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Presentations of Munn matrix algebras over *K*-algebras with *K* being a commutative ring

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Abstract. We consider the Munn matrix algebras over an associative unital K-algebra $\mathcal{A}$, where K is a commutative (unital) ring and $\mathcal{A}$ as a K-module is free (of finite or infinite rank), and, for each (not necessarily finitely defined) presentation of $\mathcal{A}$, we give presentations of the Munn matrix algebras over it.

Introduction

Presentations of algebraic objects by generators and defining relations is a convenient way to define these objects by themselves. For this reason, they have been intensively studied for different classes of groups and semigroups (see, e,g., the books [4, 6–9]). As for associative algebras over fields, the active study of presentations is connected with their representation theory and, according to the well-known methods, finite dimensional algebras (under some additional conditions) can be reduced to algebras of the paths of quivers with relations (see, e.g., the books [1, 2, 5]).

In this paper we study presentations of Munn matrix algebras over associative K-algebras with K to be a ring, which are closely related to completely 0-simple semigroups (see [3]).

1. Main results

Throughout the paper, K denotes a unital commutative ring. *Algebra* means associative (unital or non-unital) K-algebra, which is free as a

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K-module (of finite or infinite rank). If an algebra is unital, its identity is usually denoted by e (leaving the symbol 1 for the ring K).

1.1. Presentations

Let $\mathcal{B}$ be a (unital or non-unital) K-algebra. Its presentation (or, in other words, co-representation) is, by definition, a pair $\langle B|R\rangle$, where B is a set of generators and R a set of defining relations for the given set of generators (each of these sets can be both finite and infinite).[1] Then we write $\mathcal{B} = \langle B|R\rangle_K$ or simply $\mathcal{B} = \langle B|R\rangle$.

We call a presentation $\langle B|R\rangle$ of $\mathcal{B}$ *normal* if the following conditions hold:[2]

(a) B does not contain the zero element;

(b) for any relation $\{f = 0\} \in R$, the free term of the polynomial f is equal 0 or $-e$;

(c) when $e \in B$, then R contains the relations $eb - b = b$ and $be - b = 0$ for any $b \in B$.

A presentation $\langle B|R\rangle$ is said to be *unital*, if $e \in B$.

1.2. Munn matrix algebras

Let $\mathcal{A}$ be a unital algebra over a commutative ring K, m, n natural numbers and $P = (p_{ji})$ an $n \times m$ matrix over $\mathcal{A}$. The free K–nodule of all $m \times n$ matrices over the algebra $\mathcal{A}$ can be made into an K-algebra with respect to the following operation ($\circ$): $B \circ C = BPC$. This algebra is called the *Munn $m \times n$ matrix algebra over $\mathcal{A}$ with sandwich matrix P* and is denoted by $\mathcal{M}(\mathcal{A}; m, n; P)$.

By analogy with the Rees semigroups, the matrix P could be called regular if each row and each column contains at least one invertible entry. But since after replacing the matrix P by an equivalent matrix $P' = XPY$ (with invertible X and Y) we obtain an isomorphic Munn algebra, it suffices to require that p_{11} is equal to the identity e of $\mathcal{A}$ and the remaining entries of the first row and first column are equal to zero. A sandwith matrix of such form is denoted by P_0. Just such a variant is considered below.

[1]Since most Munn algebras are not unital, in order to be consistent, we consider unital algebras also as factors of the free algebras in the class of all algebras (but not only unital). In this case the identity e of a unital algebra $\mathcal{B}$ can be included in a system of its generators B and then the natural relations for e (namely, $eb - e = 0, be - e = 0$ for any $b \in B$) are defining; more precisely, either themselves enter in a fixed R or are consequences of the rest.

[2]Of course, the presence of the identity e in some condition applies only to the case when the algebra $\mathcal{B}$ is unital.

1.3. Formulation of theorems

The following theorems define presentations of the Munn matrix algebras $\mathcal{M}(\mathcal{A}; m, n; P_0)$ by a presentation $\mathcal{A} = \langle X|R\rangle$ with X and R being finite or infinite. Note that the upper $\circ$ index in the notation of presentations of Munn algebras indicates the operation of multiplication. The cases when a system of generators of a (unital) algebra $\mathcal{A}$ is unital or non-unital are considered separately.

Theorem 1. *Let $\mathcal{A}$ be a K-algebra with identity e over a commutative ring K, and $\langle A|R\rangle$ be its a normal unital presentation. For the n and m, introduce the sets*

$$\Gamma = \{\gamma_2, \dots, \gamma_m\}, \quad \Lambda = \{\lambda_2, \dots, \lambda_n\}.$$

We have the following normal presentation $\langle M|\overline{R}\rangle^\circ$ of a Munn algebra $\mathcal{M} = \mathcal{M}(\mathcal{A}; m, n; P_0)$ with a regular $P_0 = (p_{ji})$: $M = A \cup \Gamma \cup \Lambda$ and $\overline{R}$ consists of the following relations:

(1) *the relations*[3] *from R;*
(2) $e \circ \gamma_i = 0, \gamma_i \circ e - \gamma_i = 0$ $(i = 2, \dots, m)$;
(3) $e \circ \lambda_j - \lambda_j = 0, \lambda_j \circ e = 0$ $(j = 2, \dots, n)$;
(4) $\gamma_i \circ \gamma_{i'} = 0$ $(i, i' = 2, \dots, m)$;
(5) $\lambda_j \circ \lambda_{j'} = 0$ $(j, j' = 2, \dots, n)$;
(6) $\lambda_j \circ \gamma_i - p_{ji} = 0$ $(i = 2, \dots, m;\ j = 2, \dots, n)$.

Theorem 2. *Let $\mathcal{A}$ be a K-algebra with identity e over a commutative ring K and $\langle A|R\rangle$ be its a normal non-unital presentation Then R contains at least one relation $f = 0$ with the polynomial f whose free term is not equal to zero. Fix such a polynomial denoting it by $f_0 - e$.*

For the n and m, introduce the sets

$$\Gamma = \{\gamma_2, \dots, \gamma_m\}, \quad \Lambda = \{\lambda_2, \dots, \lambda_n\}.$$

We have the following normal presentation $\langle M|\overline{R}\rangle^\circ$ of a Munn algebra $\mathcal{M} = \mathcal{M}(\mathcal{A}; m, n; P_0)$ with a regular $P_0 = (p_{ji})$: $M = A \cup \Gamma \cup \Lambda$ and $\overline{R}$ consists of the following relations:

(0) *the relations*[3] *from R;*
(1) $f_0 a = a f_0$ *for any* $a \in A$;
(2) $f_0 \circ \gamma_i = 0, \gamma_i \circ f_0 - \gamma_i = 0$ $(i = 2, \dots, m)$;
(3) $f_0 \circ \lambda_j - \lambda_j = 0, \lambda_j \circ f_0 = 0$ $(j = 2, \dots, n)$;

[3] This condition means that a homomorphic embedding of $\mathcal{A}$ into $\mathcal{M}$ will be indicated, but without taking into account the identity e, i.e. as non-unital algebras.

(4) $\gamma_i \circ \gamma_{i'} = 0$ $(i, i' = 2, \dots, m)$;
(5) $\lambda_j \circ \lambda_{j'} = 0$ $(j, j' = 2, \dots, n)$;
(6) $\lambda_j \circ \gamma_i - p_{ji} = 0$ $(i = 2, \dots, m;\ j = 2, \dots, n)$.

Note that the presentations constructed in both cases are not, as rule, unital (since, by [3, Lemma 5.18], the Munn algebra don't have an identity if the sandwich matrix is singular).

2. Proofs of Theorem 1 and 2

First we prove Theorem 1.

Indicate which elements of $\mathcal{M}$ correspond to generators, which are indicated in the condition of the theorem.

For $x \in \mathcal{A}$ denote by $(x)_{ij}$, where $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$, the matrix from $\mathcal{M}$ with its (i, j)th entry being x, and its remaining entries being 0; the expression $(0)_{ij}$ means the $m \times n$ zero matrix 0. Then, by the definition of $\circ$,

$$(b)_{ij} \circ (c)_{i'j'} = (bp_{ji'}c)_{ij'}$$

whence, due to the form of the matrix P_0,

$$(bc)_{ij} = (b)_{i1} \circ (c)_{1j}. \tag{$*$}$$

Due to $(*)$, the map i_0 given by the equality $i_0(x) = (x)_{11}$ with $x \in \mathcal{A}$ is a homomorphic embedding of $\mathcal{A}$ into $\mathcal{M}$ as non-unital algebras, and we identify the corresponding elements of both the algebras. Hence, in particular, the set of generators A is a subset of the set of generators M.

As for the rest of the generators from M, we take the following matrices (as elements of $\mathcal{M}$):

$$\gamma_i = (e)_{i1}, \ \ \lambda_j = (e)_{1j}$$

for $i \in \{2, \dots, m\}, j \in \{2, \dots, n\}$.

It is easy to check that conditions (2)-(6) of the theorem are satisfied. The fact that the indicated matrices $(a)_{11}, \gamma_i, \lambda_j$, where $a \in A, i = 2, \dots, m;\ j = 2, \dots, n$, form a system of generators of the algebra $\mathcal{M}$ is a consequence of the equalities

$$(a)_{i1} = \gamma_i \circ (a)_{11}, \quad (a)_{1j} = (a)_{11} \circ \lambda_j, \quad (a)_{ij} = (a)_{i1} \circ (e)_{1j},$$

which follow from $(*)$.

It remains to prove that the set of relations $\overline{R}$ defines the algebra $\mathcal{M}$. In other words, there is no a relation between generators from the set M

which would not follow from the relations of $\overline{R}$. Assume the contrary and let F be a non-commutative polynomial (over K) in variables from M such that $F = 0$ and it is not a consequence of the relations from $\overline{R}$.

To find out what kind of the polynomial F can be, we need the some lemmas. By γ and λ we denote γ_i for some i and λ_j for some j. Condition (k) of the theorem will be denoted by T_k.

Lemma 1. *$a \circ \gamma = 0$ and $\lambda \circ a = 0$ for any $a \in A$.*

Indeed, $a \circ \gamma = (a \circ e) \circ \gamma = a \circ (e \circ \gamma)$ and $\lambda \circ a = \lambda \circ (e \circ a) = (\lambda \circ e) \circ a$ and the lemma follows from equalities T_2 and T_3.

Lemma 2. *Let $\overline{A}$ and $\overline{M}$ denote the sets of all finite words from A and M (with respect to multiplication $\circ$). A non-zero word $x \in \overline{M} \setminus \overline{A}$ can only have one of the following types: $\gamma_i \circ \overline{a}$, $\overline{a} \circ \lambda_j$, $\gamma_i \circ \overline{a} \circ \lambda_j$, where $\overline{a} \in \overline{A}$.*

Proof. Let's consider connected subwords in x. Since $\gamma = \gamma \circ e$ and $\lambda = e \circ \lambda$ (see T_2 and T_3), we can assume that in x there is a subword belonging to $\overline{A}$. Let b denotes such a subword of the greatest length. By Lemma 1, only γ can stand to the left of b, and λ to the right. If such γ really stands, then it cannot be to the left of it neither a (by Lemma 1), not γ (see T_4), not λ (otherwise, by T_6 or $\lambda \circ \gamma = 0$ or the length of b will be increased). So γ is the left end of the word x. Similarly, it can be shown that if λ to the right of b really stands, then it is the right end of x. Lemma is proved. □

Thus, by Lemma 2, the equality $F = 0$ has the form

$$f_0 + \sum_{2 \leqslant i \leqslant m} \gamma_i \circ f_{1i} + \sum_{2 \leqslant j \leqslant n} f_{2j} \circ \lambda_j + \sum_{\substack{2 \leqslant i \leqslant m \\ 2 \leqslant j \leqslant n}} \gamma_i \circ f_{3ij} \circ \lambda_j = 0,$$

where $f_0, f_{1i}, f_{2j}, f_{3ij}$ are polynomials in variables $a \in A$.

Write all the terms of this equality in matrix form (that is, as elements of the Munn algebra), using the matrix form of the generators and the definition of the multiplication $\circ$:

$$f_0 = (f_0)_{11}, \quad \gamma_i \circ f_{1i} = (f_{1i})_{i1}, \quad f_{2j} \circ \lambda_j = (f_{2j})_{1j}, \quad \gamma_i \circ f_{3ij} \circ \lambda_j = (f_{3ij})_{ij}.$$

So, all the terms of the polynomial F stand in the corresponding matrix in different places and hence the equality $F = 0$ means that

$$f_0 = 0, \quad f_{1i} = 0, \quad f_{2j} = 0, \quad f_{3ij} = 0,$$
$$i = 1, \ldots, m, \quad j = 1, \ldots, n.$$

We got that the equality $F = 0$ is equivalent to the system of the equalities in variables from A which leads to contradiction, since these relations are consequences of the relations from R. Theorem 1 is proved.

Theorem 2 follows from Theorem 1 as follows: first we add to A the element e and write out the presentation of the Munn algebra, applying Theorem 1, and then exclude the generator e and replace it in the relations by the polynomial f_0.

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