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Coarse selectors of groups

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Abstract. For a group G, $\mathcal{F}_G$ denotes the set of all non-empty finite subsets of G. We extend the finitary coarse structure of G from $G \times G$ to $\mathcal{F}_G \times \mathcal{F}_G$ and say that a macro-uniform mapping $f\colon \mathcal{F}_G \to \mathcal{F}_G$ (resp. $f\colon [G]^2 \to G$) is a finitary selector (resp. 2-selector) of G if $f(A) \in A$ for each $A \in \mathcal{F}_G$ (resp. $A \in [G]^2$). We prove that a group G admits a finitary selector if and only if G admits a 2-selector and if and only if G is a finite extension of an infinite cyclic subgroup or G is countable and locally finite. We use this result to characterize groups admitting linear orders compatible with finitary coarse structures.

1. Introduction and results

The notions of selectors came from *Topology*. Let X be a topological space, $\exp X$ denotes the set of all non-empty closed subsets of X endowed with some (initially, the Vietoris) topology, $\mathcal{F}$ be a non-empty subset of $\exp X$. A continuous mapping $f : \mathcal{F} \to X$ is called an $\mathcal{F}$-selector of X if $f(A) \in A$ for each $A \in \mathcal{F}$. The question on selectors of topological spaces was studied in a plenty of papers, we mention only [1], [4], [9], [10].

Formally, coarse spaces, introduced independently and simultaniously in [17] and [13], can be considered as asymptotic counterparts of uniform topological spaces. But actually, this notion is rooted in *Geometry, Geometric Group Theory* and *Combinatorics*, see [17, Chapter 1], [6, Chapter 4] and [13]. Every group G admits the natural finitary coarse structure which, in the case of finitely generated G, can be viewed as the metric structure of a Cayley graph of G. At this point, we need some basic definitions.

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Given a set X, a family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* on X if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X := \{(x,x) : x \in X\}$ of X;
- if E, $E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x,y) : \exists z \ \ ((x,z) \in E, \ (z,y) \in E')\}$, $E^{-1} = \{(y,x) : (x,y) \in E\}$;
- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$.

Elements $E \in \mathcal{E}$ of the coarse structure are called *entourages* on X.

For $x \in X$ and $E \in \mathcal{E}$ the set $E[x] := \{y \in X : (x,y) \in \mathcal{E}\}$ is called the *ball of radius E centered at x*. Since $E = \bigcup_{x \in X}(\{x\} \times E[x])$, the entourage E is uniquely determined by the family of balls $\{E[x] : x \in X\}$. A subfamily $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* of the coarse structure $\mathcal{E}$ if each set $E \in \mathcal{E}$ is contained in some $E' \in \mathcal{E}'$.

The pair $(X, \mathcal{E})$ is called a *coarse space* [17] or a *ballean* [13], [16].

A coarse space $(X, \mathcal{E})$ is called *connected* if, for any $x, y \in X$, there exists $E \in \mathcal{E}$ such that $y \in E[x]$.

A subset $Y \subseteq X$ is called *bounded* if $Y \subseteq E[x]$ for some $E \in \mathcal{E}$ and $x \in X$. If $(X, \mathcal{E})$ is connected then the family $\mathcal{B}_X$ of all bounded subsets of X is a bornology on X. We recall that a family $\mathcal{B}$ of subsets of a set X is a *bornology* if $\mathcal{B}$ contains the family $[X]^{<\omega}$ of all finite subsets of X and $\mathcal{B}$ is closed under finite unions and taking subsets. A bornology $\mathcal{B}$ on a set X is called *unbounded* if $X \notin \mathcal{B}$. A subfamily $\mathcal{B}'$ of $\mathcal{B}$ is called a base for $\mathcal{B}$ if, for each $B \in \mathcal{B}$, there exists $B' \in \mathcal{B}'$ such that $B \subseteq B'$.

Each subset $Y \subseteq X$ defines a *subspace* $(Y, \mathcal{E}|_Y)$ of $(X, \mathcal{E})$, where $\mathcal{E}|_Y = \{E \cap (Y \times Y) : E \in \mathcal{E}\}$. A subspace $(Y, \mathcal{E}|_Y)$ is called *large* if there exists $E \in \mathcal{E}$ such that $X = E[Y]$, where $E[Y] = \bigcup_{y \in Y} E[y]$.

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be coarse spaces. A mapping $f : X \to X'$ is called *macro-uniform* if for every $E \in \mathcal{E}$ there exists $E' \in \mathcal{E}'$ such that $f(E(x)) \subseteq E'(f(x))$ for each $x \in X$. If f is a bijection such that f and f^{-1} are macro-uniform, then f is called an *asymorphism.* If $(X, \mathcal{E})$ and $(X', \mathcal{E}')$ contain large asymorphic subspaces, then they are called *coarsely equivalent.*

Given a coarse spaces $(X, \mathcal{E})$, we denote by $\exp X$ the set of all non-empty subsets of X and endow $\exp X$ with the coarse structure $\exp \mathcal{E}$ with the base $\{\exp E : E \in \mathcal{E}\}$, where

$$(A, B) \in \exp E \Leftrightarrow A \subseteq E[B], \quad B \subseteq E[A].$$

The coarse space $(\exp X, \exp \mathcal{E})$ is called the *hyperballean* of $(X, \mathcal{E})$, for hyperballeans see [2], [3], [14], [15].

Now we are ready to the key definition. Let $(X, \mathcal{E})$ be coarse space, $\mathcal{F}$ be a non-empty subspace of $\exp X$. A macro-uniform mapping $f : \mathcal{F} \longrightarrow X$

is called an $\mathcal{F}$-*selector* of $(X, \mathcal{E})$ if $f(A) \in A$ for each $A \in \mathcal{F}$. In the case $\mathcal{F} = \exp X$, $\mathcal{F} = \mathcal{B} \setminus \{0\}$, $\mathcal{F} = [X]^2$ we get a *global selector*, a *bornologous selector* and a *2-selector* respectively. The investigation of selectors of coarse spaces was initiated in [11], [12].

Every group G with the identity e can be considered as the coarse spaces $(G, \mathcal{E})$, where $\mathcal{E}$ is the *(right) finitary coarse structure* with the base

$$\{\{(x, y) : x \in Fy\} : F \in [G]^{<\omega},\ e \in F\}.$$

We note that the bornology of $(G, \mathcal{E})$ coincides with $\mathcal{F}_G$ and use the name *finitary selector* in place the bornologous selector.

Every metric d on a set X defines the coarse structure $\mathcal{E}_d$ on X with the base $\{\{(x, y) : d(x, y) \leqslant r\} : r > 0\}$. Given a connected graph Γ, $\Gamma = \Gamma[V]$, we denote by d the path metric on the set V of vertices of Γ and consider Γ as the coarse space $(V, \mathcal{E}_d)$. We recall that Γ is *locally finite* if the set $\{y : d(x, y) \leqslant 1\}$ if finite for each $x \in V$.

Our goal is to prove the following theorem.

Theorem 1. *For a group G, the following statements are equivalent:*

(i) *G admits a finitary selector;*

(ii) *G admits a 2-selector;*

(iii) *G is a finite extension of an infinite cyclic subgroup or G is countable and locally finite (i.e. every finite subset of G generates a finite subgroup).*

In the proof of Theorem 1 we use the following characterization of locally finite graphs admitting selectors. By $\mathbb{N}$ and $\mathbb{Z}$, we denote graphs on the sets of natural and integer numbers in which two vertices a, b are incident if and only if $|a-b| = 1$. We note also that two graphs are coarsely equivalent if and only if they are quasi-isometric, see [6, Chapter 4] for quasi-isometric spaces.

Theorem 2. *For a locally finite graph Γ, the following statements are equivalent:*

(i) *Γ admits a finitary selector;*

(ii) *Γ admits a 2-selector;*

(iii) *Γ is either finite or coarsely equivalent to $\mathbb{N}$ and $\mathbb{Z}$.*

We prove Theorem 2 in Section 2 and Theorem 1 in Section 3. In Section 4, we apply Theorem 1 to characterize groups admitting linear orders compatible with finitary coarse structures.

2. Proof of Theorem 2

The implication $(i) \Rightarrow (ii)$ is evident. To prove $(ii) \Rightarrow (iii)$, we choose a 2-selector f of $\Gamma[V]$ and get (iii) at the end of some chain of elementary observations.

We define a binary relation $\prec$ on V as follows: $a \prec b$ if and only if $a \neq b$ and $f(\{a, b\}) = a$.

We use also the Hausdorff metric on the set of all non-empty finite subsets of V defined by $d_H(A, B) = \max\{d(a, B),\ d(b, A) : a \in A, b \in B\}$, $d(a, B) = min\{d(a, b) : b \in B\}$. We note that the coarse structure on $[V]^2$ is defined by d_H. Since f is macro-uniform, there exists the minimal natural number r such that if $A, B \in [V]^2$ and $d_H(A, B) \leqslant 1$ then $d(f(A), f(B)) \leqslant r$. We fix and use this r.

We recall that a sequence of vertices $a_0, \dots, a_m$ is a *geodesic path* if $d(a_0, a_m) = m$ and $d(a_i, a_{i+1}) = 1$ for each $i \in \{0, \dots, m-1\}$.

Lemma 1. *Let $a_0, \dots, a_m$ be a geodesic path in V and $m \geqslant r$. If $a_0 \prec a_r$ (resp. $a_r \prec a_0$) then $a_i \prec a_j$ (resp. $a_j \prec a_i$) for all i, j such that $j - i \geqslant r$.*

Let $a_0 \prec a_r$. By the choice of r, we have $a_0 \prec a_{r+1}, \dots a_0 \prec a_j$ and $a_1 \prec a_j, \dots a_i \prec a_j$.

Lemma 2. *Let $v \in V$, $B(v, r) = \{x \in V : d(x, v) \leqslant r\}$ and U be a subset of $V \setminus B(v, r)$ such that the graph $\Gamma[U]$ is connected. Then either $v \prec u$ for each $u \in U$ or $u \prec v$ for each $u \in U$.*

We take arbitrary $u, u' \in U$ and choose $a_0, \dots, a_k$ in U such that $a_0 = u$, $a_k = u'$ and $d(a_i, a_{i+1}) = 1$ for each $i \in \{0, \dots, k-1\}$. Let $a_0 \prec v$. By the choice of r, we have $a_1 \prec v, \dots, a_k \prec v$.

Lemma 3. *Let $u, v, v' \in V$, $d(v, v') = n$ and $d(u, v) > n + r$. If $u \prec v$ (resp. $v \prec u$) then $u \prec v'$ (resp. $v' \prec u$).*

We choose a geodesic path $a_0, \dots, a_m$ from v to v'. Let $u \prec v$. By the choice of r, $u \prec a_0$, $u \prec a_1, \dots, u \prec a_n$.

Lemma 4. *Let $a_0, \dots, a_m$ be a geodesic path in V, $v \in V$, $d(v, \{a_0, \dots, a_m\}) = d(v, a_k)$, $k > 2r + 1$, $m - k > 2r + 1$. Then $d(v, a_k) \leqslant r$.*

We take the first alternative given by Claim 1, the second is analogical. Then $a_0 \prec a_k, a_k \prec a_m$. Assuming that $d(v, a_k) > r$, we can replace v to some point on a geodesic path from v to a_k and get $d(v, a_k) = r + 1$. We take the first alternative given by Claim 2, the second is analogical. Then $v \prec a_0$, $v \prec a_m$. But $v \prec a_0$ and $a_0 \prec a_k$ contradict Claim 3.

We recall that a sequence $(a_n)_{n<\omega}$ in V is a *ray* if $d(a_i, a_j) = j - i$ for all $i < j$. Evidently, $\Gamma[\{a_n : n < \omega\}]$ is asymorphic to $\mathbb{N}$.

Lemma 5. *Let $(a_n)_{n<\omega}$, $(c_n)_{n<\omega}$ be rays in V, $A = \{a_n : n < \omega\}$, $C = \{c_n : n < \omega\}$ and $A \cap C = \varnothing$. Let $t_0, \dots, t_k$ be a geodesic path from a_0 to c_0, $T = \{t_0, \dots, t_k\}$. Assume that $T \cap \{A\} = \{a_0\}$, $T \cap C = \{c_0\}$. If there exists a finite subset H of V such that every geodesic path from a vertex $a \in A$ to a vertex $c \in C$ meets H then $(A \cup C \cup T, d)$ is asymorphic to $\mathbb{Z}$.*

We define a bijection $f : A \cup C \cup T \to \mathbb{Z}$ by

$$f(c_i) = -i - 1, \quad f(t_i) = i, \quad f(a_i) = i + k + 1$$

and show that f is an asymorphism.

If $x, y \in A \cup C \cup T$ then $|f(x) - f(y)| \leqslant d(x, y)$. Hence, f^{-1} is macro-uniform.

We denote by $p = \max\{d(a_0, h), d(b_0, h) : h \in H\}$. Then the restriction of f to $C \cup T \cup \{a_0, \dots, a_p\}$ is an asymorphism and the restriction of f to $A \cup T \cup \{c_0, \dots, c_p\}$ is an asymorphism. Let $n > p$, $m > p$. Since a geodesic path from c_n to a_m meets H, we have

$$d(a_m, c_n) \leqslant n - p + m - p = |f(a_m) - f(c_n)| - k - 2p,$$

so f is macro-uniform and the claim is proven.

We suppose that V is infinite. Since $\Gamma[V]$ is locally finite, there exists a ray $(a_n)_{n<\omega}$ in V. We put $A = \{a_n : n < \omega\}$. If $V \setminus B(A, r)$ is finite then $\Gamma[V]$ is coarsely equivalent to $\mathbb{N}$.

We suppose $V \setminus B(A, r)$ is infinite, take $u \in V \setminus B(A, r)$ and show that every path P from u to a point from $B(A, r)$ meets $B(\{a_0, \dots, a_{2r+1}\}, r + 1)$. We take a point $v \in P$ such that $d(v, A) = r + 1$ and take k such that $d(v, a_k) = r + 1$. By Claim 4, $k \leqslant 2r + 1$, so $v \in B(\{a_0, \dots, a_{2r+1}\}, r + 1)$. We choose a ray $(c_n)_{n<\omega}$ in $V \setminus B(A, r)$ and put $C = (c_n)_{n<\omega}$. We delete (if necessary) a finite number of points from A so that A, C and T satisfy the assumptions of Claim 5 with $F = B(\{a_0, \dots, a_{2r+1}\}, r + 1)$. Then $(B(A \cup C \cup T), d)$ is coarsely equivalent to $\mathbb{Z}$.

We show that $V \setminus B(A \cup C, r)$ is finite, so $\Gamma[V]$ is coasly equivalent to $\mathbb{Z}$. We suppose the contrary and choose a ray $(x_n)_{n<\omega}$ in $V \setminus B(A \cup C, r)$. Applying arguments from above paragraph, we can construct a subset X of V such that (X, d) is coarsely equivalent to a tree T which is a union of three rays with common beginning. Since (X, d) has a 2-selector, by Proposition 5 from [12], T also admits a 2-selector. On the other hand, Claim 4 states that T does not admit a 2-selector and we get a contradiction.

It remains to prove $(iii) \Rightarrow (i)$. This is evident if Γ is finite. By [12, Proposition 5], it suffices to show that $\mathbb{N}$ and $\mathbb{Z}$ admit finitary selectors. In both cases, a mapping f defined by $f(A) = \max A$ is finitary selector.

3. Proof of Theorem 1

Let G be a group with the finite system S of generators, $S = S^{-1}$. We recall that the Cayley graph $Cay(G, S)$ is a graph with the set of vertices G and the set of edges $\{(x, y) : x \neq y,\ xy^{-1} \in S\}$. We note that the finitary coarse space of G is asymorphic to the coarse space of $Cay(G, S)$.

Now let G be an arbitrary group. The implication $(i) \Rightarrow (ii)$ is evident.

We prove $(ii) \Rightarrow (iii)$. By [12, Theorem 4], G is countable. Let f be a 2-selector of G. We use the binary relation $\prec$ on G, defined in Section 2, and consider two cases.

Case 1. G has an element a of infinite order. We denote by A the subgroup of G, generated by a, and show that $|G : A|$ is finite.

On the contrary, let $|G : A|$ is infinite. We put $S = \{e, a, a^{-1}\}$, denote by $\Gamma[A]$ the graph $Cay(A, S)$ and choose a natural number r such that if $B, C \in [A]^2$ and $d_H(B, C) \leqslant 1$ then $d(f(A), f(B)) \leqslant r$. By Claim 1, either $a^m \prec a^n$ for all $m, n \in \mathbb{Z}$ such that $n - m \geqslant r$ or $a^n \prec a^m$ for all $m, n \in \mathbb{Z}$ such that $n - m \geqslant r$.

Since $f : [G]^2 \to G$ is macro-uniform, there exists a finite subset F of G such that $F = F^{-1}$, $e \in F$ and if $B, C \in [G]^2$ and $A \subseteq SB$, $B \subseteq SA$ then $f(A) \in Ff(B)$. Since $|G : A|$ is infinite, we can choose $h \in G \setminus FA$, so $Fh \cap A = \varnothing$. Then either $a^n \prec h$ for each $n \in \mathbb{Z}$ or $h \prec a^n$ for each $n \in \mathbb{Z}$. We consider the first alternative, the second is analogical.

Since f is macro-uniform, we can choose $m \in \mathbb{N}$, $m \geqslant r$ such that $e \prec a^m$ and $h \prec a^m$, but $h \prec a^m$ contradicts above paragraph.

Case 2. G is a torsion group. We suppose that G is not locally finite, choose a finite subset S of G such that the subgroup H, generated by S, is infinite. We denote $\Gamma[H] = Cay(H, S)$. By Theorem 2, $\Gamma[H]$ is coarsely equivalent to $\mathbb{N}$ or $\mathbb{Z}$.

We take $v \in \Gamma[H]$ and denote $S(v, n) = \{u \in H : d(v, u) = n\}$, $n \in \mathbb{N}$. By [8, Theorem 1] or [16, Theorem 5.4.1], there exists a natural number k such that $|S(v, n)| \leqslant k$ for each $n \in \mathbb{N}$. Hence, H is of linear growth. Applying either [5] or [7], we conclude that H has an element of infinite order, a contradiction with the choice of G.

It remains to verify $(iii) \Rightarrow (i)$. If G is a finite extension of an infinite cyclic subgroup then we apply Theorem 2. If G is countable and locally finite, one can refer to Theorem 5 in [12], but we give the following direct proof to use in the proof of Theorem 3.

We write G as the union of an increasing chain $\{G_n : n < \omega\}, G_0 = \{e\}$ of finite subgroup. For each n, we choose some system R_n, $e \in R_n$ of representatives of right cosets of G_{n+1} by G_n, so $G_{n+1} = G_n R_n$. We denote

$$X = \{(x_n)_{n<\omega} : x_n \in R_n \textit{ and } x_n = e \textit{ for all but finitery many } n\}$$

and define a bijection $h : G \to X$ as follows.

We put $h(e) = (x_n)_{n<\omega}$, $x_n = e$. Let $g \in G$, $g \neq e$. We choose n_0 such that $g \in G_{n_0+1} \setminus G_{n_0}$ and write $g = g_0 r_{n_0}$, $g_0 \in G_{g_0}$, $r_{n_0} \in R_{n_0}$. If $g_0 \neq e$ then we find n_1, $g_1 \in G_{n_1}$, $r_{n_1} \in R_{n_1}$ such that $g_0 = g_1 r_{n_1}$. After a finite number k of steps, we get $g = r_{n_k} \dots r_{n_1} r_{n_0}$. We put $h(g) = (y_n)$, where $y_n = r_n$ if $n \in \{n_k, \dots, n_0\}$, otherwise, $y_n = e$.

Now we define a linear order $\leqslant$ on X. For each $n < \omega$, we choose some linear order $\leqslant_n$ on R_n with the minimal element e. If $(x_n)_{n<\omega} \neq (y_n)_{n<\omega}$ then we choose the minimal k such that $x_n = y_n$ for each $n > k$. If $x_k <_k y_k$ then we put $(x_n)_{n<\omega} < (y_n)_{n<\omega}$.

We note that $(X, \leqslant)$ is well-ordered, so every non-empty subset of X has the minimal element. To define a finitary selector $f : \mathcal{F}_G \to G$, we take an arbitrary $A \in \mathcal{F}_G$ and put $f(A) = min\ h(A)$.

4. Linear orders

Let $(X, \mathcal{E})$ be a coarse space. We say that a linear order $\leqslant$ on X is *compatible with the coarse structure* $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there exists $F \in \mathcal{E}$ such that $E \subseteq F$ and if $\{x, y\} \in [X]^2$, $x < y$ $(y < x)$ and $y \in X \setminus F[x]$ then $x' < y$ $(y < x')$ for each $x' \in E[x]$.

Let $(X, \mathcal{E})$ be a coarse space, $\leqslant$ be a linear order on X. We say that an entourage $E \in \mathcal{E}$ is interval (with respect to $\leqslant$) if, for each $x \in X$, there exist $a_x, b_x \in X$ such that $a_x \leqslant x \leqslant b_x$ and $E[x] = [a_x, b_x]$. We say that $\mathcal{E}$ is an *interval* coarse structure if there is a base of $\mathcal{E}$ consisting of interval entourages. Clearly, if $\mathcal{E}$ is interval then $\leqslant$ is compatible with $\mathcal{E}$.

Theorem 3. *Let G be a group, $\mathcal{E}$ denotes the finitary coarse structure on G. Then the following statements are equivalent:*

(i) *there exists a linear order $\leqslant$ on G such that $\mathcal{E}$ is interval with respect to $\mathcal{E}$;*

(ii) *there exists a linear order* $\leqslant$ *on* G *compatible with* $\mathcal{E}$*;*

(iii) G *admits a 2-selector.*

Proof. The implication $(i) \Rightarrow (ii)$ is evident, $(ii) \Rightarrow (iii)$ follows from Proposition 2 in [12]. To prove $(iii) \Rightarrow (i)$, we use Theorem 1 and consider two cases.

Case 1. G is a finite extension of an infinite cyclic group A. We can suppose that A is a normal subgroup. Let $A = \{a^n : n \in \mathbb{Z}\}$, $\{f_0, \dots, f_m\}$ be a set of representatives of cosets of G by A, $f_0 = e$, $F = \{f_0, \dots, f_m\}$. We set $F_n = F\{a^{-n}, \dots, a^n\}$, $E_n = \{(x, y) : xy^{-1} \in F_n\}$ and note that $\{E_n : n \in \omega\}$ is a base for $\mathcal{E}$.

We endow G with a linear order $\leqslant$ defined by the rule: $f_i a^k < f_j a^n$ if and only if either $k < n$ or $k = n$ and $i < j$.

We choose the minimal natural number d such that $f_j f_j \in F\{a^{-d}, \dots, a^d\}$ for all $i, j \in \{0, \dots, m\}$. Since $f_i^{-1} a f_j \in \{a, a^{-1}\}$, we have

$$E_n[f_i a^k] \ \subseteq\ [f_0 a^{k-n-d},\ f_m a^{k+n+d}].$$

On the other hand, $[f_0\ a^{k-n},\ f_m\ a^{k+n}] \ \subseteq\ F_n\ a^k \ \subseteq\ F_n\ f_i^{-1}\ (f_i\ a^k)$. Hence, $\mathcal{E}$ has an interval base with respect to $\leqslant$.

Case 2. G is countable and locally finite. Then $\mathcal{E}$ is interval with respect to the linear order $\leqslant$ defined in the proof of Theorem 1. □

Let $\leqslant$ be a linear order on G compatible with $\mathcal{E}$. Does there exist a global selector of G? The following theorem gives the negative answer.

Theorem 4. *The group* $\mathbb{Z}$ *does not admit a global selector.*

Proof. We suppose the contrary and let f be a global selector. Since f is macro-uniform, there exists a natural number n such that if $X, Y \in \exp G$ and

$$X \subseteq [-1, 1] + Y,\ \ Y \subseteq [-1, 1] + X$$

then $f(Y) \in [f(X) - n,\ f(x) + n]$.

We put $A = (n+1)\mathbb{Z}$, $a = f(A)$, $A' = A \setminus \{a\}$. Then

$$\begin{gathered} f(A' \cup \{a-1\}) \in \{a-1, a-(n+1)\}, \\ f(A' \cup \{a-2\}) \in \{a-2, a-(n+1)\}, \dots, \\ f(A') = a-(n+1), \\ f(A' \cup \{a+1\}) \in \{a+1, a+n+1\}, \\ f(A' \cup \{a+2\}) \in \{a+2, a+n+1\}, \dots, \\ f(A') = a+n+1, \end{gathered}$$

and we get a contradiction. □

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