \{1, 2, \ldots, m\}$ and $V_{(m,\ldots,m)} \cap P[\widehat{\mathbb{N}}][\widehat{P}] = \emptyset$.

Theorem 1. *Let $(H, \bullet)$ be a Menger algebra of rank n. A Menger hyperalgebra $(H, \bullet_P)$ of rank n is v-regular if and only if $(H, \bullet)$ is v-regular and $V_{\bar{a}} \cap P[\widehat{H}][\widehat{P}] \neq \emptyset$.*

Proof. The proof follows from Lemma 1. □

Proposition 4. *Let $(H, \bullet)$ be a Menger algebra of rank n and P, Q be nonempty subsets of H such that $P \subseteq Q$. If an element $(a_1, \ldots, a_n) \in H^n$ is v-regular in a Menger hyperalgebra $(H, \bullet_P)$ of rank n, then the element $(a_1, \ldots, a_n) \in H^n$ is v-regular in a Menger hyperalgebra $(H, \bullet_Q)$ of rank n.*

Proof. Let $(a_1, \ldots, a_n) \in H^n$ be v-regular in a Menger hyperalgebra $(H, \bullet_P)$ of rank n. By Lemma 1, the element $(a_1, \ldots, a_n) \in H^n$ is v-regular in a Menger algebra $(H, \bullet)$ of rank n and $V_{\bar{a}} \cap P[\widehat{H}][\widehat{P}] \neq \emptyset$.

Since $P \subseteq Q$, we have $V_{\bar{a}} \cap Q[\widehat{H}][\widehat{Q}] \neq \emptyset$. Again, by Lemma 1, the element $(a_1, \ldots, a_n) \in H^n$ is v-regular in a Menger hyperalgebra $(H, \bullet_Q)$ of rank n. □

Corollary 3. *Let $(H, \bullet)$ be a Menger algebra of rank n and P, Q be nonempty subsets of H such that $P \subseteq Q$. If a Menger hyperalgebra $(H, \bullet_P)$ of rank n is v-regular, then a Menger hyperalgebra $(H, \bullet_Q)$ of rank n is v-regular.*

Proof. The proof follows from Proposition 4. □

Theorem 2. *Let $(H, \bullet)$ be a v-regular Menger algebra of rank n with a diagonal unit e. Then a Menger hyperalgebra $(H, \bullet_P)$ of rank n is v-regular if and only if P contains a left invertible element and a right invertible element of $(H, \bullet)$.*

Proof. ($\Longrightarrow$) Assume that a Menger hyperalgebra $(H, \bullet_P)$ is v-regular. Then an element $(e, e, \ldots, e) \in H^n$ is v-regular in $(H, \bullet_P)$. That means, there is $x \in H$ such that $e \in \bullet_P(\bullet_P(e, x^n), e^n) = e[\widehat{P}][x^n][\widehat{P}][e^n]$, which implies that

$$\begin{aligned} e &= e[p^n][x^n][q^n][e^n] = p[x^n][q^n][e^n] \\ &= p[x^n][q[e^n] \ldots q[e^n]] = p[x^n][q^n] \\ &= p[x[q^n] \ldots x[q^n]] \end{aligned}$$

for some $p, q \in P$. So, $e = p[x^n][q^n]$ and $e = p[x[q^n] \ldots x[q^n]]$, which means that the element $p \in P$ is a right invertible element and the element $q \in P$ is a left invertible element of $(H, \bullet)$.

($\Longleftarrow$) Let x and y be a left invertible element and a right invertible element of a Menger algebra $(H, \bullet)$ of rank n, respectively. Then there are $s, t \in H$ such that $e = s[x^n]$ and $e = y[t^n]$. Let $(a_1, \ldots a_n) \in H^n$. Since $(H, \bullet)$ is v-regular, there is $b \in H$ such that $a_i = a_i[b^n][\bar{a}]$ for all $i = 1, \ldots, n$.

Now, we have

$$\begin{aligned} a_i &= a_i[b^n][\bar{a}] = a_i[e^n][b[e^n] \ldots b[e^n]][\bar{a}] \\ &= a_i[e^n][b^n][e^n][\bar{a}] = a_i[y[t^n] \ldots y[t^n]][b^n][s[x^n] \ldots s[x^n]][\bar{a}] \\ &= a_i[y^n][t^n][b^n][s^n][x^n][\bar{a}] = a_i[y^n][t[b^n] \ldots t[b^n]][s^n][x^n][\bar{a}] \\ &= a_i[y^n][t[b^n][s^n] \ldots t[b^n][s^n]][x^n][\bar{a}] \in a_i[\widehat{P}][z^n][\widehat{P}][\bar{a}] \\ &= \bullet_P(\bullet_P(a_i, z^n), a_1, \ldots, a_n), \end{aligned}$$

where $z = t[b^n][s^n] \in H$. It means that the element $(a_1, \dots, a_n) \in H^n$ is v-regular. Consequently, $(H, \bullet_P)$ forms a v-regular Menger hyperalgebra of rank n. □

Corollary 4. *Let $(H, \bullet)$ be a v-regular Menger algebra of rank n with a diagonal unit and every one-sided invertible element of $(H, \bullet)$ is invertible. A Menger hyperalgebra $(H, \bullet_P)$ of rank n is v-regular if and only if P has an invertible element of $(H, \bullet)$.*

Proof. The proof follows from Theorem 2. □

Theorem 3. *Let P be a s-ideal and Q a v-ideal of a Menger algebra $(H, \bullet)$ of rank n such that $P \cap Q \neq \emptyset$. If an element $(a_1, \dots, a_n) \in H^n$ is v-regular in Menger hyperalgebra $(H, \bullet_P)$ and $(H, \bullet_Q)$ of rank n, then the element $(a_1, \dots, a_n) \in H^n$ is v-regular in a Menger hyperalgebra $(H, \bullet_{P \cap Q})$ of rank n.*

Proof. Let $(a_1, \dots, a_n) \in H^n$ be v-regular in $(H, \bullet_P)$ and $(H, \bullet_Q)$. Then there are $x, y \in H$ such that

$$a_i \in \bullet_P(\bullet_P(a_i, x^n), a_1, \dots, a_n) = a_i[\widehat{P}][x^n][\widehat{P}][\bar{a}] \quad \text{and}$$
$$a_i \in \bullet_Q(\bullet_Q(a_i, y^n), a_1, \dots, a_n) = a_i[\widehat{Q}][y^n][\widehat{Q}][\bar{a}].$$

So, for each $i = 1, \dots, n$, we have $a_i = a_i[k^n][x^n][l^n][\bar{a}]$ and $a_i = a_i[s^n][y^n][t^n][\bar{a}]$ for some $k, l \in P, s, t \in Q$. By the superassociative law of the $(n+1)$-ary operation $\bullet$, we obtain

$$\begin{aligned}
a_i &= a_i[k^n][x^n][l^n][\bar{a}] \\
&= a_i[k^n][x^n][l^n][a_1[s^n][y^n][t^n][\bar{a}] \dots a_n[s^n][y^n][t^n][\bar{a}]] \\
&= a_i[k^n][x^n][l^n][\bar{a}][s^n][y^n][t^n][\bar{a}] \\
&= a_i[k^n][x^n][l^n][\bar{a}][s^n][y^n][t^n][a_1[k^n][x^n][l^n][\bar{a}] \dots a_n[k^n][x^n][l^n][\bar{a}]] \\
&= a_i[k^n][x^n][l^n][\bar{a}][s^n][y^n][t^n][\bar{a}][k^n][x^n][l^n][\bar{a}] \\
&= a_i[k^n][x^n][l^n][\bar{a}][s^n][y^n][t^n][\bar{a}][k^n][x^n][l^n] \\
&\quad [a_1[s^n][y^n][t^n][\bar{a}] \dots a_n[s^n][y^n][t^n][\bar{a}]] \\
&= a_i[k^n][x^n][l^n][\bar{a}][s^n][y^n][t^n][\bar{a}][k^n][x^n][l^n][\bar{a}][s^n][y^n][t^n][\bar{a}] \\
&= a_i[(k[x^n][l^n][\bar{a}][s^n])^n][(y[t^n][\bar{a}][k^n][x^n])^n[(l[\bar{a}][s^n][y^n][t^n])^n][\bar{a}].
\end{aligned}$$

Since P is a s-ideal and Q is a v-ideal of $(H, \bullet)$, we have

$$k[x^n][l^n][\bar{a}][s^n] \in P \cap Q \text{ and } l[\bar{a}][s^n][y^n][t^n] \in P \cap Q.$$

Now, we obtain

$$\begin{aligned} a_i &\in a_i[\widehat{P \cap Q}][(y[t^n][\bar{a}][k^n][x^n])^n][\widehat{P \cap Q}][\bar{a}] \\ &= \bullet_{P\cap Q}(\bullet_{P\cap Q}(a_i, z^n), a_1, \ldots, a_n), \end{aligned}$$

where $z = y[t^n][\bar{a}][k^n][x^n] \in H$. It follows that the element $(a_1, \ldots, a_n) \in H^n$ is v-regular in a Menger hyperalgebra $(H, \bullet_{P\cap Q})$ of rank n. □

Finally, we complete this section by the theorem, which shows that the regularity of Menger hyperalgebras of rank n induced by subsets of Menger algebras of rank n can be preserved under the isomorphic algebraic structures of the Menger algebras of rank n.

Theorem 4. *Let ϕ be an isomorphism from a Menger algebra $(G, \diamond)$ of rank n onto a Menger algebra $(H, \bullet)$ of rank n. Then the following assertions hold:*

(i) a Menger hyperalgebra $(G, \diamond_P)$ of rank n and a Menger hyperalgebra $(H, \bullet_{\phi(P)})$ of rank n are isomorphic for all nonempty subsets P of G;

(ii) if an element $(a_1, \ldots, a_n) \in G^n$ in a Menger hyperalgebra $(G, \diamond_P)$ of rank n is v-regular, then an element $(\phi(a_i), \ldots, \phi(a_n)) \in H^n$ in a Menger hyperalgebra $(H, \bullet_{\phi(P)})$ of rank n is v-regular,

where $\phi(P) = \{y \in H \mid \phi(x) = y \text{ for some } x \in P\}$.

Proof. (i) Assume that ϕ is an isomorphism from $(G, \diamond)$ to $(H, \bullet)$. It immediately implies that ϕ is a bijective function from G onto H, which is the base sets of a Menger hyperalgebra $(G, \diamond_P)$ and $(H, \bullet_{\phi(P)})$ of rank n.

Indeed, for each $x, y_i \in G, i = 1, \ldots, n$, we get

$$\begin{aligned} \phi(\diamond_P(x, y_1, \ldots, y_n)) &= \phi((\diamond(x, \widehat{P}), y_1 \ldots, y_n)) \\ &= \{\phi(\diamond(\diamond(x, p^n), y_1 \ldots, y_n)) \in H \mid p \in P\} \\ &= \{\bullet(\bullet(\phi(x), (\phi(p))^n), \phi(y_1), \ldots, \phi(y_n)) \in H \mid \phi(p) \in \phi(P)\} \\ &= \bullet(\bullet(\phi(x), \widehat{\phi(P)}), \phi(y_1), \ldots, \phi(y_n)) \\ &= \bullet_{\phi(P)}(\phi(x), \phi(y_1), \ldots, \phi(y_n)). \end{aligned}$$

So, ϕ is an isomorphism from $(G, \diamond_P)$ to $(H, \bullet_{\phi(P)})$, which also implies that $(G, \diamond_P) \cong (H, \bullet_{\phi(P)})$.

(ii) Assume that an element $(a_1, \ldots, a_n) \in G^n$ in a Menger hyperalgebra $(G, \diamond_P)$ of rank n is v-regular. By our assumption, there is $x \in G$ such that

$$a_i \in \diamond_P(\diamond_P(a_i,(x)^n),a_1,\ldots,a_n) = \diamond(\diamond(\diamond(\diamond(a_i,\widehat{P}),(x)^n),\widehat{P}),a_1,\ldots,a_n)$$

for all $i = 1,\ldots,n$. Then there are $p,q \in P$ such that

$$a_i = \diamond(\diamond(\diamond(\diamond(a_i,(p)^n),(x)^n),(q)^n),a_1,\ldots,a_n).$$

Now, we have

$$\begin{aligned}\phi(a_i) &= \phi(\diamond(\diamond(\diamond(\diamond(a_i,p^n),x^n),q^n),a_1,\ldots,a_n))\\ &= \bullet(\bullet(\bullet(\bullet(\phi(a_i),(\phi(p))^n),(\phi(x))^n),(\phi(q))^n),\phi(a_1),\ldots,\phi(a_n))\\ &\in \bullet(\bullet(\bullet(\bullet(\phi(a_i),\widehat{\phi(P)}),(\phi(x))^n),\widehat{\phi(P)}),\phi(a_1),\ldots,\phi(a_n))\\ &= \bullet_{\phi(P)}(\bullet_{\phi(P)}(\phi(a_i),(\phi(x))^n),\phi(a_1),\ldots,\phi(a_n)).\end{aligned}$$

Thus, an element $(\phi(a_i),\ldots,\phi(a_n)) \in H^n$ is v-regular in a Menger hyperalgebra $(H,\bullet_{\phi(P)})$ of rank n. □

Corollary 5. *Let $(G,\diamond)$ and $(H,\bullet)$ be isomorphic Menger algebras of rank n under an isomorphism ϕ. If a Menger hyperalgebra $(G,\diamond_P)$ of rank n is v-regular, then a Menger hyperalgebra $(H,\bullet_{\phi(P)})$ of rank n is v-regular.*

Proof. The proof follows from Theorem 4 (ii). □

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