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Minimal lattice points in the Newton polyhedron with an application to normal ideals

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Abstract. Let $a_1, \ldots, a_n$ be positive integers and let $\Delta = NP(a_1, \ldots, a_n)$ be the Newton polyhedron associated to these integers, that is, the convex hull in $\mathbb{R}^n$ of the axial points that have a_i in the x_i-axis. We give some characterization of the minimal elements of Δ, and then use this characterization to give an alternative simpler proof of a main result of [7] on the normality of monomial ideals.

Introduction

Let I be an ideal in a Noetherian ring R. The integral closure of I is the ideal $\overline{I}$ that consists of all elements of R that satisfy an equation of the form

$$x^n + d_1 x^{n-1} + \cdots + d_{n-1} x + d_n = 0, \qquad d_i \in I^i \ (i = 1, \ldots, n).$$

The ideal I is said to be *integrally closed* if $I = \overline{I}$. An ideal is called *normal* if all of its powers are integrally closed. It is known that if R

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is a normal integral domain, then the Rees algebra $R[It] = \oplus_{n\in N} I^n t^n$ is normal if and only if I is a normal ideal of R [4, Proposition 2.1.2]. This brings up the importance of normality of ideals as the Rees algebra is the algebraic counterpart of blowing up a scheme along a closed subscheme [9]. There is no concise solution to the problem of when a given ideal is normal, not even in the monomial ideal case.

Let $I \subset K[x_1, \ldots, x_n]$ be a monomial ideal with K a field and let $\Gamma(I)$ denote the set of exponents of all monomials in the ideal I. The *Newton polyhedron* of I, denoted $NP(I)$, is the convex hull in $\mathbb{R}^n$ of $\Gamma(I)$; see Definition 1.4.7 in Swanson and Huneke [8]. Similarly, let $\Gamma(\overline{I})$ denote the set of exponents of all monomials in $\overline{I}$. Geometrically, finding the integral closure of the monomial ideal I is the same as finding all the integer lattice points in $NP(I)$; see Proposition 1.4.6 in [8]. That is, $\Gamma(\overline{I}) = NP(I) \cap \mathbb{N}^n$. It is a challenging problem to translate the question of normality of a monomial ideal I into a question about the exponent sets $\Gamma(I)$ and $\Gamma(\overline{I})$.

Given the ideal $I = \langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle \subset R = K[x_1, \ldots, x_n]$ with a_i positive integers. Let $\mathbf{I}(a_1, \ldots, a_n)$ denote the integral closure of the ideal I. The normality of $\mathbf{I}(a_1, \ldots, a_n)$ has been of interest for many authors [1–3, 7, 9]. Investigating this normality causes the following natural question to arise:

Question: *Giving that $\mathbf{I}(a_1, \ldots, a_n) \subset R$ is normal. What hypotheses does s need to satisfy so that $\mathbf{I}(a_1, \ldots, a_n, s) \subset R[x_{n+1}]$ stays normal?*

As partial answers to this question (generalizing the main work of [2]) it is shown in Al-Ayyoub et. al. [1] that if $\mathbf{I}(a_1, \ldots, a_n)$ is normal, then $\mathbf{I}(a_1, \ldots, a_n, s)$ is normal for any $s \in \{a_1, \ldots, a_n\}$. Also, it has been proved in [1] that if $\mathbf{I}(a_1, \ldots, a_n, l)$ is *not* normal, where $l = \operatorname{lcm}(a_1, \ldots, a_n)$, then $\mathbf{I}(a_1, \ldots, a_n, s)$ is *not* normal for any $s > l$.

Searching for an answer to the above question is reduced to consider only the values of s that lie in the interval $[l, 2l - 1]$, this is due to the main result of Reid, Roberts and Vitulli [7, Theorem 5.1] which states that if $l = \operatorname{lcm}(a_1, \ldots, a_n)$ and $L := \mathbf{I}(a_1, \ldots, a_n, a_{n+1} + l)$ is normal, then $J := \mathbf{I}(a_1, \ldots, a_n, a_{n+1})$ is normal. Conversely, if $a_{n+1} \geq l$ and J is normal, then L is normal. The method of the proof of [7, Theorem 5.1] is by comparing the minimal generators of the integral closure of the Rees algebras $R[Lt]$ and $R[Jt]$. The goal of this paper is to give an elementary and simpler proof of this main result of [7]. Our proof depends on the elementary definition of convex sets, and in particular, it depends on a simple characterization of the exponents of the *minimal* generators of $\mathbf{I}(a_1, \ldots, a_n)$, see Lemma 1 which constitutes a key lemma of this paper. This lemma provides a good tool for investigating the normality of the integral closure of the chief ideals, that is, ideals of the form $\langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle$.

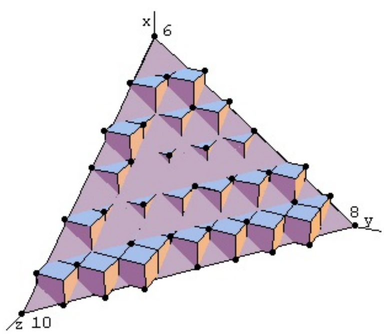
1. Characterizing the minimal elements

Throughout this paper, let $a_1 \le \cdots \le a_n \le a_{n+1}$ with a_i positive integers and let $R = K[x_1, \ldots, x_n]$ and $S = R[x_{n+1}]$. Before we proceed, we start with the following definition and emphasize the following notation.

Notation 1. Given the ideal $I = \langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle \subset R$. Let $\mathbf{I}(a_1, \ldots, a_n) = \overline{I}$ denote the integral closure of I. Let $\Gamma(a_1, \ldots, a_n)$ denote the set of exponents of all monomials in $\overline{I}$. Also, let $\Gamma'(a_1, \ldots, a_n)$ denote the set of the *minimal elements* of $\Gamma(a_1, \ldots, a_n)$, that is, the elements in $\Gamma'(a_1, \ldots, a_n)$ are in a one-to-one correspondence with the minimal generators of $\overline{I}$.

Definition 1. *The tuple* $(c_1, \ldots, c_n) \in \Gamma(a_1, \ldots, a_n)$ *is called* ***minimal*** *if whenever* $c_i > 0$*, then* $(c_1, \ldots, c_i - 1, \ldots, c_n) \notin \Gamma(a_1, \ldots, a_n)$*, where* $i \in \{1, \ldots, n\}$*.*

For instance, the black disks in the figure below give the elements of $\Gamma'(6, 8, 10)$, that is, the black disks represent the minimal set of generators of $\mathbf{I}(6, 8, 10) = \overline{\langle x^6, y^8, z^{10} \rangle}$.



Consider an n-tuple $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Q}^n_{\ge 0}$. If $\mathbf{c} \in \mathbf{\Gamma}(a_1, \ldots, a_n)$, then there are nonnegative rational numbers $\lambda_1, \ldots, \lambda_n$ with $\sum_{j=1}^{n} \lambda_j = 1$ such that $c_j \geq a_j\lambda_j$ for $j = 1, \ldots, n$; that is, $\sum_{j=1}^{n} \frac{c_j}{a_j} \geq 1$. Conversely, if $\sum_{j=1}^{n} \frac{c_j}{a_j} \geq 1$, then $\mathbf{c} \in \mathbf{\Gamma}(a_1, \ldots, a_n)$. To prove this, first note that if $\sum_{j=1}^{n} \frac{c_j}{a_j} - \frac{1}{a_i} \geq 1$ for some i with $c_i > 0$, then there must be some positive integer k such that $\sum_{j=1}^{n} \frac{c_j}{a_j} - \frac{k}{a_i} \geq 1$ and $\sum_{j=1}^{n} \frac{c_j}{a_j} - \frac{k}{a_i} - \frac{1}{a_i} < 1$. It this case we show that $(c_1, \ldots, c_i - k, \ldots, c_n) \in \mathbf{\Gamma}(a_1, \ldots, a_n)$ which in turn implies that $\mathbf{c} \in \mathbf{\Gamma}(a_1, \ldots, a_n)$. Therefore, without loss of generality, we may assume that $\sum_{j=1}^{n} \frac{c_j}{a_j} - \frac{1}{a_i} < 1$ for any i with $c_i > 0$. Fix i with $c_i > 0$. Let $\lambda_i = 1 - \sum_{j=1, j\neq i}^{n} \frac{c_j}{a_j}$ and let $\lambda_j = \frac{c_j}{a_j}$ $(j = 1, \ldots, \widehat{i}, \ldots, n)$. Then $\sum_{j=1}^{n} \lambda_j = 1$ and $c_j = a_j\lambda_j$ for $j = 1, \ldots, \widehat{i}, \ldots, n$. Also, since $\sum_{j=1}^{n} \frac{c_j}{a_j} \geq 1$, then $\frac{c_i}{a_i} \geq 1 - \sum_{j=1, j\neq i}^{n} \frac{c_j}{a_j}$; thus $c_i \geq \lambda_i a_i$. This proves the following basic lemma, yet a key lemma of this paper.

Lemma 1. *Let* $(c_1, \ldots, c_n) \in \mathbb{Q}^n_{\geq 0}$. *Then*

$$(c_1, \ldots, c_n) \in \Gamma(a_1, \ldots, a_n) \iff 1 \leq \sum_{j=1}^{n} \frac{c_j}{a_j}.$$

In particular, $(c_1, \ldots, c_n) \in \Gamma'(a_1, \ldots, a_n) \iff 1 \leq \sum_{j=1}^{n} \frac{c_j}{a_j} < 1 + \frac{1}{a_i}$ *for any* i *with* $c_i > 0$.

Remark 1. Let $(c_1, \ldots, c_n, c_{n+1}) \in \Gamma'(a_1, \ldots, a_n, a_{n+1})$ and let $l = \operatorname{lcm}(a_1, \ldots, a_n)$.
(1) If $c_{n+1} > 0$, then

$$c_{n+1} = \left\lceil a_{n+1} \left(1 - \sum_{i=1}^{n} \frac{c_i}{a_i} \right) \right\rceil.$$

(2) If $\sum_{i=1}^{n} \frac{c_i}{a_i} > 1$, then $\sum_{i=1}^{n} \frac{c_i}{a_i} \geq 1 + \frac{1}{l}$ and $c_{n+1} = 0$.

Proof. (1) Assume $c_{n+1} > 0$. Then $1 \leq \sum_{j=1}^{n+1} \frac{c_j}{a_j} < 1 + \frac{1}{a_{n+1}}$ by Lemma 1. Thus

$$a_{n+1}\left(1-\sum_{i=1}^{n}\frac{c_i}{a_i}\right)\leq c_{n+1}<a_{n+1}\left(1-\sum_{i=1}^{n}\frac{c_i}{a_i}\right)+1,$$

hence done as c_{n+1} is an integer.

(2) Note $\sum_{i=1}^{n}\frac{c_i}{a_i}=\frac{c_1l_1+\cdots+c_1l_1}{l}>1$, where $l_i=l/a_i$. Since all parameters are integers, then we must have $c_1l_1+\cdots+c_1l_1\geq l+1$, hence done. □

2. An application to normal monomial ideals

In this section, we give an alternative shorter and elementary proof of a main result of [7], see Corollaries 1 and 2.

Notation 2. Let $I_k=\left\langle x_1^{ka_1},\ldots,x_n^{ka_n}\right\rangle\subset K[x_1,\ldots,x_n]$ and $F_k=\Gamma'(ka_1,\ldots,ka_n)$. Let $J=\overline{I_1}=\langle x_1^{e_1}\cdots x_n^{e_n}\mid(e_1,\ldots,e_n)\in F_1\rangle$ and $J_k=\overline{I_k}=\langle x_1^{c_1}\cdots x_n^{c_n}\mid(c_1,\ldots,c_n)\in F_k\rangle$.

Remark 2. With notation as before, we have $J^k\subseteq J_k$.

Proof. To prove the remark we need to show that $M_1+\cdots+M_k\in\Gamma(ka_1,\ldots,ka_n)$ whenever $M_i\in F_1$. Write $M_i=(e_{i,1},\ldots,e_{i,n})\in F_1$ and let $M_1+\cdots+M_k=(c_1,\ldots,c_n)$ with $c_j=\sum_{i=1}^{k}e_{i,j}$. Since $M_i\in F_1$, then $\sum_{j=1}^{n}\frac{e_{i,j}}{a_j}\geq 1$ and hence

$$\sum_{j=1}^{n}\frac{c_j}{ka_j}=\sum_{j=1}^{n}\frac{\sum_{i=1}^{k}e_{i,j}}{ka_j}=\sum_{i=1}^{k}\frac{1}{k}\sum_{j=1}^{n}\frac{e_{i,j}}{a_j}\geq\sum_{i=1}^{k}\frac{1}{k}=1;$$

which finishes the proof. □

We adopt the following notation as given in [6].

Definition 2. *Define the k-fold Minkowski sum of F_1 to be as follows*

$$k*F_1=\{M_1+\cdots+M_k\mid M_i\in F_1\}$$

with componentwise addition.

Theorem 1 ([5, Theorem 1.4.2])**.** *Let I be a monomial ideal in a polynomial ring R. Then $\overline{I}$ is the monomial ideal generated by all monomials $u\in R$ for which there exists an integer l such that $u^l\in I^l$.*

Remark 3. With notation as before, we have $\overline{J^k} = J_k$.

Proof. Note $\langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle \subset J$; thus, $\left\langle x_1^{ka_1}, \ldots, x_n^{ka_n} \right\rangle \subset J^k$ and hence $J_k = \overline{I_k} = \overline{\left\langle x_1^{ka_1}, \ldots, x_n^{ka_n} \right\rangle} \subset \overline{J^k}$. On the other hand, assume that $\alpha \in \overline{J^k}$. Then by Theorem 1, there is an integer l such that $\alpha^l \in \left(J^k\right)^l$. This implies that $\alpha^l = m_1 m_2 \cdots m_{kl}$ with $m_i \in J$ for all i. This means that $m_i \in \overline{\langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle}$ for each i; hence, by Theorem 1, there exists an integer s_i such that $m_i^{s_i} = x_1^{t_{i,1}a_1} x_2^{t_{i,2}a_2} \cdots x_n^{t_{i,n}a_n}$ where the $t_{i,j}$ are integers with $t_{i,1} + \cdots + t_{i,n} = s_i$ for all $i = 1, \ldots, kl$. Now let $s = \operatorname{lcm}(s_1, \ldots, s_{kl})$ and $q_i = s/s_i$. Then

$$\begin{aligned}
\alpha^{ls} &= \left(\alpha^l\right)^s = m_1^s m_2^s \cdots m_{kl}^s \\
&= m_1^{s_1 q_1} m_2^{s_2 q_2} \cdots m_{kl}^{s_{kl} q_{kl}} \\
&= \prod_{i=1}^{kl} \left(x_1^{t_{i,1}a_1} x_2^{t_{i,2}a_2} \cdots x_n^{t_{i,n}a_n} \right)^{q_i} \\
&= \prod_{j=1}^{n} x_j^{(t_{1,j}q_1 + t_{2,j}q_2 + \cdots + t_{kl,j}q_{kl})a_j} \\
&\in \left(x_1^{ka_1}, \ldots, x_n^{ka_n} \right)^{ls},
\end{aligned}$$

since $\sum_{j=1}^{n} (t_{1,j}q_1 + t_{2,j}q_2 + \cdots + t_{kl,j}q_{kl}) = \sum_{i=1}^{kl} (t_{i,1} + t_{i,2} + \cdots + t_{i,n})\, q_i = \sum_{i=1}^{kl} s_i q_i = \sum_{i=1}^{kl} s = kls$; therefore, Theorem 1 implies that

$$\alpha \in \overline{\left\langle x_1^{ka_1}, \ldots, x_n^{ka_n} \right\rangle} = J_k. \qquad \square$$

Recall that an ideal is called *normal* if all of its powers are integrally closed. The ideal J as given in the above notation, is normal if J^k is integrally closed for all powers k, that is, $\overline{J^k} = J^k$. By Remark 3, we have $\overline{J^k} = J_k$. Therefore, J is normal if and only if $J^k = J_k$ for all k. But Remark 2 gives that $J^k \subseteq J_k$; therefore, we conclude that J is normal if and only if $J_k \subseteq J^k$ for all k. But $J_k \subseteq J^k$ is equivalent to $F_k \subseteq k * F_1$. Therefore, to prove J is normal, it suffices to show that $F_k \subseteq k * F_1$ for all $k \in \mathbb{N}$.

Remark 4. Let $l = \operatorname{lcm}(a_1, \ldots, a_n)$ with $a_1 < \cdots < a_n < a_{n+1}$ are positive integers.

(1) Let $(c_1, \ldots, c_n, c_{n+1}) \in \Gamma'(a_1, \ldots, a_n, a_{n+1})$ and let $\delta = 1 - \sum_{i=1}^{n} \frac{c_i}{a_i}$ or $\delta = 0$ according as $\sum_{i=1}^{n} \frac{c_i}{a_i} \leq 1$ or $\sum_{i=1}^{n} \frac{c_i}{a_i} > 1$. Then $(c_1, \ldots, c_n, c_{n+1} + l\delta) \in \Gamma'(a_1, \ldots, a_n, a_{n+1} + l)$.

(2) Let $(e_1, \ldots, e_n, e_{n+1}) \in \Gamma'(a_1, \ldots, a_n, a_{n+1} + l)$ and let $\delta = 1 - \sum_{i=1}^{n} \frac{e_i}{a_i}$ or $\delta = 0$ according as $\sum_{i=1}^{n} \frac{e_i}{a_i} \leq 1$ or $\sum_{i=1}^{n} \frac{e_i}{a_i} > 1$. Then $(e_1, \ldots, e_n, e_{n+1} - l\delta) \in \Gamma'(a_1, \ldots, a_n, a_{n+1})$.

Proof. (1) By Lemma 1, it suffices to show that $\sum_{i=1}^{n} \frac{c_i}{a_i} + \frac{c_{n+1} + l\delta}{a_{n+1} + l} < 1 + \frac{1}{a_{n+1} + l}$. By Remark 1, we have $c_{n+1} = \lceil a_{n+1}\delta \rceil < a_{n+1}\delta + 1$; thus, $c_{n+1} + l\delta < (a_{n+1} + l)\,\delta + 1$. Therefore, $\sum_{i=1}^{n} \frac{c_i}{a_i} + \frac{c_{n+1} + l\delta}{a_{n+1} + l} < \sum_{i=1}^{n} \frac{c_i}{a_i} + \delta + \frac{1}{a_{n+1} + l} = 1 + \frac{1}{a_{n+1} + l}$, as required.

(2) By Lemma 1, it suffices to show that $\sum_{i=1}^{n} \frac{e_i}{a_i} + \frac{e_{n+1} - l\delta}{a_{n+1}} < 1 + \frac{1}{a_{n+1}}$. By Remark 1, we have $e_{n+1} = \lceil (a_{n+1} + l)\,\delta \rceil < (a_{n+1} + l)\,\delta + 1$; thus, $e_{n+1} - l\delta < a_{n+1}\delta + 1$. Therefore, $\sum_{i=1}^{n} \frac{e_i}{a_i} + \frac{e_{n+1} - l\delta}{a_{n+1}} < \sum_{i=1}^{n} \frac{e_i}{a_i} + \delta + \frac{1}{a_{n+1}} = 1 + \frac{1}{a_{n+1}}$, as required. $\square$

Lemma 2. *Let $F_1 = \Gamma'(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ and let*

$$(b_1, \ldots, b_n, b_{n+1}) := \sum\nolimits_j k_j(c_{1,j}, \ldots, c_{n,j}, c_{n+1,j}) \in k * F_1, \qquad (1)$$

where $(c_{1,j}, \ldots, c_{n,j}, c_{n+1,j}) \in F_1$ and $k = \sum_j k_j$ with $k_j > 0$. Let $\delta_j = 1 - \sum_{i=1}^{n} \frac{c_{i,j}}{\alpha_i}$ or $\delta_j = 0$ according as $\sum_{i=1}^{n} \frac{c_{i,j}}{\alpha_i} \leq 1$ or $\sum_{i=1}^{n} \frac{c_{i,j}}{\alpha_i} > 1$. Also, let $\delta = 1 - \sum_{i=1}^{n} \frac{b_i}{k\alpha_i}$ or $\delta = 0$ according as $\sum_{i=1}^{n} \frac{b_i}{k\alpha_i} \leq 1$ or $\sum_{i=1}^{n} \frac{b_i}{k\alpha_i} > 1$. Then

$$k\delta \leq \sum\nolimits_j k_j\delta_j.$$

In particular,

$$k\delta = \sum_j k_j\delta_j \iff \sum_{i=1}^{n} \frac{c_{i,j}}{\alpha_i} \leq 1 \textit{ for every } j.$$

Proof. By (1) note $b_i = \sum_j k_j c_{i,j}$ for $i = 1, \ldots, n$. Consider

$$k\delta = k\left(1 - \sum_{i=1}^{n} \frac{\sum_j k_j c_{i,j}}{k\alpha_i}\right) = k - \sum_j \left(k_j \sum_{i=1}^{n} \frac{c_{i,j}}{\alpha_i}\right) = \sum_j k_j \left(1 - \sum_{i=1}^{n} \frac{c_{i,j}}{\alpha_i}\right). \quad (2)$$

But the definition of δ_j implies that $1 - \sum_{i=1}^{n} \frac{c_{i,j}}{\alpha_i} \leq \delta_j$ for all j, therefore $k\delta \leq \sum_j k_j\delta_j$. To prove the second conclusion, note that if $k\delta < \sum_j k_j\delta_j$, then (2) implies that $1 - \sum_{i=1}^{n} \frac{c_{i,j_t}}{\alpha_i} < \delta_{j_t}$ for some j_t, which in turn implies that $\sum_{i=1}^{n} \frac{c_{i,j_t}}{\alpha_i} > 1$. Conversely, if $1 < \sum_{i=1}^{n} \frac{c_{i,j_t}}{\alpha_i}$ for some j_t, then $1 - \sum_{i=1}^{n} \frac{c_{i,j_t}}{\alpha_i} < \delta_{j_t} = 0$; hence, $k\delta = \sum_j k_j \left(1 - \sum_{i=1}^{n} \frac{c_{i,j}}{\alpha_i}\right) < \sum_j k_j\delta_j$. ☐

Now, using the above two lemmas and remark, we give a direct and simpler proof for Theorem 5.1 of [7]. Let $L = \mathbf{I}(a_1, \ldots, a_n, a_{n+1} + l)$ and $J = \mathbf{I}(a_1, \ldots, a_n, a_{n+1})$, where $l = \text{lcm}(a_1, \ldots, a_n)$.

Corollary 1. *If L is normal, then J is so.*

Proof. Let $F_k = \Gamma'(ka_1, \ldots, ka_n, ka_{n+1})$ and $H_k = \Gamma'(ka_1, \ldots, ka_n, k(a_{n+1} + l))$. Our goal is to show that $F_k \subseteq k * F_1$. Let $\gamma = (b_1, \ldots, b_n, b_{n+1}) \in F_k$. We may assume $b_{n+1} > 0$. By Remark 4, $(b_1, \ldots, b_n, b_{n+1} + kl\delta) \in H_k$, where $\delta = 1 - \sum_{i=1}^{n} \frac{b_i}{ka_i} > 0$. Since L is normal, then $H_k \subseteq k * H_1$; thus, we get

$$(b_1, \ldots, b_n, b_{n+1} + kl\delta) = \sum_j k_j(e_{1,j}, \ldots, e_{n,j}, e_{n+1,j}),$$

with $(e_{1,j}, \ldots, e_{n,j}, e_{n+1,j}) \in H_1$ and $k = \sum_j k_j$. Note $b_{n+1} + kl\delta = \sum_j k_j e_{n+1,j}$. By the first conclusion of Lemma 2, we have $k\delta \leq \sum_j k_j\delta_j$, where $\delta_j = 1 - \sum_{i=1}^{n} \frac{e_{i,j}}{a_i}$ or $\delta_j = 0$ according as $\sum_{i=1}^{n} \frac{e_{i,j}}{\alpha_i} \leq 1$ or $\sum_{i=1}^{n} \frac{e_{i,j}}{\alpha_i} > 1$; thus, $b_{n+1} = \sum_j k_j e_{n+1,j} - kl\delta \geq \sum_j k_j (e_{n+1,j} - l\delta_j)$. Now, Remark 4 gives that $(e_{1,j}, \ldots, e_{n,j}, e_{n+1,j} - l\delta_j) \in F_1$ for every j; thus,

$$\gamma = (b_1, \ldots, b_n, b_{n+1}) \geq_{lex} \sum_j k_j(e_{1,j}, \ldots, e_{n,j}, e_{n+1,j} - l\delta_j) \in k * F_1,$$

where $\geq_{lex}$ is the lexicographical order, which gives that J is normal. ☐

The proof of the converse direction of the above corollary is analogue to the proof of the corollary itself but with interchanging addition and subtraction of the last coordinates of the $(n+1)$-tuples. But in order to prove this converse direction, we need the following lemma.

Lemma 3. *Let* $l = \mathrm{lcm}(\alpha_1, \ldots, \alpha_n)$ *and let* $\alpha_{n+1} \geq l$. *Let* $E_k = \Gamma'(k\alpha_1, \ldots, k\alpha_n, k\alpha_{n+1})$. *Assume that* $\mathbf{I}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ *is normal, that is,* $E_k \subseteq k * E_1$. *Let* $\gamma \in E_k$ *and write*

$$\gamma := (b_1, \ldots, b_n, b_{n+1}) = \sum_j k_j(e_{1,j}, \ldots, e_{n,j}, e_{n+1,j}) \in k * E_1,$$

where $(e_{1,j}, \ldots, e_{n,j}, e_{n+1,j}) \in E_1$ *and* $k = \sum_j k_j$ *with* $k_j > 0$. *Then*

$$b_{n+1} > 0 \implies \sum_{i=1}^{n} \frac{e_{i,j}}{\alpha_i} \leq 1 \quad \textit{for all} \quad j.$$

Proof. Since $\gamma = (b_1, \ldots, b_n, b_{n+1}) \in E_k$, then by Remark 1 we have

$$b_{n+1} = \left\lceil k\alpha_{n+1}\left(1 - \sum_{i=1}^{n} \frac{b_i}{k\alpha_i}\right)\right\rceil.$$

Assume $\sum_{i=1}^{n} \frac{e_{i,j_t}}{\alpha_i} > 1$ for some j_t; then $1 - \sum_{i=1}^{n} \frac{e_{i,j_t}}{\alpha_i} \leq \frac{-1}{l}$ and $e_{n+1,j_t} = 0$ according to Remark 1. Therefore, and since $\alpha_{n+1} \geq l$, we have

$$\alpha_{n+1}\left(1 - \sum_{i=1}^{n} \frac{e_{i,j_t}}{\alpha_i}\right) \leq l\left(1 - \sum_{i=1}^{n} \frac{e_{i,j_t}}{\alpha_i}\right) \leq -1. \tag{3}$$

For any j let $\delta_j = 1 - \sum_{i=1}^{n} \frac{e_{i,j}}{\alpha_i}$ or $\delta_j = 0$ according as $\sum_{i=1}^{n} \frac{e_{i,j}}{\alpha_i} \leq 1$ or $\sum_{i=1}^{n} \frac{e_{i,j}}{\alpha_i} > 1$. This implies that $\delta_j \geq 1 - \sum_{i=1}^{n} \frac{e_{i,j}}{\alpha_i}$. Note $\delta_{j_t} = 0$ and consider

$$\begin{aligned}
\sum_j k_j e_{n+1,j} &= \sum_j k_j \lceil \alpha_{n+1}\delta_j \rceil \geq \left\lceil \sum_j k_j \alpha_{n+1}\delta_j \right\rceil = \left\lceil \sum_{j \neq j_t} k_j \alpha_{n+1}\delta_j \right\rceil \\
\text{by (3)} &> \left\lceil k_{j_t}\alpha_{n+1}\left(1 - \sum_{i=1}^{n} \frac{e_{i,j_t}}{\alpha_i}\right) + \sum_{j \neq j_t} k_j \alpha_{n+1}\left(1 - \sum_{i=1}^{n} \frac{e_{i,j}}{\alpha_i}\right)\right\rceil \\
&= \left\lceil \sum_j k_j \alpha_{n+1}\left(1 - \sum_{i=1}^{n} \frac{e_{i,j}}{\alpha_i}\right)\right\rceil = \left\lceil k\alpha_{n+1}\left(1 - \sum_{i=1}^{n} \frac{\sum_j k_j e_{i,j}}{k\alpha_i}\right)\right\rceil \\
&= \left\lceil k\alpha_{n+1}\left(1 - \sum_{i=1}^{n} \frac{b_i}{= k\alpha_i}\right)\right\rceil = b_{n+1},
\end{aligned}$$

giving a contradiction to the hypothesis that $b_{n+1} = \sum_j k_j e_{n+1,j}$. □

Corollary 2. *If* J *is normal and* $a_{n+1} \geq l$, *then* L *is so.*

Proof. Let $H_k = \Gamma'(ka_1, \ldots, ka_n, k\,(a_{n+1} + l))$ and $F_k = \Gamma'(ka_1, \ldots, ka_n, ka_{n+1})$. Our goal is to show that $H_k \subseteq k{*}H_1$. Let $\gamma = (b_1, \ldots, b_n, b_{n+1}) \in H_k$. We may assume $b_{n+1} > 0$. By Remark 4, $(b_1, \ldots, b_n, b_{n+1} - kl\delta) \in F_k$, where $\delta = 1 - \sum_{i=1}^{n} \dfrac{b_i}{ka_i} > 0$. Since J is normal, then $F_k \subseteq k * F_1$; thus, we have

$$(b_1, \ldots, b_n, b_{n+1} - kl\delta) = \textstyle\sum_j k_j(c_{1,j}, \ldots, c_{n,j}, c_{n+1,j}),$$

with $(c_{1,j}, \ldots, c_{n,j}, c_{n+1,j}) \in F_1$ and $k = \sum_j k_j$. Note that $b_{n+1} = \sum_j k_j c_{n+1,j} + kl\delta$. By Lemma 3, we have $\sum_{i=1}^{n} \dfrac{c_{i,j}}{a_i} \leq 1$ for all j. Hence by the last conclusion of Lemma 2, we get $k\delta = \sum_j k_j \delta_j$ where $\delta_j = 1 - \sum_{i=1}^{n} \dfrac{c_{i,j}}{a_i} \geq 0$; thus, $b_{n+1} = \sum_j k_j\,(c_{n+1,j} + l\delta_j)$. Now, Remark 4 gives that $(c_{1,j}, \ldots, c_{n,j}, c_{n+1,j} + l\delta_j) \in H_1$ for every j; thus,

$$\gamma = (b_1, \ldots, b_n, b_{n+1}) = \textstyle\sum_j k_j(c_{1,j}, \ldots, c_{n,j}, c_{n+1,j} + l\delta_j) \in k * H_1,$$

which gives that L is normal. □

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