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An extension of the essential graph of a ring

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Abstract. Let A be a commutative ring with non-zero identity, and $E(A) = \{p \in A | ann_A(pq) \leq_e A, \text{ for some } q \in A^*\}$. The extended essential graph, denoted by $E_gG(A)$ is a graph with the vertex set $E(A)^* = E(A) \setminus \{0\}$. Two distinct vertices $r, s \in E(A)^*$ are adjacent if and only if $ann_A(rs) \leq_e A$. In this paper, we prove that $E_gG(A)$ is connected with $diam(E_gG(A)) \leq 3$ and if $E_gG(A)$ has a cycle, then $gr(E_gG(A)) \leq 4$. Furthermore, we establish that if A is an Artinian commutative ring, then $\omega(E_gG(A)) = \chi(E_gG(A)) = |N(A)^*| + |Max(A)|$.

Introduction

Assignment of the graph to a commutative ring help us to study the properties of commutative ring from graph theoretical aspects. The study of graph associated to a commutative ring was started by Beck [6] in which he take the set $Z(A) = \{x \in A | xy = 0, \text{ for some } y \in A^*\}$ as the vertex set and two distinct vertices are adjacent if their product is zero. He mainly interested in colouring of the graph. Further, Anderson et al. [3] slightly modified the definition of Beck by taking the set $Z(A)^* = \{x \in A^* | xy = 0, \text{ for some } y \in A^*\}$ to be vertex set and two distinct vertices are adjacent if their product is zero. It is found that the definition given by

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Anderson et al. [3] is more suitable then that of Beck. So, many of authors interested to study graph theoretical properties of ring by considering the definition given by Anderson et al. [2, 4, 8, 9, 11, 12]. In 2017, Nikmehr et al. [10] extend the zero-divisor graph by extending the edge set and taking the vertex set same as taken by Anderson et al. (i.e, $ann_A(pq)$ essential ideal of A for some $p, q \in Z(A)^*$). So, the essential graph $EG(A)$ is an edge extended graph of $\Gamma(A)$. Now, here question arises that, is there exist any further extension of an zero-divisor graph, which contains essential graph $EG(A)$ of a commutative ring? To giving answer to this question, we extend the zero-divisor graph by extending the vertex set as well as edge set of the zero-divisor graph. We denote this graph by $E_gG(A)$, in which vertex set to be $E(A)^* = \{x \in A^* | ann_A(xy) \leq_e A$, for some $y \in A^*\}$. Two distinct vertices $a, b \in E(A)^*$ are adjacent if and only if $ann_A(ab) \leq_e A$. It is easily observe that $E_gG(A)$ is an extended graph of $\Gamma(A)$ as well as $EG(A)$.

Throughout we take A be a commutative ring with unity. We denote $Z(A)$ be a set of zero-divisor elements, $U(A)$ be a set of unit elements, $N(A)$ be a set of nilpotent elements, $Max(A)$ set of maximal ideal of A and if X is any non-empty set, then the set of non-zero element is denoted by $X^* = X \setminus \{0\}$. For any $p \in A$, $ann(p) = \{q \in A : pq = 0\}$ is the annihilator ideal of p in A. An ideal I of a ring A is said to be essential, if it has non-empty intersection with every non-zero ideal of A. We denote $\leq_e A$ to be an essential ideal of A. A ring is said to be reduced if it has no nonzero nilpotent element. For more terminology and definition of the ring one can see [5].

Let us define some basic definition of graph. A graph $H = (V, E)$ is defined us the set of vertices V and edges E. A graph H is said to be connected, if every vertices of H is joined by a path, where path is the length of the shortest distance between two distinct vertices and it is denoted by $d(x, y)$, $x, y \in V(H)$. If there is no path between x and y then $d(x, y) = \infty$. If S_1 and S_2 are the subgraph of H. Then the join of S_1 and S_2, denoted by $S_1 \vee S_2$, is a graph with the vertex set $V(S_1 \vee S_2) = V(S_1) \cup V(S_2)$ and edge set $E(S_1 \vee S_2) = E(S_1) \cup E(S_2) \cup \{xy | x \in V(S_1),\ y \in V(S_2)\}$. The diameter of a graph H is denoted and given by $diam(H) = max\{d(x, y) | x, y \in V(H)\}$. A girth of a graph H is the length of the shortest cycle. We denote girth of H by $gr(H)$ and $gr(H) = \infty$, if it has no cycle. A graph is said to be complete if every vertices are adjacent to each other, we denote complete graph with n-vertices by K_n and its complement graph by $\overline{K}_n$. A clique of a graph H is defined as the maximal complete subgraph of a graph H and length of the maximal complete subgraph is called clique number and it is denoted by $\omega(H)$. We denote $\Delta(H)$, $\delta(H)$ and $d(x)$, to be maximum degree, minimum degree and the number of edges incident to x of H for some $x \in V(H)$, respectively. Assignment of colour to the vertices of H in such a manner that two adjacent vertices have distinct colour is defined as vertex colouring and minimum number of colour required to colour all the vertices of graph H is called vertex chromatic number and it is denoted by $\chi(H)$. Similarly, assignment of colour to the edges of the graph such that two incident edges assign a different colour is known to be edge colouring of the graph and the minimum number of colour required to colour all the edges of a graph H is called edge chromatic number, it is denoted by $\chi'(H)$. For more terminology and definition regarding graph one can see [7, 13].

We divide the results of this paper in two section (Section 2 and Section 3). In Section 2, we study about the connectedness, diameter, girth of the extended essential graph $E_gG(A)$, in this section, we also establish affinity between essential graph and extended essential graph. In the third section, we show that for Artinian ring, extended essential graph is weakly perfect. We also study about the edge chromatic number and show that for finite commutative ring with identity, $\chi'(EG_g(A)) = \Delta(EG_g(A))$.

1. Basic properties of extended essential graph

In this section, we study about the connectedness, diameter and girth of $E_gG(A)$ of a commutative ring. Also, we study the affinity between essential graph $EG(A)$ and extended essential graph $E_gG(A)$.

Lemma 1. *Let A be a non-reduced ring. Then the following statements hold:*

(i) For any $x \in N(A)$, $ann_A(x) \leq_e A$.

(ii) For every $u \in N(A)^*$, u is adjacent to all other vertices of $E_gG(A)$.

(iii) $E_gG(A)[N(A)^*]$ is a (induced) complete subgraph of $E_gG(A)$.

Proof. (i) Assume $x \in N(A)$. Then, we show that $ann_A(x) \leq_e A$. Let I be ideal of A and $y \in I \setminus \{0\}$. Since $x \in N(A)$, it is possible to find $n \geq 1$ such that $yx^{n-1} \neq 0$ but $yx^n = 0$. Hence $yx^{n-1} \in ann(x) \cap I \neq 0$ and hence $ann(x) \cap I \neq 0$.

(ii) As $ann_A(u) \subseteq ann_A(uy)$ for any $y \in A^*$ and $uy \in N(A)$, we deduce from part (i) u is adjacent to all other vertices of $E_gG(A)$.

(iii) Follows from part (ii). □

Lemma 2. *Let A be a reduced ring. Then $rs = 0$ if and only if $ann_A(rs) \leq_e A$ for some $r, s \in A^*$.*

Proof. If $rs = 0$, then clearly $ann_A(rs) \leq_e A$. Conversely, suppose on contrary that $ann_A(rs) \leq_e A$ but $rs \neq 0$. Since A is reduced, $ann_A(rs) \cap Ars = (0)$, a contradiction. Hence $rs = 0$. □

In the view of Lemma 1(i) and Lemma 2, we have following observations.

Observation 1. If A is a non-reduced ring, then $E(A)^* = A^*$.

Observation 2. If A is a reduced ring, then $E(A)^* = Z(A)^*$.

Theorem 1. *Let A be a reduced ring. Then $E_gG(A) = EG(A)$.*

Proof. It follows from Observation 2. □

Main results of this section

Theorem 2. *Let A be a ring. Then the following hold:*

(i) $E_gG(A)$ is connected and $diam(E_gG(A)) \leq 3$.

(ii) If $E_gG(A)$ contains a cycle, then $gr(E_gG(A)) \leq 4$. Moreover, if A is non-reduced, then $gr(E_gG(A)) = 3$.

Proof. (i) Let us take $p, q \in E(A)^*$, with $p \neq q$. Then by the definition there exist $p_1, q_1 \in E(A)^*$ such that $ann_A(pp_1)$ and $ann_A(q_1q)$ are essential ideals of A and hence $pp_1, qq_1 \in E(A)$.

Case(a): If $pq = 0$, then $ann_A(pq) \leq_e A$ and hence $d(p, q) = 1$.

Case(b): Assume that $pq \neq 0$. Now, if $p_1q_1 \neq 0$ and $p_1q_1 \in N(A)^*$, then $pp_1q_1 \in N(A)$ and $p_1q_1q \in N(A)$. Thus, by Lemma 1(i), $ann_A(pp_1q_1)$ and $ann_A(p_1q_1q)$ are essential ideals of A. Therefore, $p - p_1q_1 - p$ is a path from p to q. Hence $d(p, q) = 2$. If $p_1q_1 = 0$, then $p - p_1 - q_1 - q$ is a path from p to q and hence $d(p, q) = 3$.

(ii) Assume that $E_gG(A)$ contains a cycle. Now, consider the following cases:

Case(a): If A is reduced ring, then by Theorem 1, $E_gG(A) = EG(A)$. Therefore, by [10, Theorem 2.1], $gr(E_gG(A)) \leq 4$.

Case(b): If A is non-reduced ring, then there exists $z \in N(A)^*$. By the assumption there exists a cycle in $E_gG(A)$, that contains an edge $x_1 - y_1$, where $x_1, y_1 \in A \setminus \{0, z\}$. Then by Lemma 1(ii), $z - x_1 - y_1 - z$ has a cycle of length 3. Thus, $gr(E_gG(A)) = 3$. □

Theorem 3. *Let A be a non-reduced ring. Then $E_gG(A)$ is connected and $diam(E_gG(A)) = 2$.*

Proof. As A is a non-reduced ring, there exists $c \in N(A)^*$. We known from Lemma 1(ii), that c is adjacent to all the vertices of $E_gG(A)$. Hence, we obtain that, $E_gG(A)$ is connected and $diam(E_gG(A)) \leq 2$. As $c \in N(A)^*$, it follows that $1 + c \in A^\times$ and $1 + c \neq 1$. Let $w = 1$ and $z = 1 + c$. It is clear that $wz = 1 + c$ and so, $(0) = ann_A(wz)$ is not an essential ideal of A. Therefore, w and z are not adjacent in $E_gG(A)$. Hence, $diam(E_gG(A)) \geq 2$ and so, $diam(E_gG(A)) = 2$. □

Theorem 4. *Let A be a finite non-reduced ring. Then the followings are equivalent:*

(i) $E_gG(A)$ *is a star graph.*

(ii) $E_gG(A)$ *is a tree.*

(iii) $A \cong \mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/\langle x^2 \rangle$.

(iv) $gr(E_gG(A)) = \infty$.

Proof. (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (i), (ii), (iv) is clear, we have only to show that (ii) $\Longrightarrow$ (iii). Now, let $E_gG(A)$ is a tree. Then $E_gG(A)$ has no cycle. Let $A \cong A_1 \times A_2 \times \cdots \times A_n$, where each A_i is a local ring, $1 \leq n < \infty$, but not a field. Now, if we take $n \geq 2$ and $a = (x, y, ..., 0)$, $b = (0, 1, ..., 0), c = (1, x, 0, ..., 0) \in E(A)^*$, for some $x, y \in N(A_i)^*$, then we have a cycle $a-b-c-a$, a contradiction. Thus $n = 1$. Let $|N(A)^*| \geq 2$, then there exist $w, z \in N(A)^*$ and $u \in U(A)$. Therefore, $w - z - u - w$ is a cycle in $E_gG(A)$, again a contradiction occur. Hence $|N(A)^*| = 1$ and so $A \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\langle x^2 \rangle$. □

Corollary 1. *Let A be a finite non-reduced ring and $E_gG(A)$ is a star graph. Then the following are equivalents:*

(i) $E_gG(A) \neq EG(A)$.

(ii) $A \cong \mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/\langle x^2 \rangle$.

(iii) $gr(E_gG(A)) = \infty$.

Proof. (i) $\Longrightarrow$ (ii). Assume that $E_gG(A) \neq EG(A)$. If A is a reduced ring, then by Theorem 1, $E_gG(A) = EG(A)$, a contradiction. Now, if we consider A is a nonreduced ring and $|N(A)^*| \geq 2$, then there exist $x, y \in N(A)^*$ and $u \in U(A)$ such that $x - y - u - x$ forms a cycle, a contradiction. Thus $|N(A)^*| = 1$. Hence either $A \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\langle x^2 \rangle$.

(ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i) are follows from Theorem 4 and Observation 2. □

Theorem 5. *If A is a finite reduced ring, R is a ring which is not an integral domain and $E_gG(A) \cong E_gG(R)$, then $A \cong R$, unless $A \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ or $\mathbb{Z}_6$ and R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\langle x^2 \rangle$.*

Proof. Assume A is a reduced ring. If R is also reduced, then by [1, Theorem 5], we are done. Now, let us consider R to be non-reduced ring. As R is non-reduced, from Lemma 1(ii), there exists $x \in N(R)^*$ such that x is adjacent to all the vertices of $E_gG(R)$, and so does in $E_gG(A)$ (as $E_gG(A) \cong E_gG(R)$). Also, from Theorem 1 and [10, Theorem 2.2], we have $E_gG(A) \cong \Gamma(A)$. Therefore, by [3, Corollary 2.7], $A \cong \mathbb{Z}_2 \times F$, where F is a finite field. From Observation 1 and 2, we have $E(A)^* = Z(A)^*$ and $E(R)^* = R^*$. As $E_gG(A) \cong E_gG(R)$, we must have $|E(A)^*| = |E(R)^*|$, i.e, $|Z(A)^*| = |R^*|$. Also, from Theorem 4, $E_gG(R)$ is to be star graph, if $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\langle x^2 \rangle$. It is clear that $\Gamma(\mathbb{Z}_6)$ is a star graph and $\Gamma(\mathbb{Z}_2 \times F)$ to be a star graph such that $|Z(A)^*| = |R^*|$ is possible if $|F^*| = 3$. Hence the result. □

2. The extended essential graph of an Artinian ring is weakly perfect

The main motive of this section is to study the vertex colouring of extented essential graph of Artinian commutative ring and to study the edge colouring for a finite commutative ring. We show that the graph $E_gG(A)$ is weakly perfect if A is Artinian ring.

Main results of this section

Theorem 6. *Let A be an Artinian ring. Then*

$$\omega(E_gG(A)) = \chi(E_gG(A)) = |N(A)^*| + |Max(A)|.$$

Proof. To prove this we have the following cases:

Case(a): If A is Artinian local ring, then one can easily see that vertex set of $E_gG(A)$ is a partition of $N(A)$ and $U(A)$. By Lemma 1(iii), we have $E_gG(A)[N(A)^*]$ is a complete subgraph of $E_gG(A)$. From Lemma 1(ii), for all $a \in N(A)^*$, a is adjacent to all other vertices of A^* and the fact that there is no adjacency between two vertices of $U(A)$. Which implies that

$$E_gG(A) = E_gG(A)[N(A)^*] \vee E_gG(A)[U(A)]$$

and so

$$\omega(E_gG(A)) = \chi(E_gG(A)) =$$

$$= \omega(E_gG(A)[N(A)^*]) + \omega(E_gG(A)[U(A)]) = N(A)^* + 1.$$

Case(b): If A is an Artinian non-local ring, then from [5, Theorem 8.7], A can be written as $A = A_1 \times A_2 \times \cdots \times A_n$, where each A_i is Artinian local ring, for every $1 \leq i \leq n$. Now one can seprate the vertex of $E_gG(A)$ as follow:

$$T = \{\{(a_1, ..., a_n)|a_i \in N(A) \text{ for all } 1 \leq i \leq n\} \setminus \{(0,0,0,0)\}\},$$

$$S = \{(a_1, ..., a_n)| \ \ a_i \notin N(A), \text{ for some } i\},$$

$$U(A) = \{(a_i, ..., a_n)|a_i \in U(A), \text{ for all } \ 1 \leq i \leq n\}.$$

It is clear that there is no adjacency between W and $U(A)$ and the $V(E_gG(A)) = T \cup S \cup U(A)$. Observe that $T \cap S \cap U(A) = \emptyset$, $T \cap U(A) = \emptyset$, $S \cap U(A) = \emptyset$ and therefore the set $T, S, U(A)$ is a partition of vertex set of $E_gG(A)$. Also it is clear that $E_gG(A)[U(A)]$ is the complement of complete subgraph of $E_gG(A)$. Now we have only show that

$$E_gG(A)[T \cup S] = E_gG(A)[T] \vee E_gG(A)[S],$$

$$E_gG(A)[T \cup U(A)] = E_gG(A)[T] \vee E_gG(A)[U(A)].$$

It is clear from Lemma 1(iii) $E_gG(A)[T]$ is a complete subgraph of $E_gG(A)$, we have only show that $E_gG(A)[S]$ is an n-partite subgraph of $E_gG(A)$, which is not an (n-1)-partite subgraph of $E_gG(A)$. Now, for every $1 \leq i \leq n$, let $S_i = \{(a_1, a_2, ..., a_n) \in S | a_i \in U(A_i) \ \textit{and} \ a_j \in N(A)$ for every $1 \leq j \leq i\}$. It can be easily seen that for $1 \leq i \leq n$, there is no adjacency between two distinct vertices of S_i and the set $(1, 0, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, 0, 0, ..., 1)$ is a clique of $E_gG(A)[S]$ and hence $E_gG(A)[S]$ is n-partite subgraph which is not an (n-1)-partite subgraph of $E_gG(A)$. Therefore, we can write

$$E_gG(A)[T \cup S] = E_gG(A)[T] \vee E_gG(A)[S]$$

$$E_gG(A)[T \cup U(A)] = E_gG(A)[T] \vee E_gG(A)[U(A)],$$

and hence

$$\omega(E_gG(A)) = \chi(E_gG(A)) =$$
$$= \omega(E_gG(A)[T]) + \omega(E_gG(A)[S]) = |N(A)^*| + |Max(A)|. \qquad \square$$

Theorem 7. *Let A be a non-reduced ring. Then the following are equivalent:*

(i) $\chi(E_gG(A)) = 2$;

(ii) $\omega(E_gG(A)) = 2$;

(iii) either $E_gG(A) \cong K_{1,2}$ *or* $E_gG(A) \cong K_1 \vee \overline{K}_\infty$.

Proof. (i) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i), (ii) is clear, we only show (ii) $\Longrightarrow$ (iii). Now, let us assume that $\omega(E_gG(A)) = 2$, then we need to show that $E_gG(A) \cong K_{1,2}$ or $E_gG(A) \cong K_1 \vee K_\infty$, we first claim that $|N(A)^*| = 1$. To prove this, take $|N(A)^*| \geq 2$, then there exist $a, b \in N(A)^*$, from Lemma 1(ii), for any $u \in A^\times$, we have $a - b - u - a$, a cycle in $E_gG(A)$, a contradiction occur. Hence $N(A)^* = 1$. Now, we have following cases:

Case(a): If $Z(A) = N(A)$, then $|Z(A)^*| = 1$. If A is an Artinian ring, then from [5, Theorem 8.7], we have $A \cong A_1 \times A_2 \times \cdots \times A_n$ for some positive integer n, $1 \leq i \leq n$. Where each A_i is an Artinian local ring. Now, if $n \geq 2$, then $Z(A)^* \geq 2$, which is a contradiction. Thus $n = 1$ and A is an Artinian local ring also from Theorem 4, $A \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\left\langle x^2 \right\rangle$. Therefore, $E_gG(A) \cong K_{1,2}$.

Case(b): If $Z(A) \neq N(A)$, then we show that $E_gG(A) \cong K_1 \vee \overline{K}_\infty$. Since $\omega(E_gG(A)) = 2$ and from Lemma 1(ii), for every $a \in N(A)^*$, a is adjacent to all vertices of A^*. To show that $E_gG(A) \cong K_1 \vee \overline{K}_\infty$, we only show that $|Z(A)^*| = \infty$. On contrary suppose that $|Z(A)^*| < \infty$. From [5, Theorem 8.7], we can write A as $A \cong A_1 \times A_2 \times \cdots \times A_n$ for some positive integer n, $1 \leq i \leq n$. Since $Z(A) \neq N(A)$, we have $n \geq 2$. As A is a non-reduced ring, there exists $k \in N(A_1)^*$. Now, let $z = (k, 0, ..., 0)$, $b = (1, 0, ..., 0)$, $c = (0, 1, ..., 0) \in A^*$. Then from Lemma 1(ii), $z - b - c - z$ is a cycle, therefore induced subgraph of $E_gG(A)$ has a cycle, a contradiction. Thus, $|Z(A)^*| = \infty$. Hence $E_gG(A) \cong K_1 \vee \overline{K}_\infty$. $\square$

Lemma 3 ([7, Collorary 5.4])**.** *Let H be a simple graph. If for every vertex w of maximum degree there exists an edge $w - z$ such that $\Delta(H) - d(z) + 2$ is more than the number of vertices with maximum degree in H, then $\chi'(H) = \Delta(H)$.*

Theorem 8. *Let A be a finite non-reduced ring. Then $\chi'(E_gG(A)) = \Delta(E_gG(A))$.*

Proof. Since A is ring with unity, $A \neq N(A)$ is clear. For any $x \in N(A)^*$, x is adjacent to all the vertices of $E_gG(A)$. Now, assume $a \in A \setminus N(A)$. As $a \notin N(A)^*$, their is no adjacency between a and $a + N(A)^*$. So, $d(a) \leq |A^*| - |N(A)^*| - 1$. It is clear that each vertices in $N(A)^*$ has a maximal degree in $E_gG(A)$. Therefore, for any vertex in $N(A)^*$ has a maximum degree, now for any $x \in A \setminus N(A)$, a is adjacent to x with $d(a) \leq |A^*| - |N(A)^*| < |A^*| - |N(A)^*| + 2$. Thus from Lemma 3, $\chi'(E_gG(A)) = \Delta(E_gG(A))$. □

Since Akbari et al. [1] proved that $\chi'(\Gamma(A)) = \Delta(\Gamma(A))$ for a finite commutative ring, unless A is a complete graph. Also, if A is reduced ring, then from [10, Theorem 2.3], $\chi'(EG(A)) = \Delta(EG(A))$ and therefore from Theorem 1, $\chi'(E_gG(A)) = \Delta(E_gG(A))$. So, we end this section by stating the result.

Corollary 2. *Let A be a finite ring. Then $\chi'(E_gG(A)) = \Delta(E_gG(A))$.*

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