© Algebra and Discrete Mathematics
Volume **37** (2024). Number 1, pp. 85–105
DOI:10.12958/adm2110 RESEARCH ARTICLE

Implicit linear difference equation over residue class rings

Mykola V. Heneralov and Aleksey L. Piven'

Communicated by V. Nekrashevych

Abstract. We investigate the first order implicit linear difference equation over residue class rings modulo m. We prove an existence criterion and establish the number of solutions for this equation. We obtain analogous results for the initial problem of the considered equation. The examples which illustrate the developed theory are given.

1. Introduction

The theory of the linear difference equations is an important branch of mathematics, having a series of different applications (see, for example, [1]–[4]). The theory of implicit linear difference equations in vector spaces was developed in the 80s–90s of the 20 century (see, for example, [4]–[6]). Unlike the classical theory, the non-invertible operators have an important role in the new theory. Therefore, it appears to be interesting to investigate the problem of solving an implicit linear difference equation with non-invertible coefficients from an arbitrary commutative ring. Recently, implicit difference equations over integral domains were studied in [7], and more detailed over the ring of integers in [8]–[10].

The research was supported by the Akhiezer Foundation. Authors grateful to Sergey Gefter for useful discussions of the paper results. The authors also would like to thank the referee for the careful reading of the manuscript and helpful suggestions.

2020 Mathematics Subject Classification: *39A99, 11B37, 11B50.*

Key words and phrases: *implicit linear difference equation, ring, residue class, initial problem.*

In [11] these equations in different classes of topological vector spaces were investigated.

In this paper, the first order implicit linear difference equations over residue classes rings is investigated. Let $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ be the residue class ring modulo m, where $m \in \mathbb{N}$, $m \geq 2$. Let $A, B, Y_0 \in \mathbb{Z}_m$ and let $\{F_n\}_{n=0}^{\infty}$ be a sequence of $\mathbb{Z}_m$. Consider the initial problem

$$BX_{n+1} = AX_n + F_n, \quad n \in \mathbb{Z}_+, \tag{1.1}$$

$$X_0 = Y_0, \tag{1.2}$$

where $\mathbb{Z}_+$ denotes the set of non-negative integers. A sequence $\{X_n\}_{n=0}^{\infty}$ of elements of $\mathbb{Z}_m$ is called a solution of the initial problem (1.1), (1.2), if it satisfies Equation (1.1) and the initial condition (1.2). Equation (1.1) is called implicit, if B is a non-invertible element of the ring $\mathbb{Z}_m$. If B is an invertible element of $\mathbb{Z}_m$, then this equation is called explicit. Let a, b are representatives of classes A, B respectively. In the Section 2 we prove that if the greatest common divisor of numbers a, b, m is equal to 1, then Equation (1.1) is decomposed into the explicit equation (2.5) and the implicit equation (2.6), which has a unique solution (see lemmas 2.1, 2.2 and Theorem 2.1). Theorem 2.1 also gives the general solution for these equations. The main results of this paper are presented in Section 3 (see theorems 3.1 and 3.2). Theorem 3.1 describes necessary and sufficient conditions for the solvability, a number of solutions and the general solution for the initial problem (1.1), (1.2). This theorem gives the full description of all possible situations for the initial problem (1.1), (1.2). The analogous results for Equation (1.1) are established in Theorem 3.2. This theorem leads to the criteria of the existence and uniqueness of a solution for Equation (1.1) (see Corollaries 3.2, 3.3). As in the Fredholm theory (see, for example, [12, Chapter 7]), Corollary 3.4 shows that if corresponding to (1.1) homogeneous equation has only trivial solution then for any sequence $\{F_n\}_{n=0}^{\infty}$ of $\mathbb{Z}_m$ Equation (1.1) has a unique solution. Section 4 of the present paper contains the examples, which illustrate the constructed theory (see Examples 4.1–4.4).

Through this paper $[t]_s$ denotes the class of the element $t \in \mathbb{Z}$ of the ring $\mathbb{Z}_s$, where $s \in \mathbb{N}$. The ring $\mathbb{Z}_1$ denotes the null ring. For the numbers $n_1, n_2, \ldots, n_N \in \mathbb{Z}$ such that $|n_1| + |n_2| + \ldots + |n_N| \neq 0$ the symbol $\gcd(n_1, \ldots, n_N)$ denotes their positive greatest common divisor. If T is a nilpotent element of the ring $\mathbb{Z}_s$, then $\mathrm{ind}(T)$ denotes the nilpotency index of T.

2. Preliminary

Throughout this paper $m \geq 2$, $m \in \mathbb{N}$. Let A, B and F_n $(n \in \mathbb{Z}_+)$ be given elements of the ring $\mathbb{Z}_m$. For each of elements $A, B, Y_0, F_n, X_n \in \mathbb{Z}_m$ $(n \in \mathbb{Z}_+)$ denote, respectively, their representatives a, b, y_0, f_n, x_n.

By Fundamental Theorem of Arithmetic, there exist pairwise different primes $p_1, \ldots, p_r$ and numbers $k_1, \ldots, k_r \in \mathbb{N}$ such that $m = \prod\limits_{j=1}^{r} p_j^{k_j}$.

Denote

$$m_1 = \prod_{j\colon p_j \nmid b} p_j^{k_j}, \quad m_2 = \prod_{j\colon p_j | b} p_j^{k_j},$$

where $m_1 = 1$ in the case $p_j \mid b$ $(j = 1, \ldots, r)$ and $m_2 = 1$ in the case $p_j \nmid b$ $(j = 1, \ldots, r)$. Obviously, $m_1 \cdot m_2 = m$, and $\gcd(m_1, m_2) = 1$.

Let us introduce the natural projections $\pi_i \colon \mathbb{Z}_m \to \mathbb{Z}_{m_i}$, defined as follows:

$$\pi_i(T) = [t]_{m_i}, \ \forall T = [t]_m, \quad i = 1, 2.$$

(see [13, p. 381–382]).

For each $i = 1, 2$, according to the [13, p. 381–382] the natural projections π_i $(i = 1, 2)$ are homomorphisms.

Denote

$$A_i = \pi_i(A), \quad B_i = \pi_i(B), \quad Y_{i,0} = \pi_i(Y_0), \ F_{i,n} = \pi_i(F_n), \ i = 1, 2.$$

Let $m_1 \neq 1$, $m_2 \neq 1$. Let us introduce the isomorphism

$$\psi \colon \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \to \mathbb{Z}_m,$$

defined as follows (see, for example, [13, Section 7.6 and Exercise 5 to the Section 7.6]):

$$\psi(T_1, T_2) = [t_1 e_1 m_2 + t_2 e_2 m_1]_m, \quad \forall T_1 = [t_1]_{m_1}, \ \forall T_2 = [t_2]_{m_2}, \tag{2.1}$$

where

$$E_1 = [e_1]_{m_1} = [m_2]_{m_1}^{-1}, \quad E_2 = [e_2]_{m_2} = [m_1]_{m_2}^{-1}. \tag{2.2}$$

We see that since $\gcd(m_1, m_2) = 1$, the inverse elements E_1 and E_2 are defined.

If $T_1 \in \mathbb{Z}_{m_1}, T_2 \in \mathbb{Z}_{m_2}$, then the definition of ψ implies

$$\pi_i(\psi(T_1, T_2)) = T_i, \quad i = 1, 2. \tag{2.3}$$

Also, $\pi_1^{-1}(T_1) \cap \pi_2^{-1}(T_2)$ is a one-element set, $\psi(T_1, T_2)$ is an element of this set. This means that

$$\{\psi(T_1, T_2)\} = \pi_1^{-1}(T_1) \cap \pi_2^{-1}(T_2). \tag{2.4}$$

Consider the following equations over rings $\mathbb{Z}_{m_1}$ and $\mathbb{Z}_{m_2}$ respectively:

$$B_1 X_{1,n+1} = A_1 X_{1,n} + F_{1,n}, \quad n \in \mathbb{Z}_+, \tag{2.5}$$

$$B_2 X_{2,n+1} = A_2 X_{2,n} + F_{2,n}, \quad n \in \mathbb{Z}_+. \tag{2.6}$$

The following lemma describes the connection between solutions of Equation (1.1) and equations (2.5), (2.6).

Lemma 2.1. *Let $m_1 \neq 1, m_2 \neq 1$. The sequence*

$$X_n = \psi(X_{1,n}, X_{2,n}), \quad n \in \mathbb{Z}_+, \tag{2.7}$$

is a solution of Equation (1.1) if and only if the sequences $\{X_{1,n}\}_{n=0}^{\infty}$ and $\{X_{2,n}\}_{n=0}^{\infty}$ are solutions of equations (2.5), (2.6), respectively. Moreover, $X_{i,n} = \pi_i(X_n)$, $i = 1, 2$, $n \in \mathbb{Z}_+$.

Proof. The equalities (2.7) and (2.3) yield together the equality for $X_{i,n}$: $\pi_i(X_n) = X_{i,n}$, $i = 1, 2$.

Since π_i $(i = 1, 2)$ are homomorphisms, by the equality (2.7),

$$\pi_i(BX_{n+1} - AX_n - F_n) = B_i X_{i,n+1} - A_i X_{i,n} - F_{i,n}, \quad i = 1, 2, \quad n \in \mathbb{Z}_+.$$

By the equality (2.4), we obtain:

$$BX_{n+1} - AX_n - F_n =$$

$$= \psi(B_1 X_{1,n+1} - A_1 X_{1,n} - F_{1,n}, B_2 X_{2,n+1} - A_2 X_{2,n} - F_{2,n}), \quad n \in \mathbb{Z}_+. \tag{2.8}$$

We note that

$$\pi_1(0) = 0 \text{ and } \pi_2(0) = 0. \tag{2.9}$$

Since (2.9), (2.8) hold, we obtain that the equality (1.1) is satisfied if and only if equalities (2.5), (2.6) are satisfied. This ends the proof of the lemma. □

Let us introduce the notation:

$$d = \gcd(a, b, m).$$

Consider the equations (2.5) and (2.6). The following lemma establishes important properties for coefficients of these equations.

Lemma 2.2. *The following statements hold.*

1. *If $m_1 \neq 1$, then B_1 is invertible.*

2. *If $m_2 \neq 1$, then B_2 is nilpotent. If additionally $d = 1$, then A_2 is invertible.*

3. *B is nilpotent if and only if $m_1 = 1$.*

Proof. Proof the statement 1. The definition of m_1 and B implies the equality $\gcd(b, m_1) = 1$. Hence, the element B_1 is an invertible element of $\mathbb{Z}_{m_1}$.

Proof the statement 2. Firstly let us prove that B_2 is nilpotent. It is evident from the definition of m_2: if $k = \max\limits_{j=1,\dots,r}\{k_j\}$, then $B_2^k = [b^k]_{m_2} = 0$.

Let $d = 1$. We will prove that $\gcd(a, m_2) = 1$. Assuming the contrary, we obtain that there exists $j \in \{1, \dots, r\}$ such that $p_j \mid a$. This condition yields $p_j \mid a, p_j \mid b, p_j \mid m$. Hence $p_j \mid d$. But it contradicts $d = 1$. Therefore $\gcd(a, m_2) = 1$. This means that A_2 is an invertible element of $\mathbb{Z}_{m_2}$.

Proof the statement 3. The condition $m_1 = 1$ is equivalent to the assertion

$$\forall j = 1, \dots, r \colon p_j \mid b.$$

The last condition is equivalent to the nilpotency of the element B in $\mathbb{Z}_m$. □

Remark 2.1. Lemma 2.2 is an analogue of the spectral decomposition of a regular operator pencil in Banach spaces (see [14, Lemma 2.1]). The analogous to (2.5), (2.6) decomposition of an implicit difference equation in Banach spaces into two equations with regarded properties was obtained in [6, 15].

The following theorem is a solvability theorem for Equation (2.5) and, in the case $d = 1$, for Equation (2.6).

Theorem 2.1. *The following statements hold.*

1. *Let $m_1 \neq 1$. The general solution of Equation (2.5) is defined by the following formula:*

$$X_{1,n} = B_1^{-n} A_1^n X_{1,0} + \sum_{s=0}^{n-1} A_1^s B_1^{-s-1} F_{1,n-s-1}, \quad n \in \mathbb{N}, \tag{2.10}$$

where $X_{1,0}$ is an arbitrary element of $\mathbb{Z}_{m_1}$.

2. *Let $d = 1$ and $m_2 \neq 1$. Then Equation (2.6) has a unique solution, defined by the following formula:*

$$X_{2,n} = -\sum_{s=0}^{\operatorname{ind}(B_2)-1} A_2^{-s-1} B_2^s F_{2,n+s}, \quad n \in \mathbb{Z}_+. \tag{2.11}$$

Remark 2.2. The corresponding inverse elements exist by Lemma 2.2.

Proof. Let us prove firstly the statement 1. By the statement 1 of Lemma 2.2, B_1 is invertible. The equality (2.5) is equivalent to the equality

$$X_{1,n+1} = B_1^{-1} A_1 X_{1,n} + B_1^{-1} F_{1,n}, \quad n \in \mathbb{Z}_+. \tag{2.12}$$

According to [3, p. 4], the general solution of Equation (2.12) has the form (2.10).

Let us prove now the statement 2. Since $d = 1$ and $m_2 \neq 1$, by the statement 2 of Lemma 2.2, A_2 is an invertible element and B_2 is nilpotent.

The equality (2.6) is equivalent to the following:

$$X_{2,n} = -A_2^{-1} F_{2,n} + A_2^{-1} B_2 X_{2,n+1}, \quad n \in \mathbb{Z}_+. \tag{2.13}$$

Applying (2.13) recurrently few times, obtain the equality (2.11).

Now let $\{X_n\}_{n=0}^{\infty}$ be defined by the formula (2.11). Denote $k = \operatorname{ind}(B_2)$. Substituting (2.11) to the left part of Equation (2.6), we obtain:

$$B_2 X_{n+1} = -B_2 A_2^{-1} \sum_{s=0}^{k-1} A_2^{-s} B_2^s F_{2,n+1+s} = -\sum_{s=0}^{k-1} A_2^{-s-1} B_2^{s+1} F_{2,n+s+1} =$$
$$= -\sum_{t=1}^{k} A_2^{-t} B_2^t F_{2,n+t} = -\sum_{t=0}^{k} A_2^{-t} B_2^t F_{2,n+t} + F_{2,n} =$$
$$= -A_2 \cdot A_2^{-1} \sum_{t=0}^{k-1} A_2^{-t} B_2^t F_{2,n+t} + F_{2,n} = A_2 X_{2,n} + F_{2,n}.$$

Therefore $\{X_{2,n}\}_{n=0}^{\infty}$, defined by the formula (2.11), is the unique solution of Equation (2.6). □

Corollary 2.1. *The following statements hold.*

1. *Let* $d = 1$ *and* $m_1 = 1$. *Equation* (1.1) *has a unique solution* $\{X_n\}_{n=0}^{\infty}$, *defined by the following formula:*

$$X_n = -\sum_{s=0}^{\operatorname{ind}(B)-1} A^{-s-1}B^s F_{n+s}, \quad n \in \mathbb{Z}_+. \tag{2.14}$$

2. *Let* $m_2 = 1$. *The general solution of Equation* (1.1) *is defined by the following formula:*

$$X_n = B^{-n}A^nX_0 + \sum_{s=0}^{n-1} A^sB^{-s-1}F_{n-s-1}, \quad n \in \mathbb{N}, \tag{2.15}$$

where X_0 *is an arbitrary element of* $\mathbb{Z}_m$.

3. Main results

Here we obtain the solvability theorems over $\mathbb{Z}_m$ for Equation (1.1) and for the initial problem (1.1), (1.2).

Let us introduce the following notations:

$$m' = \frac{m}{d},\ Y_0' = [y_0]_{m'},\ A' = [a/d]_{m'},\ B' = [b/d]_{m'}.$$

Also, when $d \mid f_n$ for all $n \in \mathbb{Z}_+$, denote

$$F_n' = [f_n/d]_{m'}, \quad n \in \mathbb{Z}_+.$$

Each a prime divisor of the number m' is also a divisor of m. Then by Fundamental Theorem of Arithmetic, there exist non-negative integers $l_j \le k_j$ $(j = 1, \dots, r)$ such that $m' = \prod_{j=1}^{r} p_j^{l_j}$.

Denote also

$$m_1' = \prod_{j\colon dp_j \nmid b} p_j^{l_j}, \quad m_2' = \prod_{j\colon dp_j \mid b} p_j^{l_j},$$

$$A_i' = [a/d]_{m_i'}, \quad B_i' = [b/d]_{m_i'}, \quad Y_{i,0}' = [y_0]_{m_i'}, \quad i = 1, 2.$$

As in the definition of m_1, m_2, we assume $m_1' = 1$, in the case $dp_j \mid b$ $(j = 1, \dots, r)$ and $m_2' = 1$, in the case $dp_j \nmid b$ $(j = 1, \dots, r)$. Note that if $d = 1$, then $m_i' = m_i$, $i = 1, 2$.

Let $d \mid f_n$ for all $n \in \mathbb{Z}_+$. Denote

$$F'_{i,n} = [f_n/d]_{m'_i}.$$

and consider the initial problem

$$B'X'_{n+1} = A'X'_n + F'_n, \quad n \in \mathbb{Z}_+, \tag{3.1}$$

$$X'_0 = Y'_0 \tag{3.2}$$

over $\mathbb{Z}_{m'}$.

The following statement is a helpful lemma, which shows a connection between the equations (1.1) and (3.1).

Lemma 3.1. *Let $d \neq 1$, $d \mid f_n$ $(n \in \mathbb{Z}_+)$. The sequence $\{X_n\}_{n=0}^{\infty}$ is a solution of Equation (1.1) if and only if it admits the following representation*

$$X_n = [x'_n + \alpha_n m']_m, \quad n \in \mathbb{Z}_+, \tag{3.3}$$

where $X'_n = [x'_n]_{m'}$ $(n \in \mathbb{Z}_+)$ is a solution of Equation (3.1), and $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of $\{0, 1, \dots, d-1\}$. Moreover, the sequence $\{\alpha_n\}_{n=0}^{\infty}$ and the solution $\{X'_n\}_{n=0}^{\infty}$ of Equation (3.1) with $x'_n \in \{0, \dots, m'-1\}$ are uniquely determined by the solution $\{X_n\}_{n=0}^{\infty}$ of Equation (1.1).

Proof. Obviously, Equation (1.1) is equivalent to the congruence

$$bx_{n+1} \equiv ax_n + f_n \pmod{m}, \quad n \in \mathbb{Z}_+. \tag{3.4}$$

The congruence (3.4) is equivalent to the following condition.

$$\frac{b}{d}x_{n+1} \equiv \frac{a}{d}x_n + \frac{f_n}{d} \pmod{m'}, \quad n \in \mathbb{Z}_+. \tag{3.5}$$

The congruence (3.5) means that there exists a solution $X'_n = [x'_n]_{m'}$ $(n \in \mathbb{Z}_+)$ of Equation (3.1) such that $x_n \equiv x'_n \pmod{m'}$. Therefore $\{X_n\}_{n=0}^{\infty}$ is a solution of (1.1) if and only if $X_n = [x'_n + \alpha_n \cdot m']_m$ $(n \in \mathbb{Z}_+)$, where $\{\alpha_n\}_{n=0}^{\infty}$ is an arbitrary sequence of $\{0, \dots, d-1\}$.

Suppose that the two following representatives for the solution of Equation (1.1) hold:

$$X_n = [x'_n + \alpha_n m']_m = \left[\widehat{x'_n} + \widehat{\alpha_n} m'\right]_m, \quad n \in \mathbb{Z}_+,$$

where $X'_n = [x'_n]_{m'}$, $\widehat{X'_n} = \left[\widehat{x'_n}\right]_{m'}$ $(n \in \mathbb{Z}_+)$ are solutions of Equation (3.1), α_n, $\widehat{\alpha_n}$ $(n \in \mathbb{Z}_+)$ are numbers from $\{0, \dots, d-1\}$ and additionally $x'_n, \widehat{x'_n} \in \{0, \dots, m'-1\}$. It implies the following congruence

$$x'_n + \alpha_n m' \equiv \widehat{x'_n} + \widehat{\alpha_n} m' \pmod{m}, \quad n \in \mathbb{Z}_+. \tag{3.6}$$

Then $x'_n \equiv \widehat{x'_n} \pmod{m'}$. By the assumption $\widehat{x'_n}, x'_n \in \{0, ..., m'-1\}$, we have $x'_n = \widehat{x'_n}$, $n \in \mathbb{Z}_+$. Now the congruence (3.6) means $\alpha_n \equiv \widehat{\alpha_n} \pmod{d}$. Since $\alpha_n, \widehat{\alpha_n} \in \{0, \ldots, d-1\}$, we have $\alpha_n = \widehat{\alpha_n}$, $n \in \mathbb{Z}_+$. □

The following theorem is a solvability theorem for the initial problem (1.1), (1.2). This theorem also establishes the explicit form for the general solution of the considered initial problem, when a solution exists.

Theorem 3.1. *The following statements hold.*

1. *The initial problem (1.1), (1.2) has a unique solution if and only if $d = 1$ and one of the following conditions holds:*

   (a) $m_2 = 1$;

   (b) $m_2 \neq 1$ and the equality

$$Y_{2,0} = -\sum_{s=0}^{\operatorname{ind}(B_2)-1} A_2^{-s-1} B_2^s F_{2,s} \tag{3.7}$$

   is satisfied.

   Moreover, the unique solution of the initial problem (1.1), (1.2) is defined by the formula

$$X_n = \begin{cases} B^{-n} A^n Y_0 + \sum_{s=0}^{n-1} A^s B^{-s-1} F_{n-s-1}, & m_2 = 1, \\ -\sum_{s=0}^{\operatorname{ind}(B)-1} A^{-s-1} B^s F_{n+s}, & m_1 = 1, \\ \psi(\gamma_n, \delta_n), & m_1 \neq 1, m_2 \neq 1, \end{cases} \tag{3.8}$$

   where the isomorphism ψ is defined by the formula (2.1) and

$$\gamma_n = B_1^{-n} A_1^n Y_{1,0} + \sum_{s=0}^{n-1} A_1^s B_1^{-s-1} F_{1,n-s-1},$$

$$\delta_n = -\sum_{s=0}^{\operatorname{ind}(B_2)-1} A_2^{-s-1} B_2^s F_{2,n+s}.$$

2. *The initial problem* (1.1), (1.2) *has infinitely many solutions if and only if* $d \neq 1$, $d \mid f_n$ *for all* $n \in \mathbb{Z}_+$ *and one of the following conditions holds:*

   (a) $m_2' = 1$;

   (b) $m_2' \neq 1$ *and the equality*

$$Y_{2,0}' = - \sum_{s=0}^{\operatorname{ind}(B_2')-1} \left(A_2'\right)^{-s-1} \left(B_2'\right)^s F_{2,s} \tag{3.9}$$

   is satisfied.

   The general solution of the initial problem (1.1), (1.2) *is defined by*

$$X_n = [x_n' + \alpha_n \cdot m']_m, \quad n \in \mathbb{N}, \tag{3.10}$$

   where $X_n' = [x_n']_{m'}$ $(n \in \mathbb{Z}_+)$ *is a solution of the initial problem* (3.1), (3.2) *(this solution exists and is unique), and* $\{\alpha_n\}_{n=1}^{\infty}$ *is an arbitrary sequence of* $\{0, 1, \ldots, d-1\}$. *Moreover, the sequence* $\{\alpha_n\}_{n=1}^{\infty}$ *is uniquely determined by the solution* $\{X_n\}_{n=0}^{\infty}$ *of the initial problem* (1.1), (1.2).

3. *The initial problem* (1.1), (1.2) *has no solutions if and only if one of the following conditions holds:*

   (a) $d \nmid f_n$ *for some* $n \in \mathbb{Z}_+$;

   (b) $d \mid f_n$ $(n \in \mathbb{Z}_+)$, $m_2' \neq 1$ *and the equality* (3.9) *is not satisfied.*

Remark 3.1. In the statement 2 of Theorem 3.1 the sequence $\{X_n'\}_{n=0}^{\infty}$, when $m \neq d$, may be defined by the formula, analogous to the formula (3.8), applied to the initial problem (3.1), (3.2). When $m = d$, then evidently $X_n' = 0$ for all $n \in \mathbb{Z}_+$.

Proof. The sufficient conditions of all three statements of Theorem 3.1 are mutually exclusive and they exhaust all possibilities. Therefore, it is enough to prove the sufficiency for all of three statements of this theorem.

Let us prove the sufficiency of the statement 1. Let $d = 1$. If either $m_1 = 1$, or $m_2 = 1$, then the claimed statement follows from Corollary 2.1. Let us assume now that $m_1 \neq 1$ and $m_2 \neq 1$.

Set the initial conditions:

$$X_{1,0} = Y_{1,0} \in \mathbb{Z}_{m_1}, \tag{3.11}$$

$$X_{2,0} = Y_{2,0} \in \mathbb{Z}_{m_2} \tag{3.12}$$

for equations (2.5) and (2.6) respectively.

According to Lemma 2.1, the sequence $\{X_n\}_{n=0}^{\infty}$ is a solution of the initial problem (1.1), (1.2) if and only if the sequence $\{X_{1,n}\}_{n=0}^{\infty}$ is a solution of the initial problem (2.5), (3.11) and the sequence $\{X_{2,n}\}_{n=0}^{\infty}$ is a solution of the initial problem (2.6), (3.12). By the statement 1 of Theorem 2.1, the initial problem (2.5), (3.11) has a solution for any $Y_{1,0} \in \mathbb{Z}_{m_1}$. According to the statement 2 of Theorem 2.1, the initial problem (2.6), (3.12) has a solution if and only if $Y_{2,0}$ satisfies (3.7). Hence, the initial problem (1.1), (1.2) has a solution if and only if the condition (3.7) is satisfied, moreover this solution is unique and has the form (2.7), where $X_{1,n}$ and $X_{2,n}$ are defined by the formulas (2.10) and (2.11) respectively.

Let us prove the sufficiency of the statement 2. Let $d \neq 1$, $d \mid f_n$ for all $n \in \mathbb{Z}_+$. Additionally, let either $m_2' = 1$, or $m_2' \neq 1$ and (3.9) be satisfied. Since $\gcd(a/d, b/d, m') = 1$, we can apply the sufficiency of the statement 1 (which is already proved) to the initial problem (3.1), (3.2). Due to that statement, the initial problem (3.1), (3.2) has a unique solution $X_n' = [x_n']_{m'}$ $(n \in \mathbb{Z}_+)$. By Lemma 3.1, for any sequence $\{\alpha_n\}_{n=0}^{\infty}$ of $\{0, \ldots, d-1\}$ the formula (3.3) defines the solution of Equation (1.1).

We choose α_0 such that (1.2) is satisfied, i.e., $[x_0' + \alpha_0 m']_m = [y_0]_m$. The initial condition (3.2) implies $[x_0']_{m'} = [y_0]_{m'}$, and the following congruence holds $x_0' \equiv y_0 \pmod{m'}$. Then $\beta = \frac{y_0 - x_0'}{m'} \in \mathbb{Z}$. Divide β on d with remainder. Then there exist $q \in \mathbb{Z}$ and $\alpha_0 \in \{0, \ldots, d-1\}$ such that $\beta = qd + \alpha_0$. Therefore,

$$[x_0' + \alpha_0 m']_m = [x_0' + (\beta - qd)m']_m = [x_0' + y_0 - x_0' - qm]_m = [y_0]_m.$$

Therefore for the chosen α_0 and any sequence $\{\alpha_n\}_{n=1}^{\infty}$ of $\{0, \ldots, d-1\}$ the formula (3.10) defines a solution of the initial problem (1.1), (1.2). By Lemma 3.1, the expression (3.10) gives infinitely many solutions of this initial problem (see also (3.3)).

We prove that the general solution of the initial problem (1.1), (1.2) is defined by the formula (3.10). Let $\{X_n\}_{n=0}^{\infty}$ be an arbitrary solution of this initial problem. Then by Lemma 3.1, this solution has the form (3.3), where $\{X_n'\}_{n=0}^{\infty}$ is a solution of Equation (3.1). Moreover, $\{X_n'\}_{n=0}^{\infty}$ must satisfy the initial condition (3.2). We have proved that the initial problem (3.1), (3.2) has a unique solution. Hence, the general solution of the initial problem (1.1), (1.2) has the form (3.10).

Let us prove now the sufficiency of the statement 3. Assume $d \nmid f_n$ for some $n \in \mathbb{Z}_+$. The equality (1.1) for this n is equivalent to the congruence $bx_{n+1} - ax_n \equiv f_n \pmod{m}$. Hence,

$$f_n \equiv d \cdot \left(\frac{b}{d}x_{n+1} - \frac{a}{d}x_n\right) \pmod{m}. \tag{3.13}$$

Since $d \mid m$, the condition (3.13) means that $d \mid f_n$, which contradicts the assumption. Therefore, if $d \nmid f_n$ for some $n \in \mathbb{Z}_+$, then Equation (1.1) has no solutions. Now suppose that $d \mid f_n$, $n \in \mathbb{Z}_+$, $m_2' \neq 1$ and the equality (3.9) is not satisfied. Assume the contrary, that the initial problem (1.1), (1.2) has a solution $X_n = [x_n]_m$ ($n \in \mathbb{Z}_+$). Then the congruence (3.13) is satisfied for all $n \in \mathbb{Z}_+$ and the sequence $X_n' = [x_n]_{m'}$ ($n \in \mathbb{Z}_+$) is a solution of the initial problem (3.1), (3.2). Since $\gcd(a/d, b/d, m') = 1$, we can apply Lemma 2.1 and the statement 2 of Theorem 2.1 to this initial problem. Therefore, if $m_2' \neq 1$ and $\{X_n'\}_{n=0}^{\infty}$ is a solution of the initial problem (3.1), (3.2), then $Y_{2,0}' = [y_0]_{m_2'}$ must satisfy (3.9). This contradicts the assumption. □

The following theorem is a solvability theorem for Equation (1.1). This theorem also yields an explicit form for the general solution of Equation (1.1).

Theorem 3.2. *The following statements hold.*

1. *Equation (1.1) has a finite number of solutions if and only if $d = 1$. Moreover, the number of these solutions is equal to m_1 and in this case*

   (a) *If $m_2 = 1$, then the general solution of Equation (1.1) has the form*

   $$X_n = B^{-n}A^nX_0 + \sum_{s=0}^{n-1} A^sB^{-s-1}F_{n-s-1}, \quad n \in \mathbb{N}, \tag{3.14}$$

   where X_0 is an arbitrary element of $\mathbb{Z}_m$.

   (b) *If $m_1 = 1$, then the unique solution of Equation (1.1) has the form*

   $$X_n = -\sum_{s=0}^{\operatorname{ind}(B)-1} A^{-s-1}B^sF_{n+s}, \quad n \in \mathbb{Z}_+. \tag{3.15}$$

(c) If $m_1 \neq 1$ and $m_2 \neq 1$, then the general solution of Equation (1.1) has the form

$$X_0 = \psi\left(X_{1,0}, -\sum_{s=0}^{\mathrm{ind}(B_2)-1} A_2^{-s-1} B_2^s F_{2,s}\right),$$

$$X_n = \psi(\gamma_n, \delta_n), \quad n \in \mathbb{N}, \tag{3.16}$$

where $X_{1,0}$ is an arbitrary element of $\mathbb{Z}_{m_1}$, the isomorphism ψ is defined by the formula (2.1) and

$$\gamma_n = B_1^{-n} A_1^n X_{1,0} + \sum_{s=0}^{n-1} A_1^s B_1^{-s-1} F_{1,n-s-1},$$

$$\delta_n = -\sum_{s=0}^{\mathrm{ind}(B_2)-1} A_2^{-s-1} B_2^s F_{2,n+s}.$$

2. *Equation (1.1) has infinitely many solutions if and only if $d \neq 1$ and $d \mid f_n$ for all $n \in \mathbb{Z}_+$. The general solution in this case has the form (3.3), where $X'_n = [x'_n]_{m'}$ ($n \in \mathbb{Z}_+$) is the general solution of Equation (3.1), and $\{\alpha_n\}_{n=0}^{\infty}$ is an arbitrary sequence of $\{0, \dots, d-1\}$. Moreover, the sequence $\{\alpha_n\}_{n=0}^{\infty}$ and the solution $\{X'_n\}_{n=0}^{\infty}$ of Equation (3.1) with $x'_n \in \{0, \dots, m'-1\}$ are uniquely determined by the solution $\{X_n\}_{n=0}^{\infty}$ of Equation (1.1).*

3. *Equation (1.1) has no solutions if and only if $d \nmid f_n$ for some $n \in \mathbb{Z}_+$.*

Remark 3.2. Since $\gcd(a/d, b/d, m') = 1$, in the statement 2 of Theorem 3.2 the general solution of Equation $\{X'_n\}_{n=0}^{\infty}$ may be defined by the formula, analogous to formulas (3.14)–(3.16), applied to Equation (3.1).

Proof. The sufficient conditions of all three statements of Theorem 3.2 are mutually exclusive and they exhaust all possibilities. Therefore, it is enough to prove the sufficiency for all of three statements of this theorem.

We prove the sufficiency of the statement 1 of Theorem 3.2. Let $d = 1$. If either $m_1 = 1$ or $m_2 = 1$, then the claimed statement follows from Corollary 2.1. Let $m_1 \neq 1$ and $m_2 \neq 1$. The statement 1 of Theorem 3.1 implies that if there exists a solution of the initial problem (1.1), (1.2), then it is defined uniquely by the given $Y_{1,0}$, where $Y_{1,0}$ is an arbitrary element of the ring $\mathbb{Z}_{m_1}$. Therefore, the number of solutions of

Equation (1.1) is equal to m_1. The form (3.16) of the general solution of Equation (1.1) is obtained by using the general solution (3.8) of the initial problem (1.1), (1.2).

Let us prove the sufficiency of the statement 2 of Theorem 3.2. Let $d \neq 1$ and $d \mid f_n$ for all $n \in \mathbb{Z}_+$. Since $\gcd\left(\frac{a}{d}, \frac{b}{d}, m'\right) = 1$, we can apply the sufficiency of the statement 1 (which is already proved) to Equation (3.1). Due to that statement, Equation (3.1) has m_1' solutions. Let $X_n' = [x_n']_{m'}$ ($n \in \mathbb{Z}_+$) be the general solution of this equation. By Lemma 3.1, the general solution of Equation (1.1) has the form (3.3), where $\{\alpha_n\}_{n=0}^{\infty}$ is an arbitrary sequence of $\{0, \ldots, d-1\}$. Moreover, by Lemma 3.1, Equation (1.1) has infinitely many solutions.

Let us prove the sufficiency of the statement 3 of Theorem 3.2. Let $d \neq 1$ and $d \nmid f_n$ for some $n \in \mathbb{Z}_+$. By the statement 3 of Theorem 3.1, for any $Y_0 \in \mathbb{Z}_m$ the initial problem (1.1), (1.2) has no solutions. Hence, Equation (1.1) has no solutions. □

The following corollary of Theorem 3.2 yields the solvability of Equation (1.1) in the case of an invertible element A.

Corollary 3.1. *If A is an invertible element of $\mathbb{Z}_m$, then Equation (1.1) always has a solution. Moreover, the number of solutions for Equation (1.1) is equal to m_1.*

Theorem 3.2 also implies the following criteria of the existence and uniqueness of a solution for Equation (1.1).

Corollary 3.2. *Equation (1.1) has a unique solution if and only if $d = 1$ and $m_1 = 1$. In particular, the homogeneous equation*

$$BX_{n+1} = AX_n, \quad n \in \mathbb{Z}_+ \tag{3.17}$$

has only trivial solution if and only if $d = 1$ and $m_1 = 1$.

Corollary 3.3. *Equation (1.1) has a unique solution if and only if A is invertible and B is nilpotent. Moreover, this solution has the form (3.15).*

Proof. According to Corollary 3.2, Equation (1.1) has a unique solution if and only if $d = 1$ and $m_1 = 1$.

Hence, it suffices to prove that the conditions that B is nilpotent and A is invertible are equivalent to the conditions $d = 1$ and $m_1 = 1$.

At first, let us prove the sufficiency of the mentioned statement. Let B be nilpotent and A be invertible. We prove that $d = 1$, $m_1 = 1$. By the statement 3 of Lemma 2.2, the nilpotency of B implies $m_1 = 1$. If A is invertible, then $\gcd(a, m) = 1$, and hence $d = 1$.

Let us prove now the converse statement. Let $d = 1$ and $m_1 = 1$. By the statement 3 of Lemma 2.2, the condition $m_1 = 1$ yields B is nilpotent. By Lemma 2.2, if $d = 1$, then A_2 is an invertible element of $\mathbb{Z}_{m_2}$. Since $m_1 = 1$, this implies that $A_2 = A$ is invertible. The representation (3.15) for the unique solution of Equation (1.1) follows from Theorem 3.2. □

Corollary 3.4. *If the homogeneous equation* (3.17) *has only trivial solution, then for any sequence* $\{F_n\}_{n=0}^{\infty}$ *Equation* (1.1) *has a unique solution. Moreover, the unique solution of Equation* (1.1) *has the form* (3.15).

Proof. Let Equation (3.17) has only trivial solution. Then Corollary 3.2 implies $d = 1$, $m_1 = 1$ and, therefore, for any sequence $\{F_n\}_{n=0}^{\infty}$ of $\mathbb{Z}_m$ Equation (1.1) has a unique solution. The form (3.15) for the unique solution of Equation (1.1) follows from Corollary 3.3. □

4. Examples

Example 4.1. Consider the following equation over $\mathbb{Z}_6$:

$$[3]_6 X_{n+1} = [2]_6 X_n + F_n, \quad n \in \mathbb{Z}_+. \tag{4.1}$$

Here $A = [2]_6$, $B = [3]_6$, $m = 6$. Let $b = 3$, $a = 2$, hence $d = 1$. We have $m_1 = 2$ and $m_2 = 3$. Therefore $A_2 = [2]_3$, $B_2 = [3]_3$, $\operatorname{ind}(B_2) = 1$. Let $Y_0 = [y_0]_6$, $F_n = [f_n]_6$. By the statement 1 of Theorem 3.1, the initial problem (4.1), (1.2) has a solution if and only if

$$[y_0]_3 = -\sum_{s=0}^{\operatorname{ind}(B_2)-1} A_2^{-s-1} B_2^s F_{2,s} = -[2]_3^{-1}[f_0]_3 = [f_0]_3,$$

i.e.,

$$[y_0]_3 = [f_0]_3. \tag{4.2}$$

Further assume that the solution of the initial problem (4.1), (1.2) exists, i.e., the equality (4.2) is satisfied. This solution is unique.

The representation of this solution may be found by the formula (3.8). We obtain $A_1 = [2]_2$, $B_1 = [3]_2$, $E_1 = [3]_2^{-1} = [1]_2$, $E_2 = [2]_3^{-1} = [2]_3$ (see also the formulas (2.2)). Choose $e_1 = 1$, $e_2 = 2$. According to the formula (2.1), the isomorphism $\psi \colon \mathbb{Z}_2 \oplus \mathbb{Z}_3 \to \mathbb{Z}_6$ is defined as follows:

$$\psi(T_1, T_2) = [3t_1 + 4t_2]_6, \quad \forall T_1 = [t_1]_2, \ \forall T_2 = [t_2]_3. \tag{4.3}$$

Let us evaluate:

$$B_1^{-n}A_1^nX_{1,0}+\sum_{s=0}^{n-1}A_1^sB_1^{-s-1}F_{1,n-s-1}=$$

$$=[3]_2^{-n}[2]_2^nX_{1,0}+\sum_{s=0}^{n-1}[2]_2^s[3]_2^{-s-1}F_{1,n-s-1}=F_{1,n-1},\quad n\in\mathbb{N},\tag{4.4}$$

$$-\sum_{s=0}^{\operatorname{ind}(B_2)-1}A_2^{-s-1}B_2^sF_{2,n+s}=-[2]_3^{-1}F_{2,n}=F_{2,n},\quad n\in\mathbb{Z}_+.\tag{4.5}$$

Substituting (4.3), (4.4) and (4.5) into (3.8), we obtain the following form for the unique solution of the initial problem (4.1), (1.2):

$$X_0=Y_0,\quad X_n=\psi([f_{n-1}]_2,[f_n]_3)=$$
$$=[3f_{n-1}+4f_n]_6=3F_{n-1}+4F_n,\quad n\in\mathbb{N}.\tag{4.6}$$

By Theorem 3.2, Equation (4.1) has $m_1=2$ solutions, and the general solution of this equation has the form:

$$X_0=\psi\left([\beta]_2,[f_0]_3\right)=[3\beta+4f_0]_6,$$

$$X_n=\psi\left([f_{n-1}]_2,[f_n]_3\right)=[3f_{n-1}+4f_n]_6=3F_{n-1}+4F_n,\quad n\in\mathbb{N},$$

where β may be equal to 0 or 1.

Example 4.2. Consider the following equation over $\mathbb{Z}_9$:

$$[3]_9X_{n+1}=[2]_9X_n+F_n,\quad n\in\mathbb{Z}_+.\tag{4.7}$$

Here $A=[2]_9$, $B=[3]_9$, $m=9$. Let $b=3$, $a=2$, hence $d=1$. We have $m_1=1$ and $m_2=9$. Here B is nilpotent and A is invertible elements of $\mathbb{Z}_9$. We obtain $\operatorname{ind}(B)=2$. Let $Y_0=[y_0]_9, F_n=[f_n]_9$. By Corollary 3.3, Equation (4.2) has a unique solution. This solution has the form

$$X_n=-\sum_{s=0}^{\operatorname{ind}(B)-1}[2]_9^{-s-1}[3]_9^sF_{n+s}=-[5]_9F_n-[25]_9[3]_9F_{n+1}$$

$$=4F_n+6F_{n+1},\quad n\in\mathbb{Z}_+.\tag{4.8}$$

The initial problem (4.7), (1.2) has a solution if and only if $Y_0=4F_0+6F_1$. This solution is unique and has the form (4.8).

Example 4.3. Consider the following equation over $\mathbb{Z}_{12}$:

$$[6]_{12}X_{n+1} = [2]_{12}X_n + F_n, \quad n \in \mathbb{Z}_+. \tag{4.9}$$

Here $A = [2]_{12}$, $B = [6]_{12}$, $m = 12$, $a = 2$, $b = 6$. That implies that $d = 2$. Let $F_n = [f_n]_{12}$. If f_n is odd for some $n \in \mathbb{Z}_+$, then by the statement 3 of Theorem 3.2 Equation (4.9) has no solutions.

Further let f_n be even for all $n \in \mathbb{Z}_+$.

We obtain $m' = 6$, $m'_1 = 2$, $m'_2 = 3$, $B' = [3]_6$, $A' = [1]_6$, $F'_n = \left[\frac{f_n}{2}\right]_6$. Let $Y_0 = [y_0]_{12}$. Also, $Y'_0 = [y_0]_6$, $B'_1 = [3]_2$, $A'_1 = [1]_2$, $B'_2 = [3]_3$, $A'_2 = [2]_3$, $F'_{1,n} = \left[\frac{f_n}{2}\right]_2$, $F'_{2,n} = \left[\frac{f_n}{2}\right]_3$. Here $\mathrm{ind}(B'_2) = 1$.

By the statement 2 of Theorem 3.1, the initial problem (4.9), (1.2) has a solution if and only if

$$[y_0]_3 = -\sum_{s=0}^{\mathrm{ind}(B'_2)-1} (A'_2)^{-s-1}(B'_2)^s F'_{2,s} = -F'_{2,0} = 2\left[\frac{f_0}{2}\right]_3 = [f_0]_3,$$

i.e.,

$$[y_0]_3 = [f_0]_3. \tag{4.10}$$

The corresponding equation (3.1) over $\mathbb{Z}_6$ has the form

$$[3]_6X'_{n+1} = X'_n + F'_n, \quad n \in \mathbb{Z}_+. \tag{4.11}$$

Further let (4.10) be satisfied.

By the statement 1 of Theorem 3.1, the initial problem (4.11), (3.2) has a unique solution, which may be obtained by the formula (3.8).

Let us evaluate:

$$\left(B'_1\right)^{-n}\left(A'_1\right)^n Y'_{1,0} + \sum_{s=0}^{n-1}\left(A'_1\right)^s\left(B'_1\right)^{-s-1}F'_{1,n-s-1} =$$
$$= \left([3]_2\right)^{-n} Y'_{1,0} + \sum_{s=0}^{n-1}\left([3]_2\right)^{-s-1}F'_{1,n-s-1} = Y'_{1,0} + \sum_{s=0}^{n-1} F'_{1,n-s-1}, \tag{4.12}$$

$$-\sum_{s=0}^{\mathrm{ind}\left(B'_2\right)-1}\left(A'_2\right)^{-s-1}\left(B'_2\right)^s F'_{2,n+s} = -\left[\frac{f_n}{2}\right]_3. \tag{4.13}$$

As in Example 4.1, the isomorphism $\psi\colon \mathbb{Z}_2 \oplus \mathbb{Z}_3 \to \mathbb{Z}_6$ is defined by the formula (4.3). Substituting (4.3), (4.12) and (4.13) into (3.8), we obtain the unique solution of the initial problem (4.11), (3.2):

$$X'_0 = Y'_0, \quad X'_n = \psi\left(Y'_{1,0} + \sum_{s=0}^{n-1} F'_{1,n-s-1}, -\left[\frac{f_n}{2}\right]_2\right) =$$
$$= \left[3\left(y_0 + \sum_{s=0}^{n-1}\frac{f_s}{2}\right) + 4\left(-\frac{f_n}{2}\right)\right]_6 =$$
$$= \left[3y_0 + 3\sum_{s=0}^{n-1}\frac{f_s}{2} + 4f_n\right]_6, \quad n \in \mathbb{N}. \tag{4.14}$$

Hence, if f_n $(n \in \mathbb{Z}_+)$ is even and (4.10) is satisfied, then by the statement 2 of Theorem 3.1 the initial problem (4.9), (1.2) has infinitely many solutions. Moreover, the general solution of this initial problem has the following form (see formulas (3.10) and (4.14)).

$$X_0 = Y_0, \quad X_n = \left[3y_0 + 3\sum_{s=0}^{n-1}\frac{f_s}{2} + 4f_n + 6\alpha_n\right]_{12}, \quad n \in \mathbb{N}, \tag{4.15}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is an arbitrary sequence of the elements 0 and 1.

By the statement 2 of Theorem 3.2, Equation (4.9) has infinitely many solutions. Moreover, the general solution of Equation (4.9) has the form (see formulas (3.3) and (4.15)):

$$X_0 = [3\beta + 4f_0 + 6\alpha_0]_{12},$$

$$X_n = \left[3\beta + 3\sum_{s=0}^{n-1}\frac{f_s}{2} + 4f_n + 6\alpha_n\right]_{12}, \quad n \in \mathbb{N},$$

where β and α_n $(n \in \mathbb{Z}_+)$ are arbitrary elements of $\{0, 1\}$.

Remark 4.1. We note that the explicit equation (1.1) over the ring $\mathbb{Z}_m$ always has exactly m solutions.

Example 4.4. Consider the following equation over $\mathbb{Z}_{12}$:

$$[9]_{12}X_{n+1} = [6]_{12}X_n + F_n, \quad n \in \mathbb{Z}_+. \tag{4.16}$$

Here $A = [6]_{12}$, $B = [9]_{12}$, $m = 12$. Let $a = 6$, $b = 9$. This implies that $d = 3$. Let $Y_0 = [y_0]_{12}$, $F_n = [f_n]_{12}$. By the statement 2 of Theorem 3.1, if $3 \nmid f_n$ for some $n \in \mathbb{Z}_+$, then by the statement 3 of Theorem 3.2 Equation (4.16) has no solutions.

Further let $3 \mid f_n$ for all $n \in \mathbb{Z}_+$. We obtain $B' = [3]_4$, $A' = [2]_4$, $m' = 4$, $m_1' = 4$, $m_2' = 1$. The corresponding equation (3.1) over $\mathbb{Z}_4$ has the form:

$$[3]_4 X'_{n+1} = [2]_4 X'_n + F'_n, \quad n \in \mathbb{Z}_+, \tag{4.17}$$

where $F'_n = \left[\frac{f_n}{3}\right]_4$.

By the first statement of Theorem 3.1, for any $Y'_0 \in \mathbb{Z}_4$ the initial problem (4.17), (3.2) has a unique solution and this solution is defined by the formula:

$$\begin{aligned} X'_0 = Y'_0, \quad X'_n &= \left(B'\right)^{-n} \left(A'\right)^n Y'_0 + \sum_{s=0}^{n-1} \left(A'\right)^s \left(B'\right)^{-s-1} F'_{n-s-1} = \\ &= [3]_4^{-n}[2]_4^n Y'_0 + \sum_{s=0}^{n-1} [2]_4^s [3]_4^{-s-1} F'_{n-s-1} = \\ &= [2]_4^n Y'_0 + \sum_{s=0}^{n-1} [2]_4^s [3]_4^{-s-1} F'_{n-s-1}, \quad n \in \mathbb{N}. \end{aligned} \tag{4.18}$$

More precise,

$$X'_0 = Y'_0, \quad X'_1 = 2Y'_0 + 3F'_0, \quad X'_n = 3F'_{n-1} + 2F'_{n-2}, \quad n = 2, 3, \ldots.$$

If $3 \mid f_n$, $n \in \mathbb{Z}_+$, then by the second statement of Theorem 3.1, for any $Y_0 \in \mathbb{Z}_{12}$ the initial problem (4.16), (1.2) has infinitely many solutions. Moreover, the general solution of this initial problem has the following form (see the formula (3.10)):

$$X_0 = Y_0, \quad X_n = [x'_n + 4\alpha_n]_{12}, \quad n \in \mathbb{N},$$

i.e.,

$$X_0 = Y_0, \quad X_1 = [2y_0 + f_0 + 4\alpha_1]_{12},$$

$$X_n = \left[f_{n-1} + 2\frac{f_{n-2}}{3} + 4\alpha_n\right]_{12}, \quad n = 2, 3, \ldots. \tag{4.19}$$

Here $\{\alpha_n\}_{n=1}^{\infty}$ is an arbitrary sequence of $\{0, 1, 2\}$.

If $3 \mid f_n$, $n \in \mathbb{Z}_+$, then by the statement 2 of Theorem 3.2, Equation (4.16) has infinitely many solutions. Moreover, the general solution of this equation has the following form (see the formula (3.3)).

$$X_n = [x'_n + 4\alpha_n]_{12}, \quad n \in \mathbb{Z}_+, \tag{4.20}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is an arbitrary sequence of $\{0,1,2\}$ and the sequence $X'_n = [x'_n]_4$ ($n \in \mathbb{Z}_+$) is the general solution of Equation (4.17) which is defined as follows:

$$X'_0 = [x'_0]_4, \quad X'_1 = 2X'_0 + 3F'_0, \quad X'_n = 3F'_{n-1} + 2F'_{n-2}, \quad n = 2, 3, \ldots.$$

Now the general solution (4.20) of Equation (4.16) can be written in a more convenient form, which is similar to (4.19):

$$X_0 = [x_0]_{12}, \quad X_1 = [2x_0 + f_0 + 4\alpha_1]_{12},$$

$$X_n = \left[f_{n-1} + 2\frac{f_{n-2}}{3} + 4\alpha_n\right]_{12}, \quad n = 2, 3, \ldots,$$

where x_0 is an arbitrary integer and $\{\alpha_n\}_{n=1}^{\infty}$ is an arbitrary sequence of $\{0,1,2\}$.

References

[1] Halanay, A., Wexler, D.: Teoria calitativa a sistemelor cu impulsuri. Academiei Republicii Socialiste Romania, Bucuresti (1968).

[2] Kelley, W.G., Peterson, A.C.: Difference equation: an introduction with applications. Academic Press, New York (2001).

[3] Elaydi, S.: Introduction to difference equations. Springer-Verlag, New York (2005).

[4] Campbell, S.L.: Singular system of differential equations I. Pitman Publishing. Research Notes in Mathematics, San Fransisco, London, Mellbourne (1980).

[5] Benabdallakh, M., Rutkas, A.G., Solov'ev, A.A.: Application of asymptotic expansions to the investigation of an infinite system of equations $Ax_{n+1} + Bx_n = f_n$ in a Banach Space. J. Soviet Math. **48**(3), 124–130 (1990). https://doi.org/10.1007/BF01095789

[6] Bondarenko, M.F., Rutkas, A.G.: On a class of implicit difference equations. Dopovidi NANU of Ukraine. **7**, 11–15 (1998).

[7] Gefter, S., Goncharuk, A., Piven', A.: Implicit linear first order difference equations over commutative rings. In: Elaydi, S., Kulenović, M.R.S., Kalabušić, S. (eds.) Advances in Discrete Dynamical Systems. Difference Equations and Applications, ICDEA 2021. Springer Proceedings in Mathematics & Statistics, vol. **416**, pp. 199–216. Springer, Cham. (2023).

[8] Gerasimov, V.A., Gefter, S.L., Goncharuk, A.B.: Application of the p-adic topology on $\mathbb{Z}$ to the problem of finding solutions in integers of an implicit linear difference equation. J. Math. Sci. **235**(3), 256–261 (2018). https://doi.org/10.1007/s10958-018-4072-x

[9] Martseniuk, V.V., Gefter S.L., Piven', A.L.: Uniqueness criterion and Cramer's rule for implicit higher order linear difference equations over $\mathbb{Z}$. In: Baigent, S., Bohner, M., Elaydi, S. (eds.) Progress on Difference Equations and Discrete Dynamical Systems, ICDEA 2019. Springer Proceedings in Mathematics & Statistics, vol. **341**, pp. 311–325. Springer, Cham. (2020).

[10] Gefter, S.L., Piven', A.L.: Implicit linear nonhomogeneous difference equation over $\mathbb{Z}$ with a random right-hand Side. J. Math. Phys. Anal. Geom. **18**(1), 105–117 (2022). https://doi.org/10.15407/mag18.01.105

[11] Gefter, S.L., Piven, A.L.: Implicit linear nonhomogeneous difference equation in Banach and locally convex spaces. J. Math. Phys. Anal. Geom. **15**(3), 336–353 (2019). https://doi.org/10.15407/mag15.03.336

[12] Dunford, N., Schwartz, J.T.: Linear operators. Part I: General theory. John Wiley Sons, New York (1988).

[13] Dummit, D.S., Foote, R.M.: Abstract algebra, Wiley Hoboken (2004).

[14] Rutkas, A.G.: Spectral methods for studying degenerate differential-operator equations. I. J. Math. Sci. **144**(4), 4246–4263 (2007). https://doi.org/10.1007/s10958-007-0267-2

[15] Bondarenko, M.F., Vlasenko, L.A.: A linear quadratic regulator problem for descriptor lumped and distributed systems with discrete time. J. Autom. Inform. Sci. **42**(1), 32–41 (2010).

CONTACT INFORMATION

M. V. Heneralov,
A. L. Piven'

School of Mathematics and Computer Science,
V. N. Karazin Kharkiv National University,
4 Svobody Sq., Kharkiv, 61022, Ukraine
E-Mail: me2001.com@gmail.com,
aleksei.piven@karazin.ua

Received by the editors: 08.06.2023
and in final form 28.02.2024.