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Trivial units in commutative group rings of $G \times C_n$

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Abstract. It is known that if the unit group of an integral group ring $\mathbb{Z}G$ is trivial, then the unit group of $\mathbb{Z}(G \times C_2)$ is trivial as well [3]. The aim of this study is twofold: firstly, to identify rings R that are D-adapted for the direct product $D = G \times H$ of abelian groups G and H, such that the unit group of the ring $R(G \times H)$ is trivial. Our second objective is to investigate the necessary and sufficient conditions on both the ring R and the direct factors of D to satisfy the property that the normalized unit group $V(RD)$ is trivial in the case where D is one of the groups $G \times C_3$, $G \times K_4$ or $G \times C_4$, where G is an arbitrary finite abelian group, C_n denotes a cyclic group of order n and K_4 is Klein 4-group. Hence, the study extends the related result in [18].

1. Introduction and Preliminaries

Let R be a ring and G be a group. RG denotes the group ring of G over R. Units in RG form a multiplicative group $U(RG)$. In $U(RG)$, the set of *normalized* units which are viewed as units having augmentation 1 is exhibited as $V(RG)$ throughout the paper. An element of the form rg where $r \in U(R)$ and $g \in G$ has inverse as $r^{-1}g^{-1}$. Elements of this form

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are said to be *trivial units*. Thus, the set of trivial units is displayed as $U(R)G$. The interested reader is directed to [1, 6, 8] for additional informations and definitions about group rings and their special elements.

Let $\mathbb{P}$ denote the set of all prime integers and G and H be abelian groups (written multiplicatively) which have torsion parts $G_t = \coprod_{\forall p \in \mathbb{P}} G_p$ and $H_t = \coprod_{\forall q \in \mathbb{P}} H_q$ respectively. These co-products can be evaluated as direct products if the groups are multiplicative. Throughout the study, we take into account rings R which are commutative with unity and direct product group D of multiplicative abelian groups G and H. The maximal torsion part of a group D, as defined in [12], refers to the following subgroup:

$$D_0 = \coprod_{p \in \mathbb{P}} \coprod_{q \in \mathbb{P}} G_p \times H_q.$$

Throughout the study, $G_p \times H_q = 1$ indicates that G does not have p-primary component and H does not include q-primary component. By stating that $D_0 = 1$, we refer to the torsion-free property of either G or H. Furthermore, we denote exponent of the maximal torsion part of D as $exp(D_0) = p^\alpha q^\beta$ such that $log_p^{ord(g)} \leq \alpha$ and $log_q^{ord(h)} \leq \beta$ where $(g, h) \in D$. Considering the torsion parts of both G and H, we are able to designate the support of a direct product group in terms of composite numbers as follows:

$$supp_C(D) = \{pq \in \mathbb{P}^2 : G_p \times H_q \neq 1\}. \tag{1}$$

Besides $supp_C(D)$, we can also define the sets

$$inv_C(R) = \{pq \in \mathbb{P}^2 : pq.1_R \in U(R)\} \tag{2}$$

and

$$zd_C(R) = \{pq \in \mathbb{P}^2 : pq.(r1_R) = 0, \exists r \in R - \{0\}\}. \tag{3}$$

We generally represent the collection of all idempotent elements belonging to a ring R as $id(R)$. A commutative ring R is said to be $indecompo-sable$ if and only if $id(R)$ does not have a non-trivial idempotent element [12]. In addition to the elements that are defined as idempotents and units in a given ring R, there exists another class of elements that are specially defined as nilpotent elements which can be utilized for lifting units in RG. The set of nilpotent elements in a ring R is said to be the nil-radical of R and it is denoted by $N(R)$ in general. R is said to be *reduced* if and only if $N(R) = \{0\}$ [11].

There exist plentiful investigations concerning necessary and sufficient conditions under which the group ring RG possesses only trivial units in $V(RG)$, for certain classes of rings and groups. For instance, in the first article on this topic [3], Higman demonstrated that if R is a field and G is not a torsion group, then $V(RG) = G$. In [13], Danchev gave an extension of the result of Higman proving that if R is a field and G is an infinite abelian group, then trivialness of units in $V(RG)$ depends only on having $G_t = 1$. Moreover, Danchev proposed a necessary and sufficient condition for the triviality of $V(RG)$, given that G is an arbitrary abelian group [9]. Another relevant investigation carried out by Danchev is limited to the situation where the characteristic of a field R is positive, and in this case, necessary and sufficient conditions for $V(RG) = G$ are investigated [10]. In [5] and [6], Karpilovsky gave a criterion on when a unit of the group ring of a torsion-free group over any commutative coefficient ring is a trivial unit. In [4], he considered a torsion-free group $G \neq 1$ and extended Higman's result by proving that $V(RG)$ is trivial if and only if R is a reduced and indecomposable ring. In [11], Danchev generalized the results obtained in [4] by Karpilovsky to arbitrary abelian groups G. He formed the conditions by considering an isomorphic form of $V(RG)$ based on Mollov-Nachev's theorem in [16]. Furthermore, in [11], G is assumed to be non-R-favorable, meaning $char(R) \geq 2$ and G has a torsion element whose order is not invertible in R. We now need to remind our readers of the following definition from [1].

Definition 1. *A ring R is said to be G-adapted if R is an integral domain of characteristic 0 such that none of the prime divisors of the order of G are invertible in R.*

As a result, we can promptly state the following consequence.

Corollary 1. *Let R be an integral domain with $char(R) = 0$. Then, R is G-adapted if and only if G is R-favorable.*

Using Definition 1 and by demonstrating an extension of a well-established result for fields, it has been shown in [1] that if a ring R possesses a finite characteristic and the unit group of the group ring RG is precisely composed of the trivial units, then it follows that G can be of order at most 3. Also, some ring-theoretic necessary and sufficient conditions for a commutative ring R under which RG has non-trivial units by considering the ring R as G-adapted are given in [1] and they consider groups of exponents 2, 3 and 4 and a Hamiltonian 2-groups. Ritter and

Sehgal characterized the group rings RG with only trivial units of groups G over G-adapted rings R in [8], and also gave a rank formula for the centre of $V(\mathbb{Z}G)$. In [18], Li showed that if the unit group of integral group ring $\mathbb{Z}G$ is trivial, then unit group of $\mathbb{Z}(G \times C_2)$ is trivial without further restriction on the group G. However, it is still an open problem for a general G-adapted ring R and $G \times C_n$ where $n \geq 3$. In the next section, we implicitly provide a general structure theorem for an abelian direct product group $D = G \times H$. Afterwards, we investigate necessary and sufficient conditions for $V(RG)$ being trivial to imply $V(RD)$ is trivial, when $D = G \times C_3$, $G \times K_4$ or $G \times C_4$.

2. Main Results

Let $D = G \times H$ be the direct product group of the commutative groups G and H which are non-identity. In this section, we firstly extend the result that is attained in Theorem 1 of [12] using (1), (2) and (3) which are defined in the previous section.

Theorem 1. *Let G and H be two non-trivial abelian groups, R be a commutative unitary ring. Then, $V(RD)$ is trivial if and only if $N(R) = 0$, R is indecomposable, $V(RD_0) = D_0$ and one of the following cases holds:*

1. $D = D_0$,

2. $D \neq D_0, supp_C(D) \cap (inv_C(R) \cup zd_C(R)) = \emptyset$.

Proof. Let $V(RD)$ be trivial. It is clear from [12] that if R has non-trivial idempotent or nilpotent elements, then we can generate a non-trivial unit which is a contradiction. Thus, R has to be reduced and indecomposable. In addition, due to the fact that $V(RD_0)$ is embeddable into $V(RD)$ which is trivial, $V(RD_0)$ is precisely the maximal torsion part of D. If $D = D_0$, then we are done. Suppose that $D \neq D_0$ and D is not R-favorable. Then, there is a finitely generated subgroup $F \leq D_0$, a non-trivial idempotent $e \in RF$, a pair $(p, q) \in supp_C(D) \cap inv_C(R)$ and a non-trivial element $x_{pq} \in G_p \times H_q$ for which $u = 1 - e + ex_{pq} \in V(RD)$ is a non-trivial unit with inverse $u^{-1} = 1 - e + ex_{pq}^{-1}$. This contradiction implies that $supp_C(D) \cap inv_C(R) = \emptyset$. On the other hand, assume $supp_C(D) \cap zd_C(R) \neq \emptyset$. Then, $\exists r \in R \backslash \{0\}$ and $\exists pq \in supp_C(D)$, $pqr = 0$ and $1 \neq x_{pq} \in G_p \times H_q$ with a finite order $ord(x_{pq}) = p^i q^j$. Hence $u = 1 + r - rx_{pq} \in V(RD) \setminus D$ is a non-trivial unit for $r \neq -1, pq - 1$.

Accordingly, $supp_C(D) \cap zd_C(R) = \emptyset$. For the converse direction of the proof, one can apply [17] to achieve that

$$V(RD) = D.V(RD_0) = D.D_0 = D$$

as similarly stated in [12]. □

Defining a ring epimorphism as $\varphi : D \longrightarrow G$ by $\varphi(g,h) = g$ and extending φ linearly to the group ring RD which can be shown as $\tilde{\varphi}$, one can observe that $Ker\tilde{\varphi} = 1 + \Delta_{RG}(H)$ is a rationally closed subring of RD and $V(RD) = V(SH)$ where $S = RG$. The subgroup of units which is lifted from $Ker\tilde{\varphi}$ is $V(K) = (1 + Ker\tilde{\varphi}) \cap V(RD)$ and it is embedded into $V(SH)$ by identity map ι. It follows that

$$1 \longrightarrow V(K) \xrightarrow{\ \iota\ } V(RD) \xrightarrow{\ \tilde{\varphi}\ } V(RG) \longrightarrow 1$$

is a short exact sequence and it splits by the reverse direction of ι as

$$V(SH) = V(K) \times V(RG). \tag{4}$$

When $V(SH) = D$, we will have $V(RG) = G$ and $V(K) = H$. Besides, we know that $V(K) \subseteq V(SH)$ and $\tilde{\varphi}(V(RH)) = 1$ that is $V(RH) \subseteq V(K)$. Thus,

$$V(RH) \subseteq V(K) \subseteq V(SH).$$

It follows that $V(RH)$ is trivial as well.

Conversely, let $V(RG) = G$ and $V(RH) = H$. Hence,

$$H \subseteq V(K) \subseteq V(SH)$$

and thus $V(SH) = V(K) \times G$. One can notice that $V(K) \setminus H$ may be non-empty.

In order to investigate the necessary and sufficient conditions for $V(RD)$ to be trivial, it is important to examine the conditions that are necessary and sufficient for both $V(1 + \Delta_{RG}(H))$ and $V(1 + \Delta_{RH}(G))$ to be trivial, assuming that $V(RG) = G$ and $V(RH) = H$, respectively. In this sense, our focus is now on elucidating the circumstances under which

$$\{u \in V(RD) : u \in V(K) \setminus H\} = \emptyset$$

holds for certain types of groups H.

Theorem 2. *Let R be a commutative unitary ring, G and C_3 be finite abelian R-favorable groups and $D = G \times C_3$. Then, $V(RD)$ is trivial if and only if the following hold:*

i. R is reduced and indecomposable,

ii. $a^2 + b^2 + ab - a - b = 0$ has no non-trivial solution in RG.

Proof. $\Rightarrow$: Let $V(RD) = D$. It follows from the split extension in equation (4) that both $V(RG) = G$ and $V(K) = C_3$. Based on the assumption regarding G and the fact that $V(RD)$ is trivial, we can realize that $N(RG) = 0$ by means of Maschke's Theorem [2]. Hence, R is reduced. Furthermore, provided that G is finite group, R is an indecomposable commutative G-adapted ring, then RG is indecomposable [15]. Thus, if R is not an indecomposable ring, we have a non-trivial idempotent which can be embeddable into RG. It follows that we can generate a non-trivial unit as well [12]. As a result, R has to be indecomposable. Taking into account $D = G \times \langle x : x^3 = 1\rangle$, we also have from [11] that all the normalized units in $V(RD)$ are derived from the rationally closed subring

$$A = \langle x - 1, x^2 - 1\rangle_{RG}$$

where $V(K) = 1 + A$. Now, let $u = 1 - a - b + ax + bx^2$ in $1 + A$ for some $a, b \in RG$. Then, u has an inverse with the same form $v = 1 - c - d + cx + dx^2$ for some $c, d \in RG$ if and only if the following system $Bw = C$ of matrices as

$$\begin{bmatrix} 1-2a-b & b-a \\ a-b & 1-a-2b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} \tag{5}$$

has a unique solution $w = (c, d)^T$. This infers that

$$det(B) = 1 + 3(a^2 + b^2 + ab - a - b). \tag{6}$$

If $3(a^2 + b^2 + ab - a - b) \neq 0$, this contradicts with the fact that right-hand side of (6) is trivial. Triviality of units of RG implies $U(RG) = U(R)G$. This means that $det(B)$ is the scalar multiple of a element in G. If $det(B) = \alpha g$ for $g \neq 1$, (6) would imply $3r = -1 + \alpha g$ in RG, which is not possible. So it is the case that $det(B) = 1 + 3r$, for some $r = a^2 + b^2 - ab - a - b \in R$. However, then the same equation holds if a and b are replaced by their augmentations $\Delta(a)$, $\Delta(b)$ in the group ring RG, and this would imply $1 - \Delta(a) - \Delta(b) + \Delta(a)x + \Delta(b)x^2$ is a normalized unit of RC_3, and $V(RD) = D$ implies this has to be an element of C_3, and thus both $\Delta(a)$ and $\Delta(b)$ have to be 0. Therefore, the augmentation of $det(B)$ is 1, and therefore it is a normalized unit of RG. Hence, we deduce that $det(B) \in V(S) = G$.

Besides, since G can contain any 3-group, having $3 \in zd(R)$ would lead to a contradiction, thus we must have $a^2 + b^2 + ab - a - b = 0$. Moreover, we can observe that for any parameters a and b that render the unit u trivial, the equation

$$a^2 + b^2 + ab - a - b = 0 \tag{7}$$

is always satisfied. In other words, if there exists a non-trivial solution of (7), then $u \in V(S\langle x\rangle) \setminus \langle x\rangle$ which is a contradiction.

$\Leftarrow$: It is well-established that if RG is a group ring of a finite group defined over a commutative ring with unity R which is also G-adapted, reduced and indecomposable, RG has no non-trivial idempotent and no non-zero nilpotent element [12], [15]. Consequently, $V(RG) = G$. Besides, due to the fact that C_3 is R-favorable and so RC_3 has only trivial units, $(0,0)$, $(1,0)$, and $(0,1)$ are the only solutions to (6) in R, which correspond to u being a trivial unit in C_3. So assuming $V(RD) \neq D$ leads to a contradiction as before when one considers the non-trivial unit as an element of $V(SC_3)$.

Hence, using the assumption on equation (7) and the fact that $V(RG)$ is also trivial, one can deduce that $V(SC_3)$ is also trivial. $\square$

Certain types of rings allow us to carry out some useful constructions. One of such instances is as follows.

Corollary 2. *Let $D = G \times C_3$ and $R = \mathbb{F}_2$ which is a finite field of order 2 and G be a group as the previous theorem. Then, $V(RD)$ is trivial if and only if*

$$Sym_2^*(RG) \simeq \langle \zeta \rangle$$

where

$$Sym_2^*(RG) = \{M = \begin{bmatrix} \alpha & \alpha+\gamma \\ \alpha+\gamma & \gamma \end{bmatrix} : det(M) = \alpha^2+\gamma^2+\alpha\gamma \in U(\mathbb{F}_2G)\}$$

is a proper subgroup of the group of invertible symmetric matrices of size 2×2 in $\mathbb{F}_2G$ and ζ is the 3rd primitive root of unity.

Proof. If $R = \mathbb{F}_2$ and $V(RD) = D$, we can reduce the coefficients in the system of linear equations (5) as

$$\begin{bmatrix} 1+b & a+b \\ a+b & 1+a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}. \tag{8}$$

Notice that for any $\alpha, \beta, \gamma \in RG$, there exists for some $a, b \in RG$ such that $\alpha = 1 + b$, $\beta = a + b$ and $\gamma = 1 + a$. Using these elements, we can construct a proper subgroup of the group of invertible symmetric matrices of size 2×2 for which (8) holds as follows:

$$Sym_2^*(RG) = \{M = \begin{bmatrix} \alpha & \alpha+\gamma \\ \alpha+\gamma & \gamma \end{bmatrix} : det(M) = \alpha^2+\gamma^2+\alpha\gamma \in U(\mathbb{F}_2G)\}.$$

Precisely,

$$Sym_2^*(\mathbb{F}_2G) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

and $f : Sym_2^*(\mathbb{F}_2G) \longrightarrow \langle\zeta\rangle$, $f(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}) = \zeta$ is an isomorphism.

Conversely, consider the subgroup of the invertible symmetric matrices in $\mathbb{F}_2G$ that can be presented as in (8) and shown by $Sym_2^*(\mathbb{F}_2G)$. Let $Sym_2^*(\mathbb{F}_2G) \simeq \langle\zeta\rangle$. If, $Sym_2^*(\mathbb{F}_2G)$ has exactly three elements, then each of them corresponds to a solution of the system (8), which is equivalent to the coefficients in equation (7) being equal to $(a, b) = (0, 0)$, $(0, 1)$, or $(1, 0)$. This implies that no other solution is possible. Indeed, this is adequate to show that (7) has no non-trivial solution in $\mathbb{F}_2G$. □

Now, we give an extension of the result in [1] regarding Klein 4-group $K_4 = \langle h_1, h_2\rangle$ as follows:

Theorem 3. *Let R be commutative ring with unity for which $2 \notin zd(R)$, and let G be a finite abelian R-favorable group for which R is G-adapted. If $D = G \times K_4$, the following are equivalent:*

(1) $V(RG) = G$;
(2) $V(RD) = D$.

Proof. (1) ⇒ (2): Let $V(RG)$ be trivial. Taking into account the projections

$$\sigma_{h_1} : G \times K_4 \to G \times \langle h_2\rangle$$

by means of $\sigma_{h_1}(h_1) = 1$ and also

$$\sigma_{h_2} : G \times K_4 \to G \times \langle h_1\rangle$$

via $\sigma_{h_2}(h_2) = 1$, we achieve the following short exact sequences:

$$0 \longrightarrow \langle h_1\rangle \xrightarrow{\iota} RD \xrightarrow{\sigma_{h_1}} G \times \langle h_2\rangle \longrightarrow 0 \qquad (9)$$

and

$$0 \longrightarrow \langle h_2 \rangle \xrightarrow{\iota} RD \xrightarrow{\sigma_{h_2}} G \times \langle h_1 \rangle \longrightarrow 0. \tag{10}$$

Linearity of the above projections infers that we can move σ_{h_1} and σ_{h_2} over the ring R as $\sigma_{h_1} : RD \longrightarrow R(G \times \langle h_2 \rangle)$ through the rule

$$\sigma_{h_1}(p_0 + p_1 h_1 + p_2 h_2 + p_3 h_1 h_2) = (p_0 + p_1) + (p_2 + p_3) h_2$$

and $\sigma_{h_2} : RD \longrightarrow R(G \times \langle h_1 \rangle)$ via

$$\sigma_{h_2}(p_0 + p_1 h_1 + p_2 h_2 + p_3 h_1 h_2) = (p_0 + p_2) + (p_1 + p_3) h_1.$$

As a sequence, we have

$$Ker \ \sigma_{h_1} = \Delta_{RD}(\langle h_1 \rangle) = (h_1 - 1) R(G \times \langle h_2 \rangle) \tag{11}$$

and

$$Ker \ \sigma_{h_2} = \Delta_{RD}(\langle h_2 \rangle) = (h_2 - 1) R(G \times \langle h_1 \rangle). \tag{12}$$

Let $\sigma^r_{h_1}$ and $\sigma^r_{h_2}$ be the restrictions of σ_{h_1} and σ_{h_2} onto $Ker \ \sigma_{h_2}$ and $Ker \ \sigma_{h_1}$ respectively. Then, observe that

$$Ker \ \sigma^r_{h_1} = Ker \ \sigma^r_{h_2} = (h_1 - 1)(h_2 - 1) RG \tag{13}$$

and also the images of $\sigma^r_{h_1}$ and $\sigma^r_{h_2}$ are embeddable into $V(R(G \times \langle h_2 \rangle))$ and $V(R(G \times \langle h_1 \rangle))$ respectively.

Normalizing the statements in (9)-(13), we obtain the short exact sequences of groups as

$$\begin{array}{ccccc}
V(1 + Ker \ \sigma^r_{h_1}) & \xrightarrow{\iota} & V(1 + Ker \ \sigma_{h_2}) & \xrightarrow{\sigma^r_{h_1}} & V(1 + Im \ \sigma^r_{h_1}) \\
\iota \big\downarrow & & \iota \big\downarrow & & \iota \big\downarrow \\
V(1 + Ker \ \sigma_{h_1}) & \xrightarrow{\iota} & V(RD) & \xrightarrow{\sigma_{h_1}} & V(R(G \times \langle h_2 \rangle)) \\
\sigma^r_{h_2} \big\downarrow & & \sigma_{h_2} \big\downarrow & & \sigma_{h_2} \big\downarrow \\
V(1 + Im \ \sigma^r_{h_2}) & \xrightarrow{\iota} & V(R(G \times \langle h_1 \rangle)) & \xrightarrow{\sigma_{h_1}} & V(RG).
\end{array} \tag{14}$$

Note that a unit $u \in V(1 + Im \ \sigma^r_{h_1})$ is of the form $u = 1 + \alpha(h_1 - 1)$ if and only if $1 - 2\alpha \in U(R)G$. Similarly, $V(1 + Im \ \sigma^r_{h_2})$ is composed of the units of the form $u = 1 + \beta(h_2 - 1)$ such that $1 - 2\beta \in U(R)G$ [18]. Besides, examine that

$$V(1 + Ker \ \sigma^r_{h_1}) = \{1 + (h_1 - 1)(h_2 - 1)\gamma : 1 + 4\gamma \in V(RG)\} \tag{15}$$

generalizing [7]. Accordingly, we conclude that the sequences in (14) split $V(RD)$ into the direct factors as

$$V(RG) \times V(1 + Ker\ \sigma^r_{h_1}) \times V(1 + Im\ \sigma^r_{h_1}) \times V(1 + Im\ \sigma^r_{h_2}). \quad (16)$$

Since $V(RG) = G$ by the assumption, it is enough to show that the other factors are trivial. Notice from the definition related to the support of an element in RG that the right-hand side of $1 - 2\alpha = r_i g_i$ include only one term [2]. This means that $2\alpha = 0$ due to the fact that 1 is fixed or $\alpha = -1$. Because of the fact that $2 \notin zd(R)$, if $2\alpha = 0$ then $\alpha = 0$. Hence, $V(1 + Im\ \sigma^r_{h_1}) = \langle h_1 \rangle$ and $V(1 + Im\ \sigma^r_{h_2}) = \langle h_2 \rangle$. Similarly, observe that $1 + 4\gamma = r_i g_i$ is provided only if $\gamma = 0$. It follows that $V(1 + Ker\ \sigma^r_{h_1}) = 1$ because of (15). Thus, we deduce that

$$V(RD) = G \times \langle h_1 \rangle \times \langle h_2 \rangle$$

based on (16). Conversely, if $V(RD)$ is trivial, it can be expeditiously observed that $V(RG) = G$ by virtue of it being a direct factor, as demonstrated in (16). □

Theorem 4. *Let R be commutative ring with unity, and G be a finite abelian R-favorable group for which R is G-adapted. Let $D = G \times C_4$ where $C_4 = \langle x : x^4 = 1 \rangle$. If $V(RG) = G$ and $V(RD) \neq D$, then $2 \in zd(R)$.*

Proof. A natural map as in the previous theorem can be defined by $\varphi : D \longrightarrow G$ with $\varphi(g, h) = g$ can be lifted to the group rings linearly as $\varphi : RD \longrightarrow RG$. It follows that $\Delta_{RG}(H) = \langle 1 - x, 1 - x^2, 1 - x^3 \rangle_{RG}$ and

$$\frac{V(RD)}{V(1 + \Delta_{RG}(H))} \simeq V(RG).$$

Let $u = 1 + a(1 - x) + b(1 - x^2) + c(1 - x^3) \in V(1 + \Delta_{RG}(H))$. Then there exists the inverse $u^{-1} = 1 + d(1 - x) + e(1 - x^2) + f(1 - x^3)$ such that

$$\begin{aligned} a + d + 2ad + bd + cd + ae - ce + af - bf &= 0, \\ b + e - ad + bd + ae + 2be + ce + bf - cf &= 0, \quad (17) \\ c + f - bd + cd - ae + ce + af + bf + 2cf &= 0. \end{aligned}$$

By using the equations in (17), we can construct a non-singular matrix system $Ax = B$ as follows:

$$\begin{bmatrix} 1+2a+b+c & a-c & a-b \\ b-a & 1+a+2b+c & b-c \\ c-b & c-a & 1+a+b+2c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix}. \quad (18)$$

It is not confusing to observe that the determinant of the matrix in the system (18) is a unit and moreover, that is of the form

$$(1+2a+2c)(1+2a^2+4b^2+2c^2+4ab+4bc+2a+4b+2c). \quad (19)$$

Hence, we achieve an immediate consequence is that both $1+2a+2c$ and $1+2a^2+4b^2+2c^2+4ab+4bc+2a+4b+2c$ are belong to $V(RG)$ which is assumed to be G with respect to the hypothesis. According to the definition of support of an element in RG, this implies that

$$2(a+c) = 2(a^2+2b^2+c^2+2ab+2bc+a+2b+c) = 0. \quad (20)$$

Assuming there exist non-trivial units in $V(RD)$, we deduce that there may exist non-trivial elements of the form $a+c$ or $a^2+2b^2+c^2+2ab+2bc+a+2b+c$. This is only possible if $2 \in zd(R)$, which completes the proof. □

3. Conclusion and Discussion

One of the earliest works on trivial units in group rings was as follows: for $R = \mathbb{Z}$, if the units of the integral group ring $\mathbb{Z}G$ are only trivial units, then the same property holds for $\mathbb{Z}(G \times C_2)$ as previously emphasized [1], [3], [14], [18]. It should be noted that the direction of this result is only one way. After that work, investigating the necessary and sufficient conditions for all units of the group ring RG defined over a general commutative ring R to be trivial became an important topic for researchers interested in this area. Researchers can easily observe that Li's result was directly based on $V(\mathbb{Z}G) = G$ [18]. In this sense, we approached the question of when the equality $V(RD) = D$ holds under the condition of $V(RG) = G$ by considering larger groups of the form $D = G \times C_n$ for some particular but not general $n \in \mathbb{N}$ in this study. It remains an open problem for all general $n \in \mathbb{N}$, still shrouded in mystery in regard to conditions on the triviality of

$$V(1 + \Delta_{RG}(C_n)).$$

We possess strong indications that investigations on trivial units will remain a subject of considerable interest to researchers in the foreseeable future, owing to its interdisciplinary nature and its close relationship with algebraic topology and module theory.

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