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RESEARCH ARTICLE

# Deformations of compatible Hom-mock-Lie algebras

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Abstract. In this paper, we study Hom-mock-Lie algebras as a twisted version of mock-Lie algebras. Also we consider a pair of Hom-mock-Lie algebras structures satisfying that any linear combination of the two Hom-mock-Lie structures is still a Hom-mock-Lie structure called compatible Hom-mock-Lie algebras and exhibit some related results. Next, we introduce the notion of representation of a compatible Hom-mock-Lie algebra. Finally, we study linear deformations of a compatible Hom-mock-Lie algebra.

## Introduction

Mock-Lie algebras also known as Jacobi-Jordan algebras are commutative algebras satisfying the jacobi identity. This class of algebras was firstly introduced by D. Burde and A. Fialowski in 2014 under the name of Jacobi Jordan algebras [14]. Since then different names are used for them, reflecting the fact that they were considered from different view points by different communities. The motivation to study these algebraic structures is related to the fact that they are examples of the well referenced category of Jordan algebras which were introduced to study

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and explain some aspects of physics [25]. Observe that there is a close relationship between mock-Lie algebras and associative algebras; they can be produced from anti-associative algebras the same way as from associative algebras. By anti-associative algebra we mean algebra subject to the operation $(ab)c + a(bc) = 0$ for all $a, b, c$. Thanks to the approach developped by Eilenberg and nicely described in [24], representations of mock-Lie algebras are introduced in [42] where many results and conjectures are made. The notion of mock-Lie bialgebra was introduced in [11] as an equivalent to a Manin triple of mock-Lie algebras (see also [10]).

There have been quite much interest in (linearly) compatible products, in the sense that there are two products of a certain kind defined on the same vector space whose linear combinations are still of the same kind. For example, compatible associative algebras were studied in connection with Cartan matrices of affine Dynkin diagrams, integrable matrix equations, infinitesimal bialgebras and quiver representations [29–31] and the corresponding operad and free objects were obtained in [18]. Compatible Lie algebras were studied in [19–21, 32, 34] in the contexts of the classical Yang–Baxter equation and principal chiral field, loop algebras over Lie algebras and elliptic theta functions. In particular, there is the so-called Lenard–Magri scheme,[27, 33] which gives the construction of completely integrable Hamiltonian dynamical systems of compatible Poisson brackets whose Lie brackets are compatible [12, 13]. More general compatible products were defined in [38].

The theory of Hom-algebras was introduced by D. Larsson, S. D. Silvestrov with the introduction of Hom-Lie algebras [23]. They first arose when studying quasi-deformation of Lie algebras of vector fields. Since then, other type of Hom-algebras such that Hom-Malcev, Hom-Novikov, Hom-alternative were defined and discussed. Further development could be found in [3, 4, 9].

The famous examples are the Witt and the Virasoro algebras in which the derivations are replaced by $\sigma$-derivations, see also [2, 15]. Hom-algebras are also useful in mathematical physics. For example in [40, 41], we can see applications of the Yang Baxter Equations and in [7, 8, 39] applications to braid group representations. Hom-associative algebras were introduced by Makhlouf and Silvestrov to generalize the classical construction of Lie algebra from an associative algebra by taking commutators to Hom-Lie algebras see for instance [36, 37]. Representation theory of Hom-algebras is a subject that interests many authors because of its important role in investigating algebra properties. It was firstly defined in [28, 35]. See also [1, 26]. Hom-mock-Lie algebras is a kind of deformation of mock-Lie algebras via suitable morphisms. Many studies are done on this subject [5, 22].

The main purpose of this paper on the first hand is to develop the theory of Hom-mock-Lie algebras and their representations. On the other hand to introduce the notion of compatible Hom-moch-Lie algebras and their relations with compatible Hom-antiassociative algebras. We also define infinitesimal deformations of a compatible HM-Lie algebras and characterize equivalence classes of infinitesimal deformations in terms of zigzag cohomology.

This paper is organised as follows, in Section 1, we recall some basic results about Hom-mock-Lie algebras and Hom-anti-associativee algebras. Section 2 is devoted to introduce the notion of compatible HM-Lie algebra and study their representations. In Section 3, we study linear deformations of a compatible HM-Lie algebra and introduce Nijenhuis operators on compatible HM-Lie algebras that generate trivial linear deformations. Throughout this paper, all vector spaces are finite dimensional and over a filed $\mathbb{K}$ of characteristic zero.

## 1. Some basics on Hom-mock-Lie algebras and Hom-anti-associative algebras

In this section, we recall some basics concerning Hom mock-Lie algebras, Hom-anti-associative algebras and add some elementary new results. Our main references are [5, 14, 22].

**Definition 1.** A Hom-mock-Lie-algebra (denoted for short HM-Lie algebra) is a Hom-algebra $(\mathcal{J}, \star, \alpha)$ such that the following conditions are satisfied

$$x \star y = y \star x, \quad \text{(Commutativity condition)}, \tag{1}$$

$$\circlearrowleft_{x,y,z} \alpha(x) \star (y \star z) = 0, \quad \text{(Hom-Jacobi identity)} \tag{2}$$

for all $x, y, z \in \mathcal{J}$. Let $(\mathcal{J}, \star, \alpha)$ and $(\mathcal{J}', \star', \alpha')$ be two HM-Lie algebras. A map $f : \mathcal{J} \longrightarrow \mathcal{J}'$ is said to be a morphism of HM-Lie algebras if it is linear and satisfies:

$$f(x \star y) = f(x) \star' f(y), \quad f \circ \alpha = \alpha' \circ f.$$

**Remark 1.**

1. A HM-Lie algebra $(\mathcal{J}, \star, \alpha)$ is called a multiplicative (resp. regular) HM-Lie algebra if $\alpha$ is a morphism (resp. invertible). Throughout this paper all Hom-algebras are considered multiplicative.

2. A mock-Lie algebra (M-Lie algebra) is a HM-Lie algebra when $\alpha = id$.

**Example 1.** Let $(\mathcal{J}, \star)$ be a M-Lie algebra and $\alpha : \mathcal{J} \to \mathcal{J}$ a morphism. Then $(\mathcal{J}, \star_\alpha := \alpha \circ \star, \alpha)$ is a HM-Lie algebra.

**Definition 2.** Let $(\mathcal{J}, \star, \alpha)$ be a HM-Lie algebra.

- A subspace $\mathcal{J}_1$ of $\mathcal{J}$ is a subalgebra of $\mathcal{J}$ if $\alpha(\mathcal{J}_1) \subseteq \mathcal{J}$, $\mathcal{J}_1 \star \mathcal{J}_1 \subseteq \mathcal{J}_1$.

- A subalgebra $\mathcal{J}_1$ of $\mathcal{J}$ is an ideal of $\mathcal{J}$ if $\mathcal{J}_1 \star \mathcal{J} \subseteq \mathcal{J}_1$.

- A HM-Lie algebra is said to be simple if it has no non trivial ideals.

The following result is given in [5].

**Proposition 1.** *Any HM-Lie algebra is a Hom-Jordan algebra.*

**Definition 3.** A HM-Lie algebra $(\mathcal{J}, \star, \alpha)$ is called of mock-Lie type if there exists a mock-Lie structure $\diamond$ on $\mathcal{J}$ such that $\star = \alpha \circ \diamond$. The pair $(\mathcal{J}, \diamond)$ is called the induced M-Lie algebra.

**Proposition 2.** *Suppose $(\mathcal{J}, \star, \alpha)$ be a regular HM-Lie algebra. Then $(\mathcal{J}, \star, \alpha)$ is a HM-Lie algebra of mock-Lie type. The induced M-Lie algebra is $(\mathcal{J}, \diamond := \alpha^{-1} \circ \star)$.*

*Proof.* Straightforward. □

**Proposition 3.** *Let $(\mathcal{J}_1, \star_1, \alpha_1)$ and $(\mathcal{J}_2, \star_2, \alpha_2)$ be two regular HM-Lie algebras. Then $(\mathcal{J}_1, \star_1, \alpha_1)$ and $(\mathcal{J}_2, \star_2, \alpha_2)$ are isomorphic if and only if their induced M-Lie algebras $(\mathcal{J}_1, \diamond_1)$ and $(\mathcal{J}_2, \diamond_2)$ are isomorphic by a map $\varphi : (\mathcal{J}_1, \diamond_1) \longrightarrow (\mathcal{J}_2, \diamond_2)$ satisfying $\varphi \circ \alpha_1 = \alpha_2 \circ \varphi$.*

*Proof.* "⇒" Let $\varphi : (\mathcal{J}_1, \star_1, \alpha_1) \longrightarrow (\mathcal{J}_2, \star_2, \alpha_2)$ be an isomorphism HM-Lie algebras. Since we have $\varphi \circ \alpha_1 = \alpha_2 \circ \varphi$ and $\varphi(x \star_1 y) = \varphi(x) \star_2 \varphi(y)$. Then, for any $x, y \in \mathcal{J}_1$, we get

$$\begin{aligned}\varphi(x \diamond_1 y) = \varphi \circ \alpha_1^{-1}(x \star_1 y) &= (\alpha_2)^{-1}(\varphi(x \star_1 y)) \\ &= (\alpha_2)^{-1}(\varphi(x) \star_2 \varphi(y)) = \varphi(x) \diamond_2 \varphi(y).\end{aligned}$$

Hence, $\varphi(x \diamond_1 y) = \varphi(x) \diamond_2 \varphi(y)$ and $\varphi$ is an isomorphism of M-Lie algebras satisfying $\varphi \circ \alpha_1 = \alpha_2 \circ \varphi$.
"$\Leftarrow$" If $\varphi : \mathcal{J}_1 \longrightarrow \mathcal{J}_2$ an isomorphism of M-Lie algebras satisfying $\varphi \circ \alpha_1 = \alpha_2 \circ \varphi$ and $\varphi(x \diamond_1 y) = \varphi(x) \diamond_2 \varphi(y)$. Then, for any $x, y \in \mathcal{J}_1$, we have

$$\begin{aligned}\varphi(x \star_1 y) = \varphi(\alpha_1(x \diamond_1 y)) &= \alpha_2 \circ \varphi(x \diamond_1 y)\\ &= \alpha_2(\varphi(x) \diamond_2 \varphi(y)) = \varphi(x) \star_2 \varphi(y).\end{aligned}$$

Hence, $\varphi$ is an isomorphism of HM-Lie algebra. □

**Definition 4.** A Hom-anti-associative algebra is a triple $(A, \cdot, \alpha)$ consisting of a vector space $A$, a binary product "$\cdot$" and a linear map $\alpha : A \to A$ satisfying

$$(x \cdot y) \cdot \alpha(z) + \alpha(x) \cdot (y \cdot z) = 0 \quad \forall x, y, z \in A$$

A Hom-anti-associative algebra $(A, \cdot, \alpha)$ is called regular if $\alpha$ is invertible.

**Definition 5.** A Hom-anti-associative algebra $(A, \cdot, \alpha)$ is called of anti-associative-type if there exists an anti-associative structure $*$ on $A$ such that $\cdot = \alpha \circ *$ and $(A, *)$ is called the induced anti-associative algebra.

**Proposition 4.** *Let $(A, \cdot, \alpha)$ be a regular Hom-anti-associative algebra. Then $(A, \cdot, \alpha)$ is of anti-associative-type and the induced anti-associative algebra is given by $(A, *)$, where,*

$$x * y = \alpha^{-1}(x \cdot y) \quad \forall x, y \in A.$$

**Proposition 5.** *Let $(A, \cdot, \alpha)$ be a Hom-anti-associative algebra. Then the triple $(A, \star, \alpha)$ is a HM-Lie algebra, where*

$$x \star y = x \cdot y + y \cdot x \quad \forall x, y \in A.$$

**Corollary 1.** Let $(A, \cdot, \alpha)$ be a Hom-anti-associative algebra of anti-associative-type. Then, $(A, \star, \alpha)$ given as above Proposition is a HM-Lie algebra of mock-Lie-type.

*Proof.* Since $(A, \cdot, \alpha)$ is of anti-associative-type, then, there is an anti-associative structure $*$ on $A$ defined by $\alpha(x * y) = x \cdot y, \ \forall x, y \in A$. Consider $(A, \star, \alpha)$ the HM-Lie algebra associate to the Hom-anti-associative algebra $(A, \cdot, \alpha)$ and $(A, \diamond)$ the mock-Lie algebra associate to the anti-associative $(A, *)$ given in Proposition 5. Then we have for any $x, y \in A$,

$$\begin{aligned} x \star y &= x \cdot y + y \cdot x \\ &= \alpha(x * y) + \alpha(y * x) \\ &= \alpha(x * y + y * x) \\ &= \alpha(x \diamond y). \end{aligned}$$

Therefore, $(A, \star, \alpha)$ is of mock-Lie type. □

**Proposition 6.** *Let $(\mathcal{J}, \star, \alpha_{\mathcal{J}})$ be a HM-Lie algebra and $(A, \circ, \alpha_A)$ be commutative Hom-associative algebra. Then $(\mathcal{J} \otimes A, \diamond, \alpha_{\mathcal{J}\otimes A})$ is a HM-Lie algebra, where*

$$\begin{aligned} (x \otimes a) \diamond (y \otimes b) &= (x \star y) \otimes (a \circ b), \\ \alpha_{\mathcal{J}\otimes A} &= \alpha_{\mathcal{J}} \otimes \alpha_A. \end{aligned}$$

*Proof.* Since $(\mathcal{J}, \star, \alpha_{\mathcal{J}})$ a HM-Lie and $(A, \circ, \alpha_A)$ a commutative Hom-associative algebra. Then $\diamond$ is commutative. For the Hom-Jacobi identity we get

$$\begin{aligned} &\alpha_{\mathcal{J}\otimes A}(x \otimes a) \diamond ((y \otimes b) \diamond (z \otimes c)) + \alpha_{\mathcal{J}\otimes A}(y \otimes b) \diamond ((z \otimes c) \diamond (x \otimes a)) \\ &\qquad + \alpha_{\mathcal{J}\otimes A}(z \otimes c) \diamond ((x \otimes a) \diamond (y \otimes b)) \\ &= (\alpha_{\mathcal{J}}(x) \otimes \alpha_A(a)) \diamond ((y \star z) \otimes (b \circ c)) + (\alpha_{\mathcal{J}}(y) \otimes \alpha_A(b)) \diamond ((z \star x) \otimes (c \circ a)) \\ &\qquad + (\alpha_{\mathcal{J}}(z) \otimes \alpha_A(c)) \diamond ((x \star y) \otimes (a \circ b)) \\ &= \alpha_{\mathcal{J}}(x) \star (y \star z) \otimes \alpha_A(a) \circ (b \circ c) + \alpha_{\mathcal{J}}(y) \star (z \star x) \otimes \alpha_A(b) \circ (c \circ a) \\ &\qquad + \alpha_{\mathcal{J}}(z) \star (x \star y) \otimes \alpha_A(c) \circ (a \circ b) \\ &= \alpha_{\mathcal{J}}(x) \star (y \star z) \otimes \alpha_A(a) \circ (b \circ c) + \alpha_{\mathcal{J}}(y) \star (z \star x) \otimes \alpha_A(a) \circ (b \circ c) \\ &\qquad + \alpha_{\mathcal{J}}(z) \star (x \star y) \otimes \alpha_A(a) \circ (b \circ c) \\ &= (\alpha_{\mathcal{J}}(x) \star (y \star z) + \alpha_{\mathcal{J}}(y) \star (z \star x) + \alpha_{\mathcal{J}}(z) \star (x \star y)) \otimes \alpha_A(a) \circ (b \circ c) = 0. \end{aligned}$$

Thus, $(\mathcal{J} \otimes A, \diamond, \alpha_{\mathcal{J}\otimes A})$ is a HM-Lie algebra. □

**Definition 6.** A representation of a HM-Lie algebra $(\mathcal{J}, \star, \alpha)$ is a triple $(V, \rho, \phi)$ where $V$ is a vector space, $\phi \in gl(V)$ and $\rho : \mathcal{J} \longrightarrow gl(V)$ is a linear map such that the following equalities hold, for all $x, y \in \mathcal{J}$,

$$\phi \circ \rho(x) = \rho(\alpha(x)) \circ \phi, \tag{3}$$

$$\rho(x \star y) \circ \phi = -\rho(\alpha(x))\rho(y) - \rho(\alpha(y))\rho(x). \tag{4}$$

**Remark 2.** Note that we recover the class of representation of M-Lie algebra if we take $\alpha = id_{\mathcal{J}}$ and $\phi = id_V$

**Example 2.** Let $(\mathcal{J}, \star, \alpha)$ be a HM-Lie algebra. Define the map $\mathrm{ad}(x)(y) = x \star y$, then $(\mathcal{J}, \mathrm{ad}, \alpha)$ is a representation of $\mathcal{J}$ which is called the adjoint representation.

**Example 3.** Let $(\mathcal{J}_1, \star_1, \alpha_1)$ and $(\mathcal{J}_2, \star_2, \alpha_2)$ be two HM-Lie algebras and $f : \mathcal{J}_1 \longrightarrow \mathcal{J}_2$ a morphism of HM-Lie algebras. Then $(\mathcal{J}_2, \rho, \alpha_2)$ is a representation of $\mathcal{J}_1$ where $\rho(x)y = f(x) \star_2 y$, for all $(x, y) \in \mathcal{J}_1 \times \mathcal{J}_2$.

**Proposition 7.** *The triple $(V, \rho, \phi)$ is a representation of the HM-Lie algebra $(\mathcal{J}, \star, \alpha)$ if and only if the direct sum $\mathcal{J} \oplus V$ of vector spaces is equipped with a Hom-mock-Lie structure given by*

$$(x + u) \star_{\mathcal{J}\oplus V} (y + v) = x \cdot y + \rho(x)v + \rho(y)u,$$
$$(\alpha + \phi)(x + u) = \alpha(x) + \phi(u)$$

*for all $x, y \in \mathcal{J}, v \in V$. We denote such a HM-Lie algebra by $\mathcal{J} \ltimes_\rho V$.*

Now, we construct the dual representation of an old representation of a HM-Lie algebra without any additional condition. Let $(V, \rho, \phi)$ be a representation of a regular HM-Lie algebra $(\mathcal{J}, \star, \alpha)$ such that $\phi$ is invertible. Define, $\rho^* : \mathcal{J} \longrightarrow gl(V^*)$ by

$$\langle \rho^*(x)\xi, u\rangle = \langle \xi, \rho(x)u\rangle \quad \forall x \in \mathcal{J}, u \in V, \xi \in V^*.$$

Observe that in general $(V^*, \rho^*, \phi^*)$ is not a representation. In [22], the authors need to add strong conditions to obtain a representation on the dual space. To get rid of it, define

$$\rho^\star(x)\xi := \rho^*(\alpha(x))((\phi)^{-2})^*(\xi) \quad \forall x \in \mathcal{J}, \xi \in V^*.$$

**Proposition 8.** *Under the above notations, $\rho^\star$ is a representation of $\mathcal{J}$ on $V^*$ with respect to $(\phi^{-1})^*$.*

**Proposition 9.** *Let $(\mathcal{J}, \star)$ be a M-Lie algebra and $(V, \rho)$ be a representation. Let $\alpha : \mathcal{J} \to \mathcal{J}$ be a morphism of M-Lie algebras and $\phi \in gl(V)$. Suppose that $\phi \circ \rho(x) = \rho(\alpha(x)) \circ \phi$, then $(V, \tilde{\rho}, \phi)$ is a representation of $(\mathcal{J}, \star_\alpha, \alpha)$, where*

$$\tilde{\rho}(x)v = \rho(\alpha(x))\phi(v) \quad \forall x \in \mathcal{J}, v \in V.$$

**Definition 7.** Let $(\mathcal{J}, \star, \alpha)$ be a HM-Lie algebra of mock-Lie-type with $(\mathcal{J}, \diamond)$ the induced M-Lie algebra. A representation $(V, \rho, \phi)$ of $\mathcal{J}$ is called of mock-Lie-type if there exists $\rho'$ a representation of the induced M-Lie algebra $(\mathcal{J}, \diamond)$ such that $\rho = \phi \circ \rho'$.

**Proposition 10.** *Let $\mathcal{J} = (\mathcal{J}, \star, \alpha)$ be a HM-Lie algebra of mock-Lie type and $(V, \rho, \phi)$ be a representation such that $\phi$ is invertible. Then, $(V, \rho' := \phi^{-1}\rho)$ is a representation of mock-Lie type.*

*Proof.* By definition of representation of HM-Lie algebra we have

$$\rho(\alpha(x)) \circ \phi = \phi \circ \rho(x)$$

which is equivalent to

$$\rho(\alpha(x)) = \phi\rho(x)\phi^{-1},$$

then for $x, y \in \mathcal{J}$, we get

$$\begin{aligned} &\rho'(x \diamond y) + \rho'(x)\rho'(y) + \rho'(y)\rho'(x) \\ &= \phi^{-1}\rho(x \diamond y) + \phi^{-1}\rho(x)\phi^{-1}\rho(y) + \phi^{-1}\rho(y)\phi^{-1}\rho(x) \\ &= \phi^{-2}\phi\rho(x \diamond y) + \phi^{-2}\phi\rho(x)\phi^{-1}\rho(y) + \phi^{-2}\phi\rho(y)\phi^{-1}\rho(x) \\ &= \phi^{-2}\big(\rho(x \star y)\phi + \rho(\alpha(x))\rho(y) + \rho(\alpha(y))\rho(x)\big) = 0. \end{aligned}$$

Hence, $\rho'$ is a representation of M-Lie algebra $(\mathcal{J}, \diamond)$. □

**Definition 8.** Two representations $(V_1, \rho_1, \phi_1)$ and $(V_2, \rho_2, \phi_2)$ of a HM-Lie algebra are said to be equivalent if there exists an isomorphism $f : V_1 \longrightarrow V_2$ satisfying;

$$f \circ \phi_1 = \phi_2 \circ f, \tag{5}$$

$$f \circ \rho_1(x) = \rho_2(x) \circ f \quad \forall x \in \mathcal{J}. \tag{6}$$

**Proposition 11.** *Let $\mathcal{J}_\alpha = (\mathcal{J}, \star, \alpha)$ be a regular HM-Lie algebra of mock-Lie-type with the induced M-Lie algebra $(\mathcal{J}, \diamond)$. Two representation $(V_1, \rho_1, \phi_1)$ and $(V_2, \rho_2, \phi_2)$ of $\mathcal{J}_\alpha$ are isomorphic if and only if the induced representations $(V_1, \rho_1^{'})$ and $(V_2, \rho_2^{'})$ are isomorphic by $f$ such that $f \circ \phi_1 = \phi_2 \circ f$.*

*Proof.* " $\Rightarrow$ " Let $(V_1, \rho_1, \phi_1)$ and $(V_2, \rho_2, \phi_2)$ be two isomorphic representations of a HM-Lie algebra $\mathcal{J}_\alpha = (\mathcal{J}, \star_\alpha, \alpha)$. Then, there exists an isomorphism $f : V_1 \longrightarrow V_2$ such that $\rho_2(x) \circ f = f \circ \rho_1(x) \quad \forall x \in \mathcal{J}$ and $f \circ \phi_1 = \phi_2 \circ f$.
Since $\rho_1$ and $\rho_2$ are of mock-Lie-type, then $\rho_1' = \phi_1^{-1}\rho_1,$ and $\rho_2' = \phi_2^{-1}\rho_2$ are the induced representations of $\rho_1$ and $\rho_2$.
We have $\phi_1 \circ \rho_2'(x) \circ f = f \circ \phi_1 \circ \rho_1'(x)$, so $\rho_2'(\alpha(x)) \circ f = f \circ \rho_1'(\alpha(x))$. Since $\alpha$ is invertible, then $\rho_2' \circ f = f \circ \rho_1'$. Hence the induced representations

are isomorphic.
" $\Leftarrow$ " Suppose that the induced representations are isomorphic by an isomorphism $f$ that satisfies $\phi_2 \circ f = f \circ \phi_1$.
We have $\rho_2'(x) \circ f = f \circ \rho_1'(x)$. Then, $\phi_2^{-1} \circ \rho_2(x) \circ f = f \circ \phi_1^{-1} \circ \rho_1(x)$. Applying $\phi_2$ we get $\rho_2 \circ f = \phi_2 \circ f \circ \phi_1 \circ \rho_1$, use the fact that $\phi_2^{-1} \circ f = f \circ \phi_1^{-1}$. Then, $\rho_2 \circ f = f \circ \rho_1$. Hence, $\rho_1$ and $\rho_2$ are isomorphic. □

**Definition 9.** Let $(A, \cdot, \alpha)$ be a Hom-anti-associative algebra and $(V, \phi)$ be a Hom-module. Consider two linear maps $l, r : A \to End(V)$. The quadruple $(V, l, r, \phi)$ is called a bimodule of $A$ if

$$\phi l(x) = l(\alpha(x))\phi, \quad \phi r(x) = r(\alpha(x))\phi, \tag{7}$$
$$l(x \cdot y)\phi + l(\alpha(x))l(y) = 0, \tag{8}$$
$$r(x \cdot y)\phi + r(\alpha(y))r(x) = 0, \tag{9}$$
$$l(\alpha(x))r(y)v + r((\alpha(y))l(x) = 0, \tag{10}$$

for all $x, y \in A$.

**Proposition 12.** *The tuple $(V, l, r, \phi)$ is a bimodule of a Hom-anti-associative algebra $(A, \cdot, \alpha)$ if and only if the direct sum $A \oplus V$ of vectors spaces is turned into a Hom-anti-associative algebra by*

$$(x + u) \diamond (y + v) = x \cdot y + l(x)v + r(y)u,$$
$$(\alpha + \phi)(x + u) = \alpha(x) + \phi(u),$$

*for all $x, y \in A, v \in V$. We denote such a Hom-anti-associative algebra $A \ltimes_{l,r} V$.*

**Example 4.** Let $(A, \cdot, \alpha)$ be a Hom-anti-associative algebra. Let $L.$ and $R.$ denote the left and right multiplication operators, respectively, that is, $L.(x)(y) = R.(y)(x) = x \cdot y$, for any $x, y \in A$. Then, $(L., 0, \alpha)$, $(0, R., \alpha)$ and $(L., R., \alpha)$ are bimodules of $(A, \cdot, \alpha)$.

The following result is straightforward.

**Proposition 13.** *Let $(V, l, r, \phi)$ be a bimodule of a Hom-anti-associative algebra $(A, \cdot, \alpha)$. Then, $(V, l + r, \phi)$ is a representation of the subadjacent HM-Lie algebra $(A, \star, \alpha)$.*

## 2. Compatible Hom-mock-Lie algebras

In this section, we introduce compatible HM-Lie algebras analogously to compatible Hom-associative algebras and compatible Hom-Lie algebras introduced in [16, 17]. We end this section by defining representations of compatible HM-Lie algebras.

### 2.1. Compatible Hom-anti-associative algebras

In this section, we first introduce compatible Hom-anti-associative algebras and then define compatible bimodules over them.

**Definition 10.** A compatible Hom-anti-associative algebra is a tuple $(A, \mu_1, \mu_2, \alpha)$ in which $(A, \mu_1, \alpha)$ and $(A, \mu_2, \alpha)$ are both Hom-anti-associative algebras satisfying the following compatibility

$$(a \cdot_1 b) \cdot_2 \alpha(c) + (a \cdot_2 b) \cdot_1 \alpha(c) + \alpha(a) \cdot_1 (b \cdot_2 c) + \alpha(a) \cdot_2 (b \cdot_1 c) = 0,$$

for $a, b, c \in A$. Here $\cdot_1$ and $\cdot_2$ are used for the multiplications $\mu_1$ and $\mu_2$, respectively.

We may denote a compatible Hom-anti-associative algebra as above by $(A, \mu_1, \mu_2, \alpha)$ or simply by $A$, and say that $(\mu_1, \mu_2)$ is a compatible Hom-anti-associative algebra structure on $A$.

**Remark 3.** It follows from the above definition that $(A, \mu_1 + \mu_2, \alpha)$ is a Hom-anti-associative algebra. In fact, one can show that $(A, k\mu_1 + l\mu_2, \alpha)$ is a Hom-anti-associative algebra, for any $k, l \in \mathbb{K}$.

Let $A = (A, \mu_1, \mu_2, \alpha)$ and $A' = (A', \mu_1', \mu_2', \alpha')$ be two compatible Hom-anti-associative algebras. A morphism of compatible Hom-anti-associative algebras from $A$ to $A'$ is a linear map $\phi : A \to A'$ satisfying $\phi \circ \alpha = \alpha' \circ \phi$, $\phi \circ \mu_1 = \mu_1' \circ (\phi \otimes \phi)$ and $\phi \circ \mu_2 = \mu_2' \circ (\phi \otimes \phi)$.

**Definition 11.** A compatible Hom-anti-associative algebra $(A, \mu_1, \mu_2, \alpha)$ is called of compatible anti-associative type if $(A, \mu_1, \alpha)$ and $(A, \mu_2, \alpha)$ are both Hom-anti-associative algebras of anti-associative type with induced anti-associative algebras $(A, *_1), (A, *_2)$ and $(A, *_1, *_2)$ is called the induced compatible anti-associative-algebra.

**Proposition 14.** *Any regular compatible Hom-anti-associative algebra is of compatible anti-associative type.*

**Example 5.** Let $(A, \cdot_1, \cdot_2)$ be a compatible anti-associative algebra and $\alpha : A \to A$ be a compatible Hom-anti-associative algebra morphism. Then $(A, \alpha \circ \cdot_1, \alpha \circ \cdot_2, \alpha)$ is a compatible Hom-anti-associative algebra.

**Example 6.** Let $(A, \cdot_1, \cdot_2, \alpha)$ be a compatible Hom-anti-associative algebra. Then, for each $n \geq 0$, the quadruple $(A, \alpha^n \circ \cdot_1, \alpha^n \circ \cdot_2, \alpha^{n+1})$ is a compatible Hom-anti-associative algebra. This is the $n$-th derived compatible Hom-anti-associative algebra.

**Example 7.** Let $(A, \mu, \alpha)$ be an Hom-anti-associative algebra. A Nijenhuis operator on $A$ is a linear map $N : A \to A$ satisfying

$$N \circ \alpha = \alpha \circ N,$$
$$N(a) \cdot N(b) = N\big(N(a) \cdot b + a \cdot N(b) - N(a \cdot b)\big), \text{ for } a, b \in A.$$

A Nijenhuis operator $N$ induces a new Hom-anti-associative multiplication on $A$, denoted by $\mu_N : A \otimes A \to A, (a, b) \mapsto a \cdot_N b$ and it is defined by

$$a \cdot_N b := N(a) \cdot b + a \cdot N(b) - N(a \cdot b), \quad \text{for } a, b \in A.$$

Then it is easy to see that $(A, \mu, \mu_N, \alpha)$ is a compatible Hom-anti-associative algebra. Indeed, for $a, b, c \in A$, we have,

$$\begin{aligned}
&(a \cdot b) \cdot_N \alpha(c) + (a \cdot_N b) \cdot \alpha(c) + \alpha(a) \cdot (b \cdot_N c) + \alpha(a) \cdot_N (b \cdot c) \\
&= N(a \cdot b) \cdot (\alpha(c)) + (a \cdot b) \cdot \alpha(N(c)) - N((a \cdot b) \cdot \alpha(c)) \\
&\quad + (N(a) \cdot b) \cdot \alpha(c) + (a \cdot N(b)) \cdot \alpha(c) - N(a \cdot b) \cdot \alpha(c) \\
&\quad + \alpha(a) \cdot (N(b) \cdot c) + \alpha(a) \cdot (b \cdot N(c)) - \alpha(a) \cdot N(b \cdot c) \\
&\quad + \alpha(N(a)) \cdot (b \cdot c) + \alpha(a) \cdot N(b \cdot c) - N(\alpha(a) \cdot (b \cdot c)) \\
&= (N(a \cdot b) \cdot (\alpha(c)) - N(a \cdot b) \cdot (\alpha(c))) + (\alpha(a) \cdot N(b \cdot c) \\
&\quad - \alpha(a) \cdot N(b \cdot c)) + ((a \cdot b) \cdot \alpha(N(c)) + \alpha(a) \cdot (b \cdot N(c))) \\
&\quad - (N((a \cdot b) \cdot \alpha(c)) + N(\alpha(a) \cdot (b \cdot c))) + ((N(a) \cdot b) \cdot \alpha(c) \\
&\quad + \alpha(N(a)) \cdot (b \cdot c)) + ((a \cdot N(b)) \cdot \alpha(c) + \alpha(a) \cdot (N(b) \cdot c)) \\
&= 0.
\end{aligned}$$

The next class of examples comes from compatible Rota-Baxter operators on Hom-anti-associative algebras.

**Definition 12.** Let $(A, \mu, \alpha)$ be a Hom-anti-associative algebra. A Rota-Baxter operator on $A$ is a linear map $R : A \to A$ satisfying

$$R \circ \alpha = \alpha \circ R,$$
$$R(a) \cdot R(b) = R\big(R(a) \cdot b + a \cdot R(b)\big) \quad \forall a, b \in A.$$

A Rota-Baxter operator $R$ induces a new Hom-anti-associative multiplication $\mu_R : A \otimes A \to A,\ (a, b) \mapsto a \cdot_R b$ on the underlying vector space $A$ with respect $\alpha$ and it is given by

$$a \cdot_R b = R(a) \cdot b + a \cdot R(b) \quad \forall a, b \in A.$$

**Definition 13.** Two Rota-Baxter operators $R, S : A \to A$ on a Hom-anti-associative algebra $A$ are said to be compatible if for any $k, l \in \mathbb{K}$, the sum $kR + lS : A \to A$ is a Rota-Baxter operator on $A$. Equivalently,

$$\begin{aligned} &R(a) \cdot S(b) + S(a) \cdot R(b) \\ &= R\big(S(a) \cdot b + a \cdot S(b)\big) + S\big(R(a) \cdot b + a \cdot R(b)\big) \quad \forall a, b \in A. \end{aligned}$$

Then, we can easily check

**Proposition 15.** *Let $R, S : A \to A$ be two compatible Rota-Baxter operators on $A$. Then $(A, \cdot_R, \cdot_S, \alpha)$ is a compatible Hom-anti-associative algebra.*

**Definition 14.** Let $(A, \mu_1, \mu_2, \alpha)$ be a compatible Hom-anti-associative algebra. A compatible $A$-bimodule consists of a tuple $(V, l_1, r_1, l_2, r_2, \phi)$ in which $V$ is a vector space, $\phi \in End(V)$ and

$$\begin{cases} l_1 : A \otimes V \to V,\ (a, v) \mapsto a \cdot_1 v, \\ r_1 : V \otimes A \to V,\ (v, a) \mapsto v \cdot_1 a, \end{cases}$$

$$\begin{cases} l_2 : A \otimes V \to V,\ (a, v) \mapsto a \cdot_2 v, \\ r_2 : V \otimes A \to V,\ (v, a) \mapsto v \cdot_2 a, \end{cases}$$

are linear maps such that

1. $(V, l_1, r_1, \phi)$ is a bimodule over $(A, \mu_1, \alpha)$.
2. $(V, l_2, r_2, \phi)$ is a bimodule over $(A, \mu_2, \alpha)$.
3. The following compatibilities hold: for any $a, b \in A$ and $v \in V$,

$$(a \cdot_1 b) \cdot_2 \phi(v) + (a \cdot_2 b) \cdot_1 \phi(v) + \alpha(a) \cdot_1 (b \cdot_2 v) + \alpha(a) \cdot_2 (b \cdot_1 v) = 0, \quad (11)$$

$$(a \cdot_1 v) \cdot_2 \alpha(b) + (a \cdot_2 v) \cdot_1 \alpha(b) + \alpha(a) \cdot_1 (v \cdot_2 b) + \alpha(a) \cdot_2 (v \cdot_1 b) = 0, \quad (12)$$

$$(v \cdot_1 a) \cdot_2 \alpha(b) + (v \cdot_2 a) \cdot_1 \alpha(b) + \phi(v) \cdot_1 (a \cdot_2 b) + \phi(v) \cdot_2 (a \cdot_1 b) = 0. \quad (13)$$

A compatible $A$-bimodule as above may be simply denoted by $V$ when no confusion arises.

**Example 8.** Any compatible Hom-anti-associative algebra $A=(A,\mu_1,\mu_2,\alpha)$ is a compatible $A$-bimodule in which $l_1=r_1=\mu_1$, $l_2=r_2=\mu_2$ and $\phi=\alpha$.

**Example 9.** Let $(A,\mu_1,\mu_2,\alpha)$ be a compatible Hom-anti-associative algebra and let $(A,L_1,0,\alpha),(A,0,R_1,\alpha)$ be bimodules of $(A,\mu_1,\alpha)$ and $(A,L_2,0,\alpha),(A,0,R_2,\alpha)$ be modules of $(A,\mu_2,\alpha)$. Then $(A,L_1,0,L_2,0,\alpha)$ and $(A,0,R_1,0,R_2,\alpha)$ are bimodules of $(A,\mu_1,\mu_2,\alpha)$.

**Remark 4.** Let $A=(A,\mu_1,\mu_2,\alpha)$ be a compatible Hom-anti-associative algebra and $(V,l_1,r_1,l_2,r_2,\phi)$ be a compatible $A$-bimodule. Then it is easy to see that $(V,l_1+l_2,r_1+r_2,\phi)$ is a bimodule over the Hom-anti-associative algebra $(A,\mu_1+\mu_2,\alpha)$.

The following result generalizes the semidirect product for Hom-anti-associative algebras.

**Proposition 16.** *Let $(A,\cdot_1,\cdot_2,\alpha)$ be a compatible Hom-anti-associative algebra and $V$ be a compatible $A$-bimodule. Then the direct sum $A\oplus V$ carries a compatible Hom-anti-associative algebra structure given by*

$$\begin{aligned}(a,u)\cdot_1(b,v)&=(a\cdot_1 b,\ a\cdot_1 v+u\cdot_1 b),\\ (a,u)\cdot_2(b,v)&=(a\cdot_2 b,\ a\cdot_2 v+u\cdot_2 b),\\ (\alpha\oplus\phi)(a+u)&=\alpha(a)+\phi(u),\end{aligned}$$

*for $(a,u),(b,v)\in A\oplus V$. This is called the semidirect product.*

## 2.2. Definitions and constructions

**Definition 15.** Two HM-Lie algebras $(\mathcal{J},\star_1,\alpha)$ and $(\mathcal{J},\star_2,\alpha)$ are said to be compatible if for all $k_1,k_2\in\mathbb{K}$, the tuple $(\mathcal{J},k_1\star_1+k_2\star_2,\alpha)$ is a HM-Lie algebra.

The compatibility condition can be rewritten as

$$\begin{aligned}(x\star_1 y)\star_2\alpha(z)+(y\star_1 z)\star_2\alpha(x)+(z\star_1 x)\star_2\alpha(y)\\ +(x\star_2 y)\star_1\alpha(z)+(y\star_2 z)\star_1\alpha(x)+(z\star_2 x)\star_1\alpha(y)=0,\end{aligned}\tag{14}$$

for any $x,y,z\in\mathcal{J}$. The tuple $(\mathcal{J},\star_1,\star_2,\alpha)$ is called a compatible HM-Lie algebra.

**Remark 5.**

1. If $\alpha = \mathrm{id}_{\mathcal{J}}$, we recover the classic compatible M-Lie algebras.
2. A compatible HM-Lie algera $(\mathcal{J}, \star_1, \star_2, \alpha)$ is said to be regular if $\alpha$ is invertible.

Let $(\mathcal{J}, \star_1, \star_2, \alpha)$ and $(\mathcal{J}', \star_1{}', \star_2{}', \alpha')$ be two compatible HM-Lie algebras. A morphism between them is a linear map $f : \mathcal{J} \to \mathcal{J}'$ which is a HM-Lie algebra morphism from $(\mathcal{J}, \star_1, \alpha)$ to $(\mathcal{J}', \star_1', \alpha')$, and a HM-Lie algebra morphism from $(\mathcal{J}, \star_2, \alpha)$ to $(\mathcal{J}', \star_2', \alpha')$.

**Example 10.** Let $(\mathcal{J}, \star_1, \star_2)$ be a compatible M-Lie algebra and $\alpha : \mathcal{J} \to \mathcal{J}$ be a compatible M-Lie algebra morphism. Then the quadruple $(\mathcal{J}, \alpha \circ \star_1, \alpha \circ \star_2, \alpha)$ is a compatible HM-Lie algebra.

**Example 11.** Consider a 3-dimensional vector space $\mathcal{J}$ generated by the basis $\{e_1, e_2, e_3\}$ and define the following multiplications:

$$e_1 \bullet e_1 = e_2, \quad e_1 \star e_1 = e_2, e_3 \star e_3 = e_2.$$

By a straightforward computation, we check that $(\mathcal{J}, \bullet, \star)$ is a compatible M-Lie algebra. Define the linear map $\alpha : \mathcal{J} \to \mathcal{J}$ with respect to the basis $\{e_1, e_2, e_3\}$ by

$$\alpha(e_1) = ae_1, \quad \alpha(e_2) = a^2 e_2, \quad \alpha(e_3) = -ae_3$$

where $a \neq 0$ is a parameter. It is easy to show that $\alpha$ is a morphism of compatible M-Lie algebras. According to the twist method we can show that $(\mathcal{J}, \bullet_\alpha, \star_\alpha, \alpha)$ is a compatible HM-Lie algebra where the multiplications $\bullet_\alpha$ and $\star_\alpha$ are defined as follows

$$\begin{aligned} &e_1 \bullet_\alpha e_1 = a^2 e_2, \\ &e_1 \star_\alpha e_1 = a^2 e_2, \quad e_3 \star_\alpha e_3 = a^2 e_2. \end{aligned}$$

**Example 12.** Let $(\mathcal{J}, \star, \alpha)$ be a HM-Lie algebra. A Nijenhuis operator on $\mathcal{J}$ is a linear map $N : \mathcal{J} \to \mathcal{J}$ satisfying

$$\begin{gathered} N \circ \alpha = \alpha \circ N, \\ N(x) \star N(y) = N(N(x) \star y + x \star N(y) - N(x \star y)) \quad \forall x, y, \in \mathcal{J}. \end{gathered} \tag{15}$$

It is easy to see that the deformed product

$$x \star_N y = N(x) \star y + x \star N(y) - N(x \star y)$$

defines a HM-Lie algebra structure on $\mathcal{J}$ with respect to $\alpha$. Then the tuple $(\mathcal{J},\star,\star_N,\alpha)$ is a compatible HM-Lie algebra. In fact, let $x,y,z\in\mathcal{J}$, we have

$$\begin{aligned}
&(x\star y)\star_N\alpha(z)+(y\star z)\star_N\alpha(x)+(z\star x)\star_N\alpha(y)+(x\star_N y)\star\alpha(z)\\
&\qquad\qquad+(y\star_N z)\star\alpha(x)+(z\star_N x)\star\alpha(y)\\
&=N(x\star y)\star\alpha(z)+(x\star y)\star N(\alpha(z))-N((x\star y)\star\alpha(z))\\
&\quad+N(y\star z)\star\alpha(x)+(y\star z)\star N(\alpha(x))-N((y\star z)\star\alpha(x))\\
&\quad+N(z\star x)\star\alpha(y)+(z\star x)\star N(\alpha(y))-N((z\star x)\star\alpha(y))\\
&\quad+(N(x)\star y)\star\alpha(z)+(x\star N(y))\star\alpha(z)-N(x\star y)\star\alpha(z)\\
&\quad+(N(y)\star z)\star\alpha(x)+(y\star N(z))\star\alpha(x)-N(y\star z)\star\alpha(x)\\
&\quad+(N(z)\star x)\star\alpha(y)+(z\star N(x))\star\alpha(y)-N(z\star x)\star\alpha(y)\\
&=[(x\star y)\star\alpha(N(z))+(y\star N(z))\star\alpha(x)+(N(z)\star x)\star\alpha(y)]\\
&\quad+[(y\star z)\star\alpha(N(x))+(z\star N(x))\star\alpha(y)+(N(x)\star y)\star\alpha(z)]\\
&\quad+[(z\star x)\star\alpha(N(y))+(x\star N(y))\star\alpha(z)+(N(y)\star z)\star\alpha(x)]\\
&\quad-N[(x\star y)\star\alpha(z)+(y\star z)\star\alpha(x)+(z\star x)\star\alpha(y)]\\
&=0.
\end{aligned}$$

The next class of examples come from compatible Rota-Baxter operators on HM-Lie algebras.

**Definition 16.** Let $(\mathcal{J},\star,\alpha)$ be a HM-Lie algebra. A Rota-Baxter operator on $\mathcal{J}$ is a linear map $R:\mathcal{J}\to\mathcal{J}$ satisfying

$$R\circ\alpha=\alpha\circ R,$$
$$R(x)\star R(y)=R\big(R(x)\star y+x\star R(y)\big),\quad\text{for } x,y\in\mathcal{J}.$$

A Rota-Baxter operator $R$ induces a new HM-Lie multiplication $\star_R:\mathcal{J}\otimes\mathcal{J}\to\mathcal{J},\ (x,y)\mapsto x\star_R y$ on the underlying vector space $\mathcal{J}$ and it is given by

$$x\star_R y=R(x)\star y+x\star R(y),\quad\text{for } x,y\in\mathcal{J}.$$

**Definition 17.** Two Rota-Baxter operators $R,S\colon J\to J$ on a HM-Lie algebra $\mathcal{J}$ are said to be compatible if for any $k,l\in\mathbb{K}$, the sum $kR+lS:\mathcal{J}\to\mathcal{J}$ is a Rota-Baxter operator on $\mathcal{J}$. Equivalently,

$$\begin{aligned}
&R(x)\star S(y)+S(x)\star R(y)\\
&=R\big(S(x)\star y+x\star S(y)\big)+S\big(R(x)\star y+x\star R(y)\big),\quad\text{for } x,y\in\mathcal{J}.
\end{aligned}$$

The following result is straightforward.

**Proposition 17.** *Let $R, S : \mathcal{J} \to \mathcal{J}$ be two compatible Rota-Baxter operators on $\mathcal{J}$. Then $(\mathcal{J}, \star_R, \star_S, \alpha)$ is a compatible HM-Lie algebra.*

*Proof.* Let $x, y, z \in \mathcal{J}$, then we have

$$\begin{aligned}
&(x \star_R y) \star_S \alpha(z) + (y \star_R z) \star_S \alpha(x) + (z \star_R x) \star_S \alpha(y) + (x \star_S y) \star_R \alpha(z) \\
&\qquad + (y \star_S z) \star_R \alpha(x) + (z \star_S x) \star_R \alpha(y) \\
&= (R(x) \star y + x \star R(y)) \star_S \alpha(z) + (R(y) \star z + y \star R(z)) \star_S \alpha(x) \\
&\quad + (R(z) \star x + z \star R(x)) \star_S \alpha(y) + (S(x) \star y + x \star S(y)) \star_R \alpha(z) \\
&\quad + (S(y) \star z + y \star S(z)) \star_R \alpha(x) + (S(z) \star x + z \star S(x)) \star_R \alpha(y) \\
&= S(R(x) \star y + x \star R(y)) \star \alpha(z) + (R(x) \star y + x \star R(y)) \star S(\alpha(z)) \\
&\quad + S(R(y) \star z + y \star R(z)) \star \alpha(x) + (R(y) \star z + y \star R(z)) \star S(\alpha(x)) \\
&\quad + S(R(z) \star x + z \star R(x)) \star \alpha(y) + (R(z) \star x + z \star R(x)) \star S(\alpha(y)) \\
&\quad + R(S(x) \star y + x \star S(y)) \star \alpha(z) + (S(x) \star y + x \star S(y)) \star R(\alpha(z)) \\
&\quad + R(S(y) \star z + y \star S(z)) \star \alpha(x) + (S(y) \star z + y \star S(z)) \star R(\alpha(x)) \\
&\quad + R(S(z) \star x + z \star S(x)) \star \alpha(y) + (S(z) \star x + z \star S(x)) \star R(\alpha(y)) \\
&= (R(x) \star S(y) + S(x) \star R(y)) \star \alpha(z) + (R(y) \star S(z) + S(y) \star R(z)) \star \alpha(x) \\
&\quad + (R(z) \star S(x) + S(z) \star R(x)) \star \alpha(y) + (R(x) \star y) \star \alpha(S(z)) \\
&\quad + (x \star R(y)) \star \alpha(S(z)) + (R(y) \star z) \star \alpha(S(x)) + (y \star R(z)) \star \alpha(S(x)) \\
&\quad + (R(z) \star x) \star \alpha(S(y)) + (z \star R(x)) \star \alpha(S(y)) + (S(x) \star y) \star \alpha(R(z)) \\
&\quad + (x \star S(y)) \star \alpha(R(z)) + (S(y) \star z) \star \alpha(R(x)) + (y \star S(z)) \star \alpha(R(x)) \\
&\quad + (S(z) \star x) \star \alpha(R(y)) + (z \star S(x)) \star \alpha(R(y)) \\
&= (R(x) \star S(y)) \star \alpha(z) + (S(y) \star z) \star \alpha(R(x)) + (z \star R(x)) \star \alpha(S(y)) \\
&\quad + (S(x) \star R(y)) \star \alpha(z) + (R(y) \star z) \star \alpha(S(x)) + (z \star S(x)) \star \alpha(R(y)) \\
&\quad + (R(y) \star S(z)) \star \alpha(x) + (S(z) \star x) \star \alpha(R(y)) + (x \star R(y)) \star \alpha(S(z)) \\
&\quad + (S(y) \star R(z)) \star \alpha(x) + (R(z) \star x) \star \alpha(S(y)) + (x \star S(y)) \star \alpha(R(z)) \\
&\quad + (R(z) \star S(x)) \star \alpha(y) + (S(x) \star y) \star \alpha(R(z)) + (y \star R(z)) \star \alpha(S(x)) \\
&\quad + (S(z) \star R(x)) \star \alpha(y) + (R(x) \star y) \star \alpha(S(z)) + (y \star S(z)) \star \alpha(R(x)) \\
&= 0.
\end{aligned}$$

Hence, the proof is finished. □

**Proposition 18.** *Let $(\mathcal{J}, \star_1, \star_2, \alpha_{\mathcal{J}})$ be a compatible HM-Lie algebra and $(A, \circ, \alpha_A)$ be a commutative Hom-associative algebra. Then, $(\mathcal{J} \otimes A, \diamond_1, \diamond_2, \alpha_{\mathcal{J} \otimes A})$ is a compatible HM-Lie algebra, where*

$$(x \otimes a) \diamond_1 (y \otimes b) = (x \star_1 y) \otimes (a \circ b),$$

$$(x \otimes a) \diamond_2 (y \otimes b) = (x \star_2 y) \otimes (a \circ b),$$
$$\alpha_{\mathcal{J} \otimes A} = \alpha_{\mathcal{J}} \otimes \alpha_A.$$

*Proof.* By Proposition 6, we have $(\mathcal{J} \otimes A, \diamond_i, \alpha_{\mathcal{J}\otimes A})$ is a HM-Lie algebra for $i=1,2$. Applying the compatibility condition, for $x, y, z \in \mathcal{J}, a, b, c \in A$, we have

$$\begin{aligned}
&((x \otimes a) \diamond_1 (y \otimes b)) \diamond_2 \alpha_{\mathcal{J}\otimes A}(z \otimes c) + ((y \otimes b) \diamond_1 (z \otimes c)) \diamond_2 \alpha_{\mathcal{J}\otimes A}(x \otimes a) \\
&+ ((z \otimes c) \diamond_1 (x \otimes a)) \diamond_2 \alpha_{\mathcal{J}\otimes A}(y \otimes b) + ((x \otimes a) \diamond_2 (y \otimes b)) \diamond_1 \alpha_{\mathcal{J}\otimes A}(z \otimes c) \\
&+ ((y \otimes b) \diamond_2 (z \otimes c)) \diamond_1 \alpha_{\mathcal{J}\otimes A}(x \otimes a) + ((z \otimes c) \diamond_2 (x \otimes a)) \diamond_1 \alpha_{\mathcal{J}\otimes A}(y \otimes b) \\
&= ((x \star_1 y) \otimes (a \circ b)) \diamond_2 \alpha_{\mathcal{J}\otimes A}(z \otimes c) + ((y \star_1 z) \otimes (b \circ c)) \diamond_2 \alpha_{\mathcal{J}\otimes A}(x \otimes a) \\
&+ ((z \star_1 x) \otimes (c \circ a) \diamond_2 \alpha_{\mathcal{J}\otimes A}(y \otimes b) + (x \star_2 y) \otimes (a \circ b) \diamond_1 \alpha_{\mathcal{J}\otimes A}(z \otimes c) \\
&+ ((y \star_2 z) \otimes (b \circ c)) \diamond_1 \alpha_{\mathcal{J}\otimes A}(x \otimes a) + ((z \star x) \otimes (c \circ a)) \diamond_1 \alpha_{\mathcal{J}\otimes A}(y \otimes b) \\
&= ((x \star_1 y) \star_2 \alpha_{\mathcal{J}}(z)) \otimes ((a \circ b) \circ \alpha_A(c)) + ((y \star_1 z) \star_2 \alpha_{\mathcal{J}}(x)) \otimes ((b \circ c) \circ \alpha_A(a)) \\
&+ ((z \star_1 x) \star_2 \alpha_{\mathcal{J}}(y)) \otimes ((c \circ a) \circ \alpha_A(b)) + ((x \star_2 y) \star_1 \alpha_{\mathcal{J}}(z)) \otimes ((a \circ b) \circ \alpha_A(c)) \\
&+ ((y \star_2 z) \star_1 \alpha_{\mathcal{J}}(x)) \otimes ((b \circ c) \circ \alpha_A(a)) + ((z \star_2 x) \star_1 \alpha_{\mathcal{J}}(y)) \otimes ((c \circ a) \circ \alpha_A(b)) \\
&= ((x \star_1 y) \star_2 \alpha_{\mathcal{J}}(z)) \otimes ((a \circ b) \circ \alpha_A(c)) + ((y \star_1 z) \star_2 \alpha_{\mathcal{J}}(x)) \otimes ((b \circ c) \circ \alpha_A(a)) \\
&+ ((z \star_1 x) \star_2 \alpha_{\mathcal{J}}(y)) \otimes ((c \circ a) \circ \alpha_A(b)) + ((x \star_2 y) \star_1 \alpha_{\mathcal{J}}(z)) \otimes ((a \circ b) \circ \alpha_A(c)) \\
&+ ((y \star_2 z) \star_1 \alpha_{\mathcal{J}}(x)) \otimes ((b \circ c) \circ \alpha_A(a)) + ((z \star_2 x) \star_1 \alpha_{\mathcal{J}}(y)) \otimes ((c \circ a) \circ \alpha_A(b)) \\
&= [(x \star_1 y) \star_2 \alpha_{\mathcal{J}}(z) + (y \star_1 z) \star_2 \alpha_{\mathcal{J}}(x) + (z \star_1 x) \star_2 \alpha_{\mathcal{J}}(y) + (x \star_2 y) \star_1 \alpha_{\mathcal{J}}(z) \\
&+ (y \star_2 z) \star_1 \alpha_{\mathcal{J}}(x) + (z \star_2 x) \star_1 \alpha_{\mathcal{J}}(y)]((a \circ b) \circ \alpha_A(c)) = 0.
\end{aligned}$$

Hence, we get the desired result. ∎

**Proposition 19.** *Let $(A, \cdot_1, \cdot_2, \alpha)$ be a compatible Hom-anti-associative algebra. Then $(A, \star_1, \star_2, \alpha)$ is a compatible HM-Lie algebra, where*

$$x \star_i y = x \cdot_i y + y \cdot_i x, \;\; i = 1, 2 \quad \forall x, y \in A. \tag{16}$$

*Proof.* Let $(A, \cdot_1, \cdot_2, \alpha)$ be a compatible Hom-anti-associative algebra. Then $(A, \cdot_1, \alpha)$ and $(A, \cdot_2, \alpha)$ are Hom-anti-associative algebras. Hence, by Proposition 5, $(A, \star_1, \alpha)$ and $(A, \star_2, \alpha)$ are HM-Lie algebras. It remains to prove the compatibility condition (14). For any $x, y, z \in A$, we have

$$\begin{aligned}
&(x \star_1 y) \star_2 \alpha(z) + (y \star_1 z) \star_2 \alpha(x) + (z \star_1 x) \star_2 \alpha(y) + (x \star_2 y) \star_1 \alpha(z) \\
&\qquad + (z \star_2 x) \star_1 \alpha(x) + (z \star_2 x) \star_1 \alpha(y) \\
&= (x \cdot_1 y) + y \cdot_1 x) \cdot_2 \alpha(z) + \alpha(z) \cdot_2 \big(x \cdot_1 y + y \cdot_1 x)\big) + \big(y \cdot_1 z + z \cdot_1 y\big) \cdot_2, \alpha(x) \\
&\quad + \alpha(x) \cdot_2 \big(y \cdot_1 z + z \cdot_1 y\big) + \big(z \cdot_1 x + x \cdot_1 z\big) \cdot_2 \alpha(y) + \alpha(y) \cdot_2 \big(z \cdot_1 x + x \cdot_1 z\big) \\
&\quad + \big(x \cdot_2 y + y \cdot_2 x\big) \cdot_1 \alpha(z) + \alpha(z) \cdot_1 \big(x \cdot_2 y + y \cdot_2 x\big) + \big(y \cdot_2 z + z \cdot_2 y\big) \cdot_1 \alpha(x) \\
&\quad + \alpha(x) \cdot_1 \big(y \cdot_2 z + z \cdot_2 y\big) + \big(z \cdot_2 x + x \cdot_2 z\big) \cdot_1 \alpha(y) + \alpha(y) \cdot_1 \big(z \cdot_2 x + x \cdot_2 z\big)
\end{aligned}$$

$$
\begin{aligned}
&= (x \cdot_1 y) \cdot_2 \alpha(z) + (x \cdot_2 y) \cdot_1 \alpha(z) + \alpha(x) \cdot_2 (y \cdot_1 z) + \alpha(x) \cdot_1 (y \cdot_2 z)\\
&\quad + (y \cdot_1 x) \cdot_2 \alpha(z) + (y \cdot_2 x) \cdot_1 \alpha(z) + \alpha(y) \cdot_2 (x \cdot_1 z) + \alpha(y) \cdot_1 (x \cdot_2 z)\\
&\quad + (y \cdot_1 z) \cdot_2 \alpha(x) + (y \cdot_2 z) \cdot_1 \alpha(x) + \alpha(y) \cdot_2 (z \cdot_1 x) + \alpha(y) \cdot_1 (z \cdot_2 x)\\
&\quad + (z \cdot_1 y) \cdot_2 \alpha(x) + (z \cdot_2 y) \cdot_1 \alpha(x) + \alpha(z) \cdot_2 (y \cdot_1 x) + \alpha(z) \cdot_1 (y \cdot_2 x)\\
&\quad + (z \cdot_1 x) \cdot_2 \alpha(y) + (z \cdot_2 x) \cdot_1 \alpha(y) + \alpha(z) \cdot_2 (x \cdot_1 y) + \alpha(z) \cdot_1 (x \cdot_2 y)\\
&\quad + (x \cdot_1 z) \cdot_2 \alpha(y) + (x \cdot_2 z) \cdot_1 \alpha(y) + \alpha(x) \cdot_2 (z \cdot_1 y) + \alpha(x) \cdot_1 (z \cdot_2 y)\\
&= 0,
\end{aligned}
$$

and the proof is achieved. □

**Definition 18.** The compatible HM-Lie algebra $(\mathcal{J}, \star_1, \star_2, \alpha)$ is called of compatible mock-Lie type if $(\mathcal{J}, \star_1, \alpha)$ and $(\mathcal{J}, \star_2, \alpha)$ are both of mock-Lie type with induced M-Lie algebras $(\mathcal{J}, \diamond_1)$ and $(\mathcal{J}, \diamond_2)$. In this case, $(\mathcal{J}, \diamond_1, \diamond_2)$ is called the induced compatible mock-Lie algebra.

According to Proposition 1 we have the following result.

**Proposition 20.** *Let $(A, \cdot_1, \cdot_2, \alpha)$ be a compatible Hom-anti-associative algebra of compatible anti-associative type. Then $(A, \star_1, \star_2, \alpha)$ given in Proposition 19 is a compatible HM-Lie algebra of compatible mock-Lie type.*

## 2.3. Representations of compatible Hom-mock-Lie algebras

In this section, we introduce the notion of a representation of a compatible HM-Lie algebra and provide some construction results about the dual representation.

**Definition 19.** A representation of a compatible HM-Lie algebra $(\mathcal{J}, \bullet, \star, \alpha)$ on a vector space $V$ is a triple of linear maps $\rho_1, \rho_2 : \mathcal{J} \to gl(V)$ and $\phi : V \to V$ such that for any $k_1, k_2 \in \mathbb{K}$, $(V, k_1\rho_1 + k_2\rho_2, \phi)$ is a representation of the HM-Lie algebra $(\mathcal{J}, k_1 \bullet + k_2 \star, \alpha)$. We denote it by $(V, \rho_1, \rho_2, \phi)$.

It follows that any compatible HM-Lie algebra $(\mathcal{J}, \bullet, \star, \alpha)$ is a representation on itself, via the following action: $ad_{1\mathcal{J}}(x)y = x \bullet y$ and $ad_{2\mathcal{J}}(x)y = x \star y$. This is called the adjoint representation.

**Proposition 21.** *Let $(\mathcal{J}, \bullet, \star, \alpha)$ be a compatible HM-Lie algebra. Let $V$ be a vector space and $\rho_1, \rho_2 : \mathcal{J} \to gl(V)$, $\phi : V \to V$ be a triple of linear maps. Then the following conditions are equivalent:*

*1.* $(V, \rho_1, \rho_2, \phi)$ *is a representation of* $(\mathcal{J}, \bullet, \star, \alpha)$.

*2.* $(V, \rho_1, \phi)$ *is a representation of the HM-Lie algebra* $(\mathcal{J}, \bullet, \alpha)$, $(V, \rho_2, \phi)$ *is a representation of the HM-Lie algebra* $(\mathcal{J}, \star, \alpha)$ *and for any* $x, y \in \mathcal{J}$ *we have the following compatibility condition:*

$$\begin{aligned} &\rho_1(x \star y)\phi(v) + \rho_2(x \bullet y)\phi(v) \\ =& -\rho_1(\alpha(x))\rho_2(y) - \rho_2(\alpha(y))\rho_1(x) - \rho_2(\alpha(x))\rho_1(y) - \rho_1(\alpha(y))\rho_2(x). \end{aligned} \tag{17}$$

Define two symmetric bilinear operations $\bullet_{\mathcal{J}\oplus V}, \star_{\mathcal{J}\oplus V} : \otimes^2(\mathcal{J}\oplus V) \to (\mathcal{J}\oplus V)$ by

$$\begin{aligned} (x+u) \bullet_{\mathcal{J}\oplus V} (y+v) &= x \bullet y + \rho_1(x)v + \rho_1(y)u, \\ (x+u) \star_{\mathcal{J}\oplus V} (y+v) &= x \star y + \rho_2(x)v + \rho_2(y)u, \\ (\alpha \oplus \phi)(x+u) &= \alpha(x) + \phi(u), \end{aligned}$$

for any $x, y \in \mathcal{J}$ and $u, v \in V$. Then $(\mathcal{J} \oplus V, \bullet_{\mathcal{J}\oplus V}, \star_{\mathcal{J}\oplus V}, \alpha \oplus \phi)$ is a compatible HM-Lie algebra called the semidirect product of a compatible HM-Lie algebra $(\mathfrak{g}, \bullet, \star, \alpha)$ and a representation $(V, \rho, \mu, \beta)$, and denote it by $\mathfrak{g} \ltimes_{\rho_1,\rho_2} V$ , or simply $\mathfrak{g} \ltimes V$.

Define $\rho_1^\star : \mathcal{J} \longrightarrow gl(V^*)$ and $\rho_2^\star : \mathcal{J} \longrightarrow gl(V^*)$ as in Proposition 8, we get the following result for compatible HM-Lie algebras.

**Proposition 22.** *Let* $(V, \rho_1, \rho_2, \phi)$ *be a representation of the compatible HM-Lie algebra* $(\mathcal{J}, \bullet, \star, \alpha)$. *Then,* $(V^*, \rho_1^\star, \rho_2^\star, (\phi^{-1})^*)$ *is a representation of* $(\mathcal{J}, \bullet, \star, \alpha)$ *on* $V^*$ *with respect to* $(\phi^{-1})^*$.

*Proof.* Let $(V, \rho_1, \rho_2, \phi)$ be a representation of $(\mathcal{J}, \bullet, \star, \alpha)$. Then, $(V^*, \rho_1^\star, (\phi^{-1})^*)$ and $(V^*, \rho_2^\star, (\phi^{-1})^*)$ are respectively representations of $(\mathcal{J}, \bullet, \alpha)$ and $(\mathcal{J}, \star, \alpha)$ on $V^*$ with respect to $(\phi^{-1})^*$. It remains to prove that

$$\begin{aligned} &\rho_1^*(x \bullet y)(\phi^{-1})^* + \rho_2^*(x \star y)(\phi^{-1})^* \\ =& -\rho_1^*(\alpha(x))\rho_1^*(y) - \rho_1^*(\alpha(y))\rho_1^*(x) - \rho_2^*(\alpha(x))\rho_2^*(y) - \rho_2^*(\alpha(y))\rho_2^*(x). \end{aligned}$$

In fact, we have

$$\begin{aligned} &\langle \rho_1^*(x \bullet y)(\phi^{-1})^*\xi + \rho_2^*(x \star y)(\phi^{-1})^*, u\rangle \\ &= \langle \rho_1^*(\alpha(x) \bullet \alpha(y))((\phi)^{-3})^*(\xi) + \rho_2^*(\alpha(x) \star \alpha(y))((\phi)^{-3})^*(\xi), u\rangle \\ &= \langle ((\phi)^{-3})^*, (-\rho_1(\alpha^2(x)\rho_1(\alpha(y)) - \rho_1(\alpha^2(y)\rho_1(\alpha(x) - \rho_2(\alpha^2(x)\rho_2(\alpha(y)) \\ &\quad -\rho_2(\alpha^2(y)\rho_2(\alpha(x)))((\phi)^{-1})^*\rangle \\ &= \langle ((\phi)^{-3})^*, \phi^{-1}(-\rho_1(\alpha^3(x)\rho_1(\alpha^2(y)) - \rho_1(\alpha^3(y)\rho_1(\alpha^2(x)) - \rho_2(\alpha^3(x)\rho_2(\alpha^2(y)) \\ &\quad -\rho_2(\alpha^3(y)\rho_2(\alpha^2(x)))\rangle \end{aligned}$$

$$= \langle ((\phi)^{-4})^*, (-\rho_1(\alpha^3(x)\rho_1(\alpha^2(y)) - \rho_1(\alpha^3(y)\rho_1(\alpha^2(x)) - \rho_2(\alpha^3(x)\rho_2(\alpha^2(y)) \\ -\rho_2(\alpha^3(y)\rho_2(\alpha^2(x)))\rangle$$
$$= \langle(-\rho_1^*(\alpha^3(y)\rho_1^*(\alpha^2(x)) - \rho_1^*(\alpha^3(x)\rho_1^*(\alpha^2(y)) - \rho_2^*(\alpha^3(y)\rho_2^*(\alpha^2(x)) \\ -\rho_2^*(\alpha^3(x)\rho_2^*(\alpha^2(y)))((\phi)^{-4})^*, u\rangle$$
$$= \langle -\rho_1^*(\alpha(y))\rho_1^*(x) - \rho_1^*(\alpha(x))\rho_1^*(y) - \rho_2^*(\alpha(y))\rho_2^*(x) - \rho_2^*(\alpha(x))\rho_2^*(y), u\rangle,$$

which achieve the proof. □

Using Proposition 9 and Proposition 10 we get the following result

**Proposition 23.** *Let $(V, \rho_1, \rho_2)$ be a representation of the compatible M-Lie algebra $(\mathcal{J}, \bullet, \star, \alpha)$. Suppose there exists a morphism $\alpha : \mathcal{J} \longrightarrow \mathcal{J}$ and $\beta \in gl(V)$ such that*

$$\rho_i(\alpha(x)) \circ \beta = \beta \circ \rho_i(x), \quad for \quad i = 1, 2. \tag{18}$$

*Then, $(V, \widehat{\rho_1} := \beta \circ \rho_1, \widehat{\rho_2} := \beta \circ \rho_2, \beta)$ is a representation of the compatible HM-Lie algebra $(\mathcal{J}, \alpha \circ \bullet, \alpha \circ \star, \alpha)$.*

## 3. Deformations of compatible Hom-mock-Lie algebras

In this section, we follow the cohomology theory for M-Lie algebras introduced in [6] called zigzag cohomology to define the second zigzag cohomology of compatible HM-Lie algebras. Next, we study linear deformations of a compatible HM-Lie algebra and introduce Nijenhuis operators on compatible HM-Lie algebras that generate trivial linear deformations. We also define infinitesimal deformations of a compatible HM-Lie algebras and characterize equivalence classes of infinitesimal deformations in terms of zigzag cohomology.

### 3.1. The second cohomology group of compatible Hom-mock-Lie algebras

In this section, we construct a cochain complex that defines a Hom-type cohomology of a compatible HM-Lie algebra in a suitable representation in low degree. Let $(V, \rho_1, \rho_2, \phi)$ be a representation of a compatible HM-Lie algebra $(\mathcal{J}, \bullet, \star, \alpha)$. Define the spaces

$$L^n_{Hom}(\mathcal{J}, V) = \{f : \times^n \mathcal{J} \to V; \phi \circ f = f \circ \alpha^{\times n}\},\ n \geq 1.$$

and

$$A^n_{Hom}(\mathcal{J}, V) = \{f : \wedge^n \mathcal{J} \to V; \phi \circ f = f \circ \alpha^{\wedge n}\},\ n \geq 1.$$

Set $d^n_{\bullet Hom} : L^n_{Hom}(\mathcal{J}, V) \to L^{n+1}_{Hom}(\mathcal{J}, V)$ and $\delta^n_{\bullet Hom} : A^n_{Hom}(\mathcal{J}, V) \to L^{n+1}_{Hom}(\mathcal{J}, V)$, for $n \geq 1$, be the coboundaries operators for the zigzag cohomology of the HM-Lie algebra $(\mathcal{J}, \bullet, \alpha)$ with coefficients in the representation $(V, \rho, \phi)$. Let us define $d^1_{\bullet Hom} : L^1_{Hom}(\mathcal{J}, V) \to L^2_{Hom}(\mathcal{J}, V)$ by

$$d^1_{\bullet Hom}(f)(x_1, x_2) = \rho(x_1)f(x_2) + \rho(x_2)f(x_1) + f(x_1 \bullet x_2),$$

$d^2_{\bullet Hom} : L^2_{Hom}(\mathcal{J}, V) \to L^3_{Hom}(\mathcal{J}, V)$ by

$$\begin{aligned} d^2_{\bullet Hom}(f)(x_1, x_2, x_3) = &\rho(\alpha(x_1))f(x_2, x_3) + \rho(\alpha(x_2))f(x_1, x_3) \\ &+ \rho(\alpha(x_3))f(x_1, x_2) + f(x_1 \bullet x_2, \alpha(x_3)) \\ &+ f(x_1 \bullet x_3, \alpha(x_2)) + f(x_2 \bullet x_3, \alpha(x_1)) \end{aligned}$$

and $\delta^1_{\bullet Hom} : A^1_{Hom}(\mathcal{J}, V) \to L^2_{Hom}(\mathcal{J}, V)$ by

$$\delta^1_{\bullet Hom}(f)(x_1, x_2) = \rho(x_1)f(x_2) + \rho(x_2)f(x_1) - f(x_1 \bullet x_2).$$

**Lemma 1.** *With the above discussion,* $d^2_{\bullet Hom} \circ \delta^1_{\bullet Hom} = 0$.

*Proof.* Let $x_1, x_2, x_3 \in \mathcal{J}$ and $f \in A^1_{Hom}(\mathcal{J}, V)$, then we have

$$\begin{aligned} &d^2_{\bullet Hom} \circ \delta^1_{\bullet Hom}(f)(x_1, x_2, x_3) \\ = &\rho(\alpha(x_1))(\delta^1_{\bullet Hom}(f))(x_2, x_3) + \rho(\alpha(x_2))(\delta^1_{\bullet Hom}(f))(x_1, x_3) \\ &+ \rho(\alpha(x_3))(\delta^1_{\bullet Hom}(f))(x_1, x_2) + \delta^1_{\bullet Hom}(f)(x_1 \bullet x_2, \alpha(x_3)) \\ &+ \delta^1_{\bullet Hom}(f)(x_1 \bullet x_2, \alpha(x_2)) + \delta^1_{\bullet Hom}(f)(x_2 \bullet x_3, \alpha(x_1)) \\ = &\rho(\alpha(x_1))\rho(x_2)f(x_3) + \rho(\alpha(x_1))\rho(x_3)f(x_2) - \rho(\alpha(x_1))f(x_2 \bullet x_3) \\ &+ \rho(\alpha(x_2))\rho(x_1)f(x_3) + \rho(\alpha(x_2))\rho(x_3)f(x_1) - \rho(\alpha(x_2))f(x_1 \bullet x_3) \\ &+ \rho(\alpha(x_3))\rho(x_1)f(x_2) + \rho(\alpha(x_3))\rho(x_2)f(x_1) - \rho(\alpha(x_3))f(x_1 \bullet x_2) \\ &+ \rho(x_1 \bullet x_2)f(\alpha(x_3)) + \rho(\alpha(x_3))f(x_1 \bullet x_2) - f((x_1 \bullet x_2) \bullet \alpha(x_3)) \\ &+ \rho(x_1 \bullet x_3)f(\alpha(x_2)) + \rho(\alpha(x_2))f(x_1 \bullet x_3) - f((x_1 \bullet x_3) \bullet \alpha(x_2)) \\ &+ \rho(x_2 \bullet x_3)f(\alpha(x_1)) + \rho(\alpha(x_1))f(x_2 \bullet x_3) - f((x_2 \bullet x_3) \bullet \alpha(x_1)). \end{aligned}$$

Since $\phi \circ f = f \circ \alpha$, then by using Eqs. (2) +(4), we get that $d^2_{\bullet Hom} \circ \delta^1_{\bullet Hom} = 0$. □

We denote by $Z^2_{\bullet Hom}(\mathcal{J}, V) = Ker(d^2_{\bullet Hom})$ the space of 2-cocycles, and by

$$\begin{aligned} B^2_{\bullet Hom}(\mathcal{J}, V) &= Im(\delta^1_{\bullet Hom}) \\ &= \big\{f \in L^2_{\bullet Hom}(\mathcal{J}, V); \exists g \in A^1_{\bullet Hom}(\mathcal{J}, V) : f = \delta^1_{\bullet Hom}(g)\big\}, \end{aligned}$$

the space of 2-coboundaries. Hence, we can define the second zigzag cohomology space of HM-Lie algebras $(\mathcal{J}, \bullet, \alpha)$ with coefficients in the representation $(V, \rho, \phi)$ as

$$H^2_{\bullet Hom}(\mathcal{J}, V) = \frac{Z^2_{\bullet Hom}(\mathcal{J}, V)}{B^2_{\bullet Hom}(\mathcal{J}, V)}.$$

We are now in a position to define the second zigzag cohomology group of a compatible HM-Lie algebra $(\mathcal{J}, \bullet, \star, \alpha)$ with coefficients in a representation $(V, \rho_1, \rho_2, \phi)$. For each $n \geq 1$, we define the abelian groups $L^n_{cHom}(\mathcal{J}, V)$ and $A^n_{cHom}(\mathcal{J}, V)$ as follows:

$$L^n_{cHom}(\mathcal{J}, V) = \underbrace{L^n_{Hom}(\mathcal{J}, V) \oplus \cdots \oplus L^n_{Hom}(\mathcal{J}, V)}_{n\ copies},$$
$$A^n_{cHom}(\mathcal{J}, V) = \underbrace{A^n_{Hom}(\mathcal{J}, V) \oplus \cdots \oplus A^n_{Hom}(\mathcal{J}, V)}_{n\ copies}.$$

Define $d^1_{cHom} : L^1_{cHom}(\mathcal{J}, V) \to L^2_{cHom}(\mathcal{J}, V)$ by

$$d^1_{cHom}(f) = \big(d^1_{\bullet Hom}(f), d^1_{\star Hom}(f)\big),$$

$d^2_{cHom} : L^2_{cHom}(\mathcal{J}, V) \to L^3_{cHom}(\mathcal{J}, V)$ by

$$d^2_{cHom}(f_1, f_2) = \Big(d^2_{\bullet Hom}(f_1), d^2_{\bullet Hom}(f_2) + d^2_{\star Hom}(f_1), d^2_{\star Hom}(f_2)\Big),$$

and $\delta^1_{cHom} : A^1_{cHom}(\mathcal{J}, V) \to L^2_{cHom}(\mathcal{J}, V)$ by

$$\delta^1_{cHom}(f) = \big(\delta^1_{\bullet Hom}(f), \delta^1_{\star Hom}(f)\big).$$

**Proposition 24.** *The composite $d^2_{cHom} \circ \delta^1_{cHom}$ is zero.*

*Proof.* Let $x_1, x_2, x_3 \in \mathcal{J}$ and $f \in A^1_{cHom}(\mathcal{J}, V)$, then we have

$$\begin{aligned} & d^2_{cHom} \circ \delta^1_{cHom}(f)(x_1, x_2, x_3) \\ =& d^2_{cHom}\big(\delta^1_{\bullet Hom}(f), \delta^1_{\star Hom}(f)\big)(x_1, x_2, x_3) = \Big(d^2_{\bullet Hom}(\delta^1_{\bullet Hom}(f)), d^2_{\bullet Hom}(\delta^1_{\star Hom}(f)) \\ & + d^2_{\star Hom}(\delta^1_{\bullet Hom}(f)), d^2_{\star Hom}(\delta^1_{\star Hom}(f))\Big)(x_1, x_2, x_3). \end{aligned}$$

Using Lemma 1, we have $d^2_{\bullet Hom} \circ \delta^1_{\bullet Hom}(f) = d^2_{\star Hom} \circ \delta^1_{\star Hom}(f) = 0$. Next, we need to show that $d^2_{\bullet Hom}(\delta^1_{\star Hom}(f)) + d^2_{\star Hom}(\delta^1_{\bullet Hom}(f)) = 0$.

For any $x_1, x_2, x_3 \in \mathcal{J}$ and $f \in A^1_{cHom}(\mathcal{J}, V)$, we have

$$\begin{aligned}
&d^2_{\bullet Hom}(\delta^1_{\star Hom}(f))(x_1,x_2,x_3) + d^2_{\star Hom}(\delta^1_{\bullet Hom}(f))(x_1,x_2,x_3)\\
&= \rho_1(\alpha(x_1))\rho_2(x_2)f(x_3) + \rho_1(\alpha(x_1))\rho_2(x_3)f(x_2) - \rho_1(\alpha(x_1))f(x_2 \star x_3)\\
&+ \rho_1(\alpha(x_2))\rho_2(x_1)f(x_3) + \rho_1(\alpha(x_2))\rho_2(x_3)f(x_1) - \rho_1(\alpha(x_2))f(x_1 \star x_3)\\
&+ \rho_1(\alpha(x_3))\rho_2(x_1)f(x_2) + \rho_1(\alpha(x_3))\rho_2(x_2)f(x_1) - \rho_1(\alpha(x_3))f(x_1 \star x_2)\\
&+ \rho_2(x_1 \bullet x_2)f(\alpha(x_3)) + \rho_2(\alpha(x_3))f(x_1 \bullet x_2) - f((x_1 \bullet x_2) \star \alpha(x_3))\\
&+ \rho_2(x_1 \bullet x_3)f(\alpha(x_2)) + \rho_2(\alpha(x_2))f(x_1 \bullet x_3) - f((x_1 \bullet x_3) \star \alpha(x_2))\\
&+ \rho_2(x_2 \bullet x_3)f(\alpha(x_1)) + \rho_2(\alpha(x_1))f(x_2 \bullet x_3) - f((x_2 \bullet x_3) \star \alpha(x_1))\\
&+ \rho_2(\alpha(x_1))\rho_1(x_2)f(x_3) + \rho_2(\alpha(x_1))\rho_1(x_3)f(x_2) - \rho_2(\alpha(x_1))f(x_2 \bullet x_3)\\
&+ \rho_2(\alpha(x_2))\rho_1(x_1)f(x_3) + \rho_2(\alpha(x_2))\rho_1(x_3)f(x_1) - \rho_2(\alpha(x_2))f(x_1 \bullet x_3)\\
&+ \rho_2(\alpha(x_3))\rho_1(x_1)f(x_2) + \rho_2(\alpha(x_3))\rho_1(x_2)f(x_1) - \rho_2(\alpha(x_3))f(x_1 \bullet x_2)\\
&+ \rho_1(x_1 \star x_2)f(\alpha(x_3)) + \rho_1(\alpha(x_3))f(x_1 \star x_2) - f((x_1 \star x_2) \bullet \alpha(x_3))\\
&+ \rho_1(x_1 \star x_3)f(\alpha(x_2)) + \rho_1(\alpha(x_2))f(x_1 \star x_3) - f((x_1 \star x_3) \bullet \alpha(x_2))\\
&+ \rho_1(x_2 \star x_3)f(\alpha(x_1)) + \rho_1(\alpha(x_1))f(x_2 \star x_3) - f((x_2 \star x_3) \bullet \alpha(x_1)).
\end{aligned}$$

Since $\phi \circ f = f \circ \alpha$, then by using Eqs. (14) +(17), we get that

$$d^2_{\bullet Hom}(\delta^1_{\star Hom}(f)) + d^2_{\star Hom}(\delta^1_{\bullet Hom}(f)) = 0.$$

Then the composite $d^2_{cHom} \circ \delta^1_{cHom}$ is zero. □

**Notation**. We denote by $Z^2_{cHom}(\mathcal{J}, V) = Ker(d^2_{cHom})$ the space of 2-cocycles, and by

$$B^2_{cHom}(\mathcal{J}, V) = \{f \in L^2_{cHom}(\mathcal{J}, V); \exists g \in A^1_{cHom}(\mathcal{J}, V) : f = \delta^1_{cHom}(g)\},$$

the space of 2-coboundaries. Since $d^2_{cHom} \circ \delta^1_{cHom} = 0$, then $B^2_{cHom}(\mathcal{J}, V)$ is a subspace of $Z^2_{cHom}(\mathcal{J}, V)$. Hence, we can define the second zigzag cohomology space of a compatible HM-Lie algebra $(\mathcal{J}, \bullet, \star, \alpha)$ as

$$H^2_{cHom}(\mathcal{J}, V) = \frac{Z^2_{cHom}(\mathcal{J}, V)}{B^2_{cHom}(\mathcal{J}, V)}.$$

## 3.2. Linear deformations and Nijenhuis operators

In this section, we study linear deformations of a compatible HM-Lie algebra. We introduce Nijenhuis operators that generate trivial linear deformations. We also define infinitesimal deformations of a compatible HM-Lie algebra and characterize equivalence classes of infinitesimal deformations in terms of cohomology.

Let $(\mathcal{J}, \bullet, \star, \alpha)$ be a compatible HM-Lie and $(\omega_1, \omega_2) \in L^2_{cHom}(\mathcal{J}, \mathcal{J})$ be a 2-cochain. Define two bilinear operations on $\mathcal{J}$ depending on the parameter $t$ as follows:

$$x \bullet^t y = x \bullet y + t\omega_1(x, y), \quad x \star^t y = x \star y + t\omega_2(x, y) \quad \forall x, y \in \mathcal{J}.$$

**Definition 20.** We say that $(\omega_1, \omega_2)$ generates a linear deformation of the compatible HM-Lie algebra $(\mathcal{J}, \bullet, \star, \alpha)$ if for all $t$, the quadruple $(\mathcal{J}, \bullet^t, \star^t, \alpha)$ is a compatible HM-Lie algebra.

Denote by $\mu_1, \mu_2 \in L^2_{Hom}(\mathcal{J}, \mathcal{J})$, the elements corresponding to the HM-Lie multiplications $\bullet$ and $\star$, respectively, then the above definition is equivalent to saying that

$$\circlearrowleft_{x,y,z} \alpha(x) \bullet^t (y \bullet^t z) = 0, \qquad \circlearrowleft_{x,y,z} \alpha(x) \star^t (y \star^t z) = 0,$$
$$\circlearrowleft_{x,y,z} \Big\{(x \bullet^t y) \star^t \alpha(z) + (x \star^t y) \bullet^t \alpha(z)\Big\} = 0.$$

In other words, the followings are hold

$$\circlearrowleft_{x,y,z} \Big\{\mu_1(\alpha(x), \omega_1(y, z)) + \omega_1(\alpha(x), \mu_1(y, z))\Big\} = 0,$$
$$\circlearrowleft_{x,y,z} \Big\{\mu_2(\alpha(x), \omega_2(y, z)) + \omega_2(\alpha(x), \mu_2(y, z))\Big\} = 0,$$
$$\circlearrowleft_{x,y,z} \Big\{\mu_2(\omega_1(x, y), \alpha(z)) + \omega_2(\mu_1(x, y), \alpha(z)) + \mu_1(\omega_2(x, y), \alpha(z))$$
$$+\omega_1(\mu_2(x, y), \alpha(z))\Big\} = 0,$$
$$\circlearrowleft_{x,y,z} \omega_1(\alpha(x), \omega_1(y, z)) = 0, \qquad \circlearrowleft_{x,y,z} \omega_2(\alpha(x), \omega_2(y, z)) = 0,$$
$$\circlearrowleft_{x,y,z} \Big\{\omega_1(\omega_2(x, y), \alpha(z)) + \omega_2(\omega_1(x, y), \alpha(z))\Big\} = 0.$$

The first three condition implies that $(\omega_1, \omega_2) \in L^2_{cHom}(\mathcal{J}, \mathcal{J})$ is a 2-cocycle in the zigzag cohomology of the compatible HM-Lie algebra $\mathcal{J}$ with coefficients in the adjoint representation, that is,

$$d^2_{cHom}(\omega_1, \omega_2) = \Big(d^2_{\bullet Hom}(\omega_1), d^2_{\bullet Hom}(\omega_2) + d^2_{\star Hom}(\omega_1), d^2_{\star Hom}(\omega_2)\Big) = 0.$$

Moreover, the last three conditions implies that $(\mathcal{J}, \omega_1, \omega_2, \alpha)$ is a compatible HM-Lie algebra.

**Definition 21.** Let $(\omega_1, \omega_2)$, $(\omega'_1, \omega'_2) \in L^2_{cHom}(\mathcal{J}, \mathcal{J})$ generate linear deformations $(\mathcal{J}, \bullet^t, \star^t, \alpha)$ and $(\mathcal{J}, \bullet'^t, \star'^t, \alpha)$ of a compatible HM-Lie algebra $\mathcal{J}$. They are said to be equivalent if there exists a linear map $N : \mathcal{J} \to \mathcal{J}$ such that

$$Id + tN : (\mathcal{J}, \bullet^t, \star^t, \alpha) \to (\mathcal{J}, \bullet'^t, \star'^t, \alpha)$$

is a morphism of compatible HM-Lie algebras.

One can equivalently write the explicit identities as follows:

$$\begin{aligned}
\alpha \circ N &= N \circ \alpha, \\
\omega_1(x, y) - \omega'_1(x, y) &= x \bullet Ny + Nx \bullet y - N(x \bullet y), \\
\omega_2(x, y) - \omega'_2(x, y) &= x \star Ny + Nx \star y - N(x \star y), \\
N\omega_1(x, y) &= \omega'_1(x, Ny) + \omega'_1(Nx, y) + Nx \bullet Ny, \\
N\omega_2(x, y) &= \omega'_2(x, Ny) + \omega'_2(Nx, y) + Nx \star Ny, \\
\omega'_i(Nx, Ny) &= 0, \quad i = 1, 2,
\end{aligned}$$

for any $x, y \in \mathcal{J}$. Note that from the first identity, we get

$$(\omega_1, \omega_2) - (\omega'_1, \omega'_2) = \delta_{cHom}(N) = \Big(\delta^1_{\bullet Hom}(N), \delta^1_{\star Hom}(N)\Big),$$

where $N$ is considered as an element in $A^1_{cHom}(\mathcal{J}, \mathcal{J})$. Hence, summarizing the above discussions, we get the following.

**Theorem 1.** *Let $\mathcal{J}$ be a compatible HM-Lie algebra. Then there is a map from the set of equivalence classes of linear deformations of $\mathcal{J}$ to the second cohomology group $H^2_{cHom}(\mathcal{J}, \mathcal{J})$.*

Next, we introduce trivial linear deformations of a compatible HM-Lie algebra and introduce Nijenhuis operators that generate trivial linear deformations.

**Definition 22.** A linear deformation $(\bullet^t = \bullet + t\omega_1, \star^t = \star + t\omega_2)$ of a compatible HM-Lie algebra with respect to $\alpha$ is said to be trivial if the deformation is equivalent to the undeformed one $(\bullet, \star)$.

Thus, a linear deformation is trivial if and only if there exists a linear map $N : \mathcal{J} \to \mathcal{J}$ satisfying

$$\begin{aligned}
\alpha \circ N &= N \circ \alpha, \\
\omega_1(x, y) &= x \bullet Ny + Nx \bullet y - N(x \bullet y), \\
\omega_2(x, y) &= x \star Ny + Nx \star y - N(x \star y), \\
N\omega_1(x, y) &= Nx \bullet Ny, \qquad N\omega_2(x, y) = Nx \star Ny,
\end{aligned}$$

for any $x, y \in \mathcal{J}$. This motivates us to introduce Nijenhuis operators on a compatible HM-Lie algebra

**Definition 23.** A Nijenhuis operator on a compatible HM-Lie algebra $(\mathcal{J}, \bullet, \star, \alpha)$ is a linear map $N : \mathcal{J} \to \mathcal{J}$ which is a Nijenhuis operator for both the HM-Lie algebras $(\mathcal{J}, \bullet, \alpha)$ and $(\mathcal{J}, \star, \alpha)$, that is, $N$ satisfies

$$\alpha \circ N = N \circ \alpha,$$
$$Nx \bullet Ny = N\Big(x \bullet Ny + Nx \bullet y - N(x \bullet y)\Big),$$
$$Nx \star Ny = N\Big(x \star Ny + Nx \star y - N(x \star y)\Big) \ \forall x, y \in \mathcal{J}.$$

It follows that any trivial linear deformation of a compatible HM-Lie algebra induces a Nijenhuis operator. The converse is given by the next result whose proof is straightforward.

**Proposition 25.** *Let $N : \mathcal{J} \to \mathcal{J}$ be a Nijenhuis operator on a compatible HM-Lie algebra $(\mathcal{J}, \bullet, \star, \alpha)$. Then $(\omega_1, \omega_2)$ generates a trivial linear deformation of the compatible HM-Lie algebra $(\mathcal{J}, \bullet, \star, \alpha)$, where*

$$\omega_1(x, y) = x \bullet Ny + Nx \bullet y - N(x \bullet y),$$
$$\omega_2(x, y) = x \star Ny + Nx \star y - N(x \star y), \ \forall x, y \in \mathcal{J}.$$

In the following, we introduce infinitesimal deformations of a compatible HM-Lie algebra as a generalization of linear deformations.

**Definition 24.** An infinitesimal deformation of a compatible HM-Lie algebra $(\mathcal{J}, \bullet, \star, \alpha)$ is a linear deformation over $\mathbb{K}[[t]]/(t^2)$.

One can similarly define equivalences between two infinitesimal deformations. It is easy to see that any 2-cocycle $(\omega_1, \omega_2) \in Z^2_{cHom}(\mathcal{J}, \mathcal{J})$ induces an infinitesimal deformation and cohomologous 2-cocycles give rise to equivalent infinitesimal deformations. Summarizing this fact with Theorem 1, we get the following.

**Theorem 2.** *Let $(\mathcal{J}, \bullet, \star, \alpha)$ be a compatible HM-Lie algebra. Then the equivalence classes of infinitesimal deformations are in one-to-one correspondence with $H^2_{cHom}(\mathcal{J}, \mathcal{J})$.*

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