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# On the LS-category of homomorphisms of groups with torsion

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Abstract. We prove the equality $\mathrm{cat}(\phi) = \mathrm{cd}(\phi)$ for homomorphisms $\phi : \Gamma \to \Lambda$ of any finitely generated abelian group $\Gamma$. In addition, we prove that the Lusternik-Schnirelmann category and the cohomological dimension of any nonzero homomorphism of a torsion group cannot be finite.

## 1. Introduction

The (reduced) *Lusternik-Schnirelmann category* (for short, LS-category), $\mathrm{cat}(X)$, of a topological space $X$ is the minimal number $k$ such that there is an open cover $\{U_0, U_1, \dots, U_k\}$ of $X$ by $k+1$ contractible in $X$. The LS-category gives a lower bound on the number of critical points for a smooth real-valued function on a closed manifold [13], [3].

Since it is a homotopy invariant, it can be defined for discrete groups $\Gamma$ as $\mathrm{cat}(\Gamma) = \mathrm{cat}(B\Gamma)$ where $B\Gamma = K(\Gamma, 1)$ is a classifying space. Computation of the LS-category of spaces presents a great challenge even spaces are nice such as manifolds [8]. In the 50s Eilenberg and Ganea [9] proved that the LS-category of a discrete group equals its cohomological dimension, $\mathrm{cat}(\pi) = \mathrm{cd}(\pi)$. We recall that the cohomological dimension of a group $\Gamma$ is defined as follows,

$$\mathrm{cd}(\Gamma) = \sup\{k \mid H^k(\Gamma, M) \neq 0\}$$

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where $M$ is some $\mathbb{Z}\Gamma$-module [2]. From the work of Dranishnikov and Rudyak [7] it follows that

$$\mathrm{cd}(\Gamma) = \max\{k \mid (\beta_\Gamma)^k \neq 0\}$$

where $\beta_\Gamma \in H^1(\Gamma, I(\Gamma))$ is the Berstein-Schwarz class of $\Gamma$ [1].

The *LS-category of the map* $f : X \to Y$, $\mathrm{cat}(f)$, is the minimal number $k$ such that X admits an open cover by $k+1$ open sets $U_0, U_2, ..., U_k$ with nullhomotopic restrictions $f|_{U_i} : U_i \to Y$ for all $i$. The LS-category $\mathrm{cat}(\phi)$ of a group homomorphism $\phi : \Gamma \to \pi$ is defined as $\mathrm{cat}(f)$ where the map $f : B\Gamma \to B\pi$ induces the homomorphism $\phi$ for the fundamental groups.

The *cohomological dimension* $\mathrm{cd}(\phi)$ of a group homomorphism $\phi : \Gamma \to \Lambda$ was introduced by Mark Grant [10] as the maximum of $k$ such that there is a $\mathbb{Z}\Lambda$-module $M$ with the nonzero induced homomorphism $\phi^* : H^k(\Lambda, M) \to H^k(\Gamma, M)$. In view of universality of the Berstein-Schwarz class [7], for any homomorphism $\phi : \Gamma \to \Lambda$

$$\mathrm{cd}(\phi) = \max\{k \mid \phi^*(\beta_\Lambda)^k \neq 0\}.$$

This together with the cup-length lower bound for the LS-category brings the inequality $\mathrm{cd}(\phi) \leqslant \mathrm{cat}(\phi)$ for all group homomorphisms.

In view of the Eilenberg-Ganea equality $\mathrm{cd}(\Gamma) = \mathrm{cat}(\Gamma)$, the following conjecture seems to be natural:

**Conjecture 1.1.** For any group homomorphism $\phi : \Gamma \to \Lambda$ always $\mathrm{cat}(\phi) = \mathrm{cd}(\phi)$.

In [15] Jamie Scott considered this conjecture for geometrically finite groups and he proved it for monomorphisms of any groups and for homomorphisms of free and free abelian groups. In [10] Tom Goodwillie gave an example of an epimorphism of an infinitely generated group $\phi : G \to \mathbb{Z}^2$ with $\mathrm{cd}(\phi) = 1$ that disproves the conjecture.

In the joint paper with Dranishnikov [6] we reduced the conjecture from arbitrary homomorphisms to epimorphisms and we gave a counterexample to the conjecture with epimorphism between geometrically finite groups. Also, we proved the conjecture for epimorphisms between finitely generated, torsion-free nilpotent groups. It is a natural question to ask if one can remove the torsion-free restriction in our result. In this paper we do it in the cases abelian groups.

**Theorem 1.2** (Theorem 4.4). *Let $\phi : \Gamma \to \Lambda$ be any homomorphism between finitely generated abelian groups. Then* $\mathrm{cat}(\phi) = \mathrm{cd}(\phi)$.

Moreover, in the above case the number $\operatorname{cat}(\phi)$ can explicitely computed. For that one needs to present the above homomorphism as a direct sum $\phi = \phi_1 \oplus \phi_2$ where $\phi_1$ is a homomorphism between free abelian groups and $\phi_2$ is from free abelian to finite abelian group $T(\Lambda) = Torsion(\Lambda)$. Then $\operatorname{cat}(\phi) = \operatorname{cd}(\phi) = rank(\phi_1) + k(T(\Lambda))$, where $k(T(\Lambda))$ is the Smith Normal number of $T(\Lambda)$.

Dealing with torsion we investigated a satellite question whether it is possible to have a homomorphism of finite groups $\phi : \Gamma \to \Lambda$ with $\operatorname{cd}(\phi) < \infty$. We give a negative answer to this question in the following theorem.

**Theorem 1.3** (Theorem 3.2)**.** *Let $\phi : G \to H$ be a nonzero homomorphism of a torsion group $G$. Then $\operatorname{cd}(\phi) = \infty$. In particular, $\operatorname{cat}(\phi) = \operatorname{cd}(\phi)$.*

## 2. Preliminaries

In this section we recall some classic theorems used in the paper.

Given positive integers $m, n \geqslant 1$, we denote by $M_{m\times n}(\mathbb{Z})$the set of $m \times n$ matrices with integer entries.

**Theorem 2.1** (The Smith Normal Form)**.** *Given a nonzero matrix $A \in M_{m\times n}(\mathbb{Z})$, there exist invertible matrices $P \in M_{m\times m}(\mathbb{Z})$ and $Q \in M_{n\times n}(\mathbb{Z})$ such that*

$$PAQ = \begin{bmatrix} n_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & n_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

*where the integer $n_i \geqslant 1$ are unique up to sign and satisfy $n_1|n_2|\cdots|n_k$. Further, one can compute the integers $n_i$ by the recursive formula $n_i = \dfrac{d_i}{d_{i-1}}$, where $d_i$ is the greatest common divisor of all $i \times i$-minors of the matrix $A$ and $d_0$ is defined to be 1.*

The proof of Theorem 2.1 can be found in [12, Proposition 2.11, p. 339].

**Corollary 2.2.** *Given a matrix $A \in M_{n\times n}(\mathbb{Z})$ with $det(A) \neq 0$, there exist invertible matrices $P \in M_{n\times n}(\mathbb{Z})$ and $Q \in M_{n\times n}(\mathbb{Z})$ such that*

$$PAQ = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & n_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & n_k \end{bmatrix}$$

*where the integer $n_i \geqslant 2$ are unique up to sign and satisfy $n_1|n_2|\cdots|n_k$.*

**Theorem 2.3** (Invariant Factor Decomposition (IDF) for Finite Abelian Groups)**.** *Every finite abelian group $G$ can be written uniquely as $G = Z_{n_1} \times ... \times Z_{n_k}$ where the integers $n_i \geqslant 2$ are the invariant factors of $G$ that satisfy $n_1|n_2|...|n_k$ and $\mathbb{Z}_{n_i}$ are cyclic group of order $n_i, i = 1, \cdots, k$.*

The proof of Theorem 2.3 can be found in [4, Theorem 3, p. 158]. Alternatively, one can apply Corollary 2.2 and get the result.

**Definition 2.4.** Given a finite abelian group $G$, the Smith Normal number $k(G)$ of $G$ is the number $k$ from Theorem 2.3.

In the proof of our main result about homomorphism between finite groups we use Shapiro's Lemma [2, Proposition 6.2, p. 73].

**Theorem 2.5** ("Shapiro's Lemma")**.** *If $i : H \to G$ is a monomorphism and $M$ is an $H$-module, then the through homomorphism*

$$H^*(G, Coind_H^G M) \xrightarrow{i^*} H^*(H, Coind_H^G M) \xrightarrow{\alpha_*} H^*(H, M)$$

*is an isomorphism, where $Coind_H^G M = Hom_{\mathbb{Z}H}(\mathbb{Z}G, M)$ and the homomorphism of coefficients $\alpha : Hom_{\mathbb{Z}H}(\mathbb{Z}G, M) \to M$ is defined as $\alpha(f) = f(e)$.*

In the paper we use the notation $H^*(\Gamma, A)$ for the cohomology of a group $\Gamma$ with coefficient in a $\Gamma$-module $A$. The cohomology groups of a space $X$ with the fundamental group $\Gamma$ we denote as $H^*(X; A)$. Thus, $H^*(\Gamma, A) = H^*(B\Gamma; A)$ where $B\Gamma = K(\Gamma, 1)$.

We say that a CW-complex $X$ is of *finite type* if each $n$-skeleton $X^{(n)}$ of $X$ is finite.

The Kunneth Formula Theorem for cohomology of product of two spaces with field coefficient $F$ can be founded in Spanier [16, Theorem 5.5.11] or Dold [5, Proposition VI.12.16]. We state a special case, which will be used in the paper.

**Theorem 2.6** (The Kunneth Formula Theorem). *Let $F$ be a field. If the CW-complexes $X$, $Y$ are of finite type, then the cross product*

$$\bigoplus_k H^k(X;F) \otimes H^{n-k}(Y;F) \xrightarrow{\times} H^n(X \times Y;F)$$

*is an isomorphism.*

We recall the Universal Coefficient Formula (UCF) for cohomology (see [16, Theorem 5.5.10]).

**UCF**: *Let the CW-complex $X$ be of finite type and a coefficient group $G$, then for each $n$ there is the short exact sequence*

$$0 \to H^n(X) \otimes G \to H^n(X;G) \to Tor(H^{n+1}(X), G) \to 0,$$

*which splits.*

## 3. Finite Groups

A group $G$ is called a *torsion group* if every element $g \in G$ has finite order.

For the proof of the main result of this section (Theorem 3.2) we need the following easy lemma.

**Lemma 3.1.** *Let $\phi : G \to \mathbb{Z}_p$ be an epimorphism where $p$ is prime and $G$ is a torsion group. Then $G$ contains $\mathbb{Z}_{p^k}$ for some $k$ such that the restriction $\psi = \phi|_{\mathbb{Z}_{p^k}} : \mathbb{Z}_{p^k} \to \mathbb{Z}_p$ is surjective.*

*Proof.* Let $a$ be a generator of $\mathbb{Z}_p$. We pick $g \in G$ with $\phi(g) = a$. Then $g$ has a finite order $n$. Since $g^n = 0$, it follows that $\phi(g)^n = na = 0$. Hence, $n$ is divisible by $p$. Let $n = p^k m$ where $m$ is not divisible by $p$. Then the order of element $g^m$ is $p^k$. Thus, $g^m$ generates a subgroup $\mathbb{Z}_{p^k} \subset G$. Since $\phi(g^m) = ma \neq 0$ in $\mathbb{Z}_p$, the element $ma$ generates $\mathbb{Z}_p$. Therefore, $\psi$ is surjective, $\phi(\mathbb{Z}_{p^k}) = \mathbb{Z}_p$. □

**Theorem 3.2.** *Let $\phi : G \to H$ be a nonzero homomorphism of a torsion group $G$. Then $cat(\phi) = \mathrm{cd}(\phi) = \infty$.*

*Proof.* In view of Theorem 3.2 and Theorem 3.3 [6] we may assume that $\phi$ is an epimorphism.

We do the proof in two steps:

*Step 1.* Suppose that $H$ is a cyclic group of order $p$, say $H \cong \mathbb{Z}_p$, where $p$ is prime. Then by Lemma 3.1 the group $G$ contains $\mathbb{Z}_{p^k}$ for some $k$ such that $\psi = \phi|_{\mathbb{Z}_{p^k}}$ maps $\mathbb{Z}_{p^k}$ onto $\mathbb{Z}_p$.

We claim that $\mathrm{cd}(\psi) = \infty$ with $\mathbb{Z}$ coefficient. It suffices to show $\psi^* : H^n(B\mathbb{Z}_p; \mathbb{Z}) \to H^n(B\mathbb{Z}_{p^k}; \mathbb{Z})$ is not trivial for all even numbers $n$. The reduced integral cohomology groups of $B\mathbb{Z}_p$ and $B\mathbb{Z}_{p^k}$ are $\mathbb{Z}_p$ and $\mathbb{Z}_{p^k}$ respectively in even cases and zero in odd cases [11]. So, we consider the case of even $n$.

Since for any $i$ the CW-complex $B\mathbb{Z}_{p^i}$ can be taken to be the infinite lens space $S^\infty/\mathbb{Z}_{p^i}$, we may assume that the $(n+1)$-dimensional lens space $L^{n+1}_{p^i} = S^{n+1}/\mathbb{Z}_{p^i}$ is the $(n+1)$-skeleton of $B\mathbb{Z}_{p^i}$. Hence it suffices to show that the map $f : L^{n+1}_{p^k} \to L^{n+1}_p$ induces nonzero homomorphism

$$f^* : H^n(L^{n+1}_p; \mathbb{Z}) \to H^n(L^{n+1}_{p^k}; \mathbb{Z}).$$

First we note that the map $f$ has degree one. We may assume that each of our lens spaces $L^{n+1}_{p^i}$, $i = 1, k$ is the orbit space of a free $\mathbb{Z}_{p^i}$-action on the odd dimensional sphere $S^{n+1} = S^1 * \cdots * S^1$ which is presented as the join product of circles. We may assume the $\mathbb{Z}_{p^i}$-action takes place only on the first factor. Let $\pi_i : S^{n+1} \to L^{n+1}_{p^i}$ be corresponding maps. Let

$$\bar{f} : S^1 * \cdots * S^1 \to S^1 * \cdots * S^1$$

be defined as $\xi * 1 * \cdots * 1$ where $\xi : S^1 \to S^1$ is taking unit complex numbers to the $p^{k-1}$ power, $\xi : z \mapsto z^{p^{k-1}}$. The map $\bar{f}$ defines the map of the orbit spaces $f : L^{n+1}_{p^k} \to L^{n+1}_p$ with $f_* = \psi$ such that the following diagram

$$\begin{array}{ccc} S^{n+1} & \xrightarrow{\bar{f}} & S^{n+1} \\ \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi_1} \\ L^{n+1}_{p^k} & \xrightarrow{f} & L^{n+1}_p \end{array}$$

is commutative. The degrees of maps $\pi_i$ and $\bar{f}$ equal the degree of their restrictions to the first circle in $S^1 * \ldots S^1$ and they are $\deg(\pi_i) = p^i$, $\deg(\bar{f}) = p^{k-1}$. It follows from the equality

$$\deg(\pi_1)\deg(\bar{f}) = \deg(f)\deg(\pi_k)$$

that $\deg(f) = 1$.

For nonzero $\alpha \in H^n(L_p^{n+1};\mathbb{Z})$ by the Poincare Duality and the naturality of the cap product it follows that

$$0 \neq [L_p^{n+1}] \cap \alpha = f_*([L_{p^k}^{n+1}]) \cap \alpha = f_*([L_p^{n+1}] \cap f^*(\alpha))$$

where $[L_p^{n+1}]$ and $[L_{p^k}^{n+1}]$ are the fundamental classes. Thus, we obtain that $f^*(\alpha) \neq 0$.

Claim: If $\mathrm{cd}(\psi) = \infty$, then $\mathrm{cd}(\phi) = \infty$ where $\psi := \phi|_{\mathbb{Z}_{p^k}}$.

Suppose $\mathrm{cd}(\phi) = n < \infty$. Since $\phi^*(a) = 0$ for all $a \in H^{2n}(B\mathbb{Z}_p;\mathbb{Z})$, the restriction homomorphism $i^* : H^{2n}(BG;\mathbb{Z}) \to H^{2n}(B\mathbb{Z}_{p^k};\mathbb{Z})$ is not trivial, but $i^*(\phi^*(a)) = i^*(0) = 0$. Since $\psi^* = i^* \circ \phi^*$, we get $\psi^*(a) = 0$ for all $a \in H^{2n}(B\mathbb{Z}_p;\mathbb{Z})$. This is contradiction, hence we prove the claim.

*Step 2.* Let $H$ be an arbitrary torsion group. Pick a nonzero element $h \in H$ generating a cyclic group $\mathbb{Z}_p$ for some prime $p$. The preimage of the subgroup $\mathbb{Z}_p$ of $H$ is a torsion subgroup of $G$. We apply Lemma 3.1 to find a cyclic subgroup $\mathbb{Z}_{p^k} \subset G$ that maps onto $\mathbb{Z}_p$. We have the following commutative diagrams:

$$\begin{array}{ccccccc} G & \xrightarrow{\phi} & H & \quad & BG & \xrightarrow{B\phi} & BH \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathbb{Z}_{p^k} & \xrightarrow{\psi} & \mathbb{Z}_p & & B\mathbb{Z}_{p^k} & \xrightarrow{B\psi} & B\mathbb{Z}_p \end{array}$$

where $\phi$ and $\psi$ are the homomorphisms of the fundamental groups induced by the maps $B\phi$ and $B\psi$.

Let $\alpha : Coind_{\mathbb{Z}_p}^H \mathbb{Z} = Hom_{\mathbb{Z}\mathbb{Z}_p}(\mathbb{Z}H, \mathbb{Z}) \to \mathbb{Z}$ be the canonical $\mathbb{Z}\mathbb{Z}_p$-homomorphism from Theorem 2.5. Consider the following commutative diagram

$$\begin{array}{ccccc} H^*(BH; Coind_{\mathbb{Z}_p}^H \mathbb{Z}) & \xrightarrow{i^*} & H^*(B\mathbb{Z}_p; Coind_{\mathbb{Z}_p}^H \mathbb{Z}) & \xrightarrow{\alpha_*} & H^*(B\mathbb{Z}_p;\mathbb{Z}) \\ \downarrow \phi^* & & \downarrow \psi^* & & \downarrow \psi^* \\ H^*(BG; Coind_{\mathbb{Z}_p}^H \mathbb{Z}) & \xrightarrow{i^*} & H^*(B\mathbb{Z}_{p^k}; Coind_{\mathbb{Z}_p}^H \mathbb{Z}) & \xrightarrow{\alpha_*} & H^*(B\mathbb{Z}_{p^k};\mathbb{Z}) \end{array}$$

where $\alpha_*$ is the coefficient homomorphism generated by $\alpha$.

We claim that $\mathrm{cd}(\phi) = \infty$. Assume the contrary, $\mathrm{cd}(\phi) < n$ for some even number $n$. By *Step 1,* $\psi$ is a nonzero homomorphism in dimension

$n$. We pick a nonzero element $a \in H^n(B\mathbb{Z}_p;\mathbb{Z})$ with $\psi^*(a) \neq 0$. By Theorem 2.5, the top row through homomorphism

$$\alpha_* i^* : H^*(BH; Coind^H_{\mathbb{Z}_p}\mathbb{Z}) \to H^*(B\mathbb{Z}_p;\mathbb{Z})$$

is an isomorphism. Let $a = \alpha_* i^*(b)$. Since by the assumption $\phi^*(b) = 0$, we obtain a contradiction:

$$0 \neq \psi^*(a) = \psi^*\alpha_* i^*(b) = \alpha_* i^* \phi^*(b) = \alpha_* i^*(0) = 0.$$

□

## 4. Finitely Generated Abelian Groups

**Lemma 4.1.** *Given an epimorphism $\phi : \mathbb{Z}^n \to G$ with a finite group $G$ there is an epimorphism $\pi : \mathbb{Z}^n \to \mathbb{Z}^k$ such that $\phi = \psi \circ \pi$ where $k = k(G)$ is the Smith normal number for $G$,*

$$\psi = \prod_{i=1}^{k} (\psi_i : \mathbb{Z} \to \mathbb{Z}_{n_i}),$$

*and the numbers $n_1 | \dots | n_k$ are taken from IFD for $G$ from Theorem 2.3.*

*Proof.* Being a subgroup of $\mathbb{Z}^n$, the kernel $\ker\phi$ is a free abelian group. Since $G$ is finite, $\ker\phi$ is isomorphic to $\mathbb{Z}^n$. We fix a basis in $\ker\phi$. Let $A : \mathbb{Z}^n \to \ker\phi$ be an isomorphism. We regard $A : \mathbb{Z}^n \to \mathbb{Z}^n$ as the embedding. Then $A$ is given by $n \times n$ matrix the columns of which form our basis. We apply Corollary 2.2 (Smith Normal Form) to get matrices $Q$ and $P$ that change in a special way the bases in the domain of $A$ and the range of $\phi$ respectively. Thus, $AQ(\mathbb{Z}^n) = A(\mathbb{Z}^n) = \ker\phi$. Then $PAQ(\mathbb{Z}^n) = \ker(\phi P^{-1})$. Then

$$G \cong (\phi P^{-1})(\mathbb{Z}^n) = \mathbb{Z}^n \Big/ \ker(\phi P^{-1}) = \mathbb{Z}^n \Big/ PAQ(\mathbb{Z}^n) =$$

$$= \left(\mathbb{Z}^{n-k} \Big/ \langle 1 \rangle \times \cdots \times \langle 1 \rangle\right) \times \left(\mathbb{Z}^k \Big/ \langle n_1 \rangle \times \langle n_2 \rangle \times \cdots \times \langle n_k \rangle\right) =$$

$$= \left(\mathbb{Z}\big/\mathbb{Z} \times \cdots \mathbb{Z}\big/\mathbb{Z}\right) \times \left(\mathbb{Z}\big/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}\big/n_k\mathbb{Z}\right) =$$

$$= pr_k(\mathbb{Z}^n) \Big/ \langle n_1 \rangle \times \cdots \times \langle n_k \rangle = \psi pr_k(\mathbb{Z}^n)$$

where $pr_k : \mathbb{Z}^n \to \mathbb{Z}^k$ is the projection onto the last k coordinates. Thus, $\phi P^{-1} = \psi pr_k$. Then $\phi = \psi\pi$ with $\pi = pr_k P$. □

**Lemma 4.2.** *Let* $\phi : \mathbb{Z}^n \to G$ *be an epimorphism, where* $G$ *is a finite abelian group. Then* $\mathrm{cat}(\phi) = \mathrm{cd}(\phi)$*. In particular,* $\mathrm{cat}(\phi) = \mathrm{cd}(\phi) = k(G)$ *where* $k$ *is the Smith Normal number for given a finite abelian group* $G$*.*

*Proof.* Since $\mathrm{cd}(\phi) \leqslant \mathrm{cat}(\phi)$ for any group homomorphism [6], we just need to show two inequalities, i.e $\mathrm{cat}(\phi) \leqslant k(G)$ and $k(G) \leqslant \mathrm{cd}(\phi)$. Then, observing the chain inequalities $k(G) \leqslant \mathrm{cd}(\phi) \leqslant \mathrm{cat}(\phi) \leqslant k(G)$, we obtain the conclusion of Lemma 4.2.

Let us show the first inequality $\mathrm{cat}(\phi) \leqslant k(G)$. Let $k = k(G)$. By Lemma 4.1, there exists an epimorphisms $\pi : \mathbb{Z}^n \to \mathbb{Z}^k$ and $\psi : \mathbb{Z}^k \to G$ such that we have the following commutative diagram:

$$\begin{array}{ccccc} & & \mathbb{Z}^n & & \\ & \swarrow^{\pi} & \downarrow{\scriptstyle \phi} & & \\ \mathbb{Z}^k & \xrightarrow{\psi} & G & \longrightarrow & 0. \end{array}$$

Using well-known facts on the LS-category cat [3], we obtain:

$$\mathrm{cat}(\phi) \leqslant \min\{\mathrm{cat}(\psi), \mathrm{cat}(\pi))\} \leqslant \mathrm{cat}(\psi) \leqslant \min\{\mathrm{cat}(T^k),$$

$$\mathrm{cat}(BG)\} \leqslant \mathrm{cat}(T^k) = k$$

where $T^k$ is the $k$ dimensional torus.

Since $B\pi : T^n \to T^k$ is a retraction, $\pi$ is injective on cohomology, so we have $\mathrm{cd}(\phi) = \mathrm{cd}(\psi)$. Then to prove the second inequality $k(G) \leqslant \mathrm{cd}(\phi)$, it suffices to show $k(G) \leqslant \mathrm{cd}(\psi)$.

We do it by induction on $k = k(G)$. When k=1 we have $G = \mathbb{Z}_{n_1}$. Then the homomorphism $\psi^* = \psi_1^* : H^1(B\mathbb{Z}_{n_1}; Z_{n_1}) \to H^1(B\mathbb{Z}; Z_{n_1})$ is nonzero, since $\psi : \mathbb{Z} \to \mathbb{Z}_{n_1}$ is surjective.

Suppose the result holds true for all $l \leqslant k$. First we note that by Theorem 2.3 the group $G$ for $k(G) = k + 1$ is written uniquely as $G = \mathbb{Z}_{n_1} \times \cdots \mathbb{Z}_{n_{k+1}}$ with $n_1 | \cdots | n_{k+1}$. Note also that $BG$ can be presented as the product $B\mathbb{Z}_{n_1} \times \cdots \times B\mathbb{Z}_{n_{k+1}}$. Let $p$ be a prime that divides $n_1$ and, hence, all $n_i$. We show that the induced homomorphism

$$\psi^* : H^{k+1}(B\mathbb{Z}_{n_1} \times ... \times B\mathbb{Z}_{n_{k+1}}; \mathbb{Z}_p) \to H^{k+1}(T^{k+1}; \mathbb{Z}_p)$$

is a nonzero homomorphism.

It is known that the integral cohomology groups $H^j(B\mathbb{Z}_m;\mathbb{Z})$ are $\mathbb{Z}_m$ if $j$ is even and zero otherwise [11]. Note that for prime $p$ dividing $m$ by the Universal Coefficient Formula $H^j(B\mathbb{Z}_m;\mathbb{Z}_p) = \mathbb{Z}_p$ for all $j$, since $\mathbb{Z}_m \otimes \mathbb{Z}_p = \mathbb{Z}_p$ and $\mathrm{Tor}(\mathbb{Z}_m,\mathbb{Z}_p) = \mathbb{Z}_p$. Thus, for prime $p$ dividing $n_1$ we obtain $H^j(B\mathbb{Z}_{n_i};\mathbb{Z}_p) = \mathbb{Z}_p$ for all $i$ and $j$. Since for each $n_i$ the CW-complex $B\mathbb{Z}_{n_i}$ are of finite type, we can apply the Kunneth Formula 2.6. By the Kunneth Formula with a field coefficient $\mathbb{Z}_p$, and induction, we get that $H^j(B(\mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_{k+1}});\mathbb{Z}_p)$ is not zero for all $j$. Clearly for the $(k+1)$-torus $T^{k+1}$ we have $H^{k+1}(T^{k+1};\mathbb{Z}_p) = \mathbb{Z}_p$. Using commutative diagram below, we get that $\psi$ is a nonzero homomorphism for the mod $p$ cohomology in dimension $k+1$.

$$\begin{array}{ccc} H^k(B\mathbb{Z}_{n_1} \times ... \times B\mathbb{Z}_{n_k});\mathbb{Z}_p) \otimes H^1(B\mathbb{Z}_{n_{k+1}};\mathbb{Z}_p) & \xrightarrow{\times} & H^{k+1}(B\mathbb{Z}_{n_1} \times ... \times B\mathbb{Z}_{n_{k+1}});\mathbb{Z}_p) \\ \downarrow{\scriptstyle \psi^*\otimes\psi^*} & & \downarrow{\scriptstyle \psi^*} \\ H^k(T^k;\mathbb{Z}_p) \otimes H^1(S^1;\mathbb{Z}_p) & \xrightarrow{\times} & H^{k+1}(T^{k+1};\mathbb{Z}_p) \end{array}$$

Indeed, the horizontal maps are isomorphism, by the Kunneth Theorem. Thus the Kunneth Formula isomorphism takes the tensor product to the cross product, $a \otimes b \xrightarrow{\times} a \times b$. Here the cross product is defined as $a \times b = p_1^*(a) \cup p_2^*(b)$ where $p_1$ and $p_2$ are the projections of the product $X \times Y$ onto $X$and $Y$ respectively. Using the naturality of the cup product and the induction assumption, we obtain:

$$\psi^*(a \times b) = \psi^*(p_1^*(a) \cup p_2^*(b)) = (\psi^* \circ p_1^*)(a) \cup (\psi^* \circ p_2^*)(b) =$$

$$= \psi^*(a) \otimes \psi^*(b) \neq 0$$

Hence, $\mathrm{cd}(\psi) = k + 1$. □

We use the notation $T(A)$ for the torsion subgroup of an abelian group $A$. For finitely generated abelian groups we define the rank $rank(A) = rank(A/T(A))$.

**Lemma 4.3.** *Every epimorphism $\phi : \mathbb{Z}^n \to \mathbb{Z}^m \oplus G$ splits as the direct sum*

$$\phi = \psi_1 \oplus \psi_2 : \mathbb{Z}^m \oplus \mathbb{Z}^{n-m} \to \mathbb{Z}^m \oplus G.$$

*Proof.* The epimorphism $\phi$ as a map to the product is defined by coordinate functions, $\phi = (\phi_1, \phi_2)$ which are also epimorphisms. There exists a section $s : \mathbb{Z}^m \to \mathbb{Z}^n$ of the epimorphism $\phi_1$, since $\mathbb{Z}^m$ is free abelian. We show that $\mathbb{Z}^n$ splits as the direct sum $s(\mathbb{Z}^m) \oplus \phi^{-1}(G)$. For each element $x \in \mathbb{Z}^n$ we have $\phi_1(x - s\phi_1(x)) = \phi_1(x) - \phi_1 s(\phi_1(x)) = 0$. Therefore,

$x - s\phi_1(x) \in \phi^{-1}(G)$. Thus, every element $x \in \mathbb{Z}^n$ can be written as $s\phi_1(x) + (x - s\phi_1(x))$. Suppose that $y \in s(\mathbb{Z}^m) \cap \phi^{-1}(G)$. Since $\phi(y) \in G$, we obtain $\phi_1(y) = 0$. Since $y \in s(\mathbb{Z}^m)$, we have $s\phi_1(y) = y$. Hence $y = 0$ and the sum $s(\mathbb{Z}^m) + \phi^{-1}(G) = \mathbb{Z}^n$ is a direct sum.

Note that $s(\mathbb{Z}^m) \cong \mathbb{Z}^m$. In view of the splitting $\mathbb{Z}^n = \mathbb{Z}^m \oplus \phi^{-1}(G)$ it follows that $rank(\phi^{-1}(G)) = n - m$. We define $\psi_1$ and $\psi_2$ to be the restrictions of $\phi_1$ and $\phi_2$ to corresponding direct summands. i.e., $\psi_1 := \phi_1|_{s(\mathbb{Z}^m)}$ and $\psi_2 := \phi_2|_{\phi^{-1}(G)}$. □

**Theorem 4.4.** *Let $\phi : \Gamma \to \Lambda$ be an epimorphism between finitely generated abelian groups. Then*

$$\mathrm{cat}(\phi) = \mathrm{cd}(\phi).$$

*If $\Gamma$ is torsion free, then*

$$\mathrm{cat}(\phi) = \mathrm{cd}(\phi) = rank(\Lambda) + k(T(\Lambda)),$$

*where $k(T(\Lambda))$ is the Smith Normal number of $T(\Lambda)$.*

*Proof.* Since the groups $\Gamma$ and $\Lambda$ are finitely generated abelian groups, we may assume $\Gamma = \mathbb{Z}^n \oplus T(\Gamma)$ and $\Lambda = \mathbb{Z}^m \oplus T(\Lambda)$ for some $m$ and $n$.

By Theorem 3.2, if both groups $\Gamma$ have torsion and $\phi(T(\Gamma)) \neq 0$, then $\mathrm{cd}(\phi) = \infty$, so $\mathrm{cat}(\phi) = \mathrm{cd}(\phi)$. Thus, we consider $\phi$ with $\phi(T(\Lambda)) = 0$. Such $\phi$ factors through the epimorphism $\bar{\phi} : \mathbb{Z}^n \to \Lambda$. In view of the retraction $B\Gamma \to B\mathbb{Z}^n$ we obtain that $\mathrm{cd}(\bar{\phi}) = \mathrm{cd}(\phi)$ and $\mathrm{cat}(\bar{\phi}) = \mathrm{cat}(\phi)$. Thus, we may assume that $\Gamma = \mathbb{Z}^n$.

If $\Lambda$ has no torsion, then $\mathrm{cat}(\phi) = \mathrm{cd}(\phi) = rank(\phi) = rank(\Lambda) = m$ by Jamie Scott's result [15]. We consider the case $T(\Lambda) \neq 0$. Let $\phi : \mathbb{Z}^n \to \mathbb{Z}^m \oplus T(\Lambda)$ be an epimorphism. By Lemma 4.3 $\phi$ breaks into the direct sum $\psi_1 \oplus \psi_2$ where $\psi_1 : \mathbb{Z}^m \to \mathbb{Z}^m$ is an isomorphism and an epimorphism $\psi_2 : \mathbb{Z}^{n-m} \to T(\Lambda)$.

By the well-known inequality for LS category of product of maps in [3], we obtain

$$\mathrm{cat}(\psi_1 \oplus \psi_2) \leqslant \mathrm{cat}(\psi_1) + \mathrm{cat}(\psi_2) = m + k,$$

where $k = k(T(\Lambda))$, since by Lemma 4.2 $\mathrm{cat}(\psi_2) = \mathrm{cd}(\psi_2) = k(T(\Lambda))$.

To complete the proof, we show that $m + k \leqslant \mathrm{cd}(\phi)$. By Theorem 2.3, the torsion group $T(\Lambda)$ admits a decomposition $T(\Lambda) = \mathbb{Z}_{n_1} \times .. \times \mathbb{Z}_{n_k}$ where $n_1, \cdots, n_k$ are natural numbers with $n_1 | \cdots | n_k$. The proof of Lemma 4.2 gives a nonzero homomorphism

$$\psi_2^* : H^k(BT(\Lambda); \mathbb{Z}_p) \to H^k(T^{n-m}; \mathbb{Z}_p).$$

We apply the Kunneth Formula (Theorem 2.6) with $\mathbb{Z}_p$ coefficients for $p|n_1$ to obtain a nonzero homomorphism

$$\phi^* : H^{m+k}(T^m \times BT(\Lambda); \mathbb{Z}_p) \to H^{m+k}(T^m \times T^{n-m}; \mathbb{Z}_p).$$

Therefore, $\mathrm{cd}(\phi) \geqslant m + k$. □

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