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RESEARCH ARTICLE

$(\mathcal{T}_{\mathsf{Lie}})$-Leibniz algebras and related properties

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Abstract. In [6], it is studied properties of the class $\mathfrak{D}_{\mathsf{Lie}}$ of non-perfect Leibniz algebras with non-trivial Lie-centers. In this paper, we study properties of Leibniz algebras of class $\mathcal{T}_{\mathsf{Lie}}$, a subclass of both the class $\mathfrak{D}_{\mathsf{Lie}}$ and the class of Lie-stem Leibniz algebras. We determine necessary and sufficient conditions under which a non-Lie Leibniz algebra is of class $\mathcal{T}_{\mathsf{Lie}}$ and study their relationship with pseudo-abelian Leibniz algebras. We also show that Leibniz algebras of class $\mathcal{T}_{\mathsf{Lie}}$ have semi-simple central Lie-derivations.

1. Introduction

Leibniz algebras were introduced in papers published by Bloh [7] in the sixties, and were rediscovered by Jean-Louis Loday [12] in his study of periodicity phenomenons in algebraic K-theory. Essentially, Leibniz algebras generalize Lie algebras, and are usually considered as non-commutative Lie algebras. Studies of Leibniz algebras have been associated to various areas such as non-commutative geometry, differential geometry, and mathematical physics. As in several papers in this research area, our aim is to investigate certain results on Leibniz algebras that are known to hold on derivations of Lie algebras. Our study particularly focusses on the class $\mathfrak{D}_{\mathsf{Lie}}$ of non-perfect Leibniz algebras whose center is not contained in the Leibniz kernel. Also, our study relies on the notions of derivations relative to the *Liezation functor* $(-)_{\mathsf{Lie}} : \mathsf{Leib} \to \mathsf{Lie}$, which

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assigns to a Leibniz algebra $\mathfrak{g}$ the Lie algebra $\mathfrak{g}_{_{\rm Lie}}$, where Leib denotes the category of Leibniz algebras and Lie denotes the category of Lie algebras. Studying notions of Leibniz algebras in this relative setting fits on a line of research studied in [1, 2, 8, 9].

Recently, several results in studies of derivations of Lie algebras have been extended on Leibniz algebras [4–6, 13]. Our aim in this paper is to study properties of a subclass of Lie-stem Leibniz algebras and their relationship with the properties of the Lie algebra of Lie-derivations. We organize the paper as follows: In Section 2, we provide some background on relative notions with respect to the Liezation functor. We recall definitions of the set of Lie-derivations $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ and central Lie-derivations $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ for a non-Lie Leibniz algebra $\mathfrak{g}$. In Section 3, we introduce the concept of Leibniz algebras of class $\mathcal{T}_{\mathsf{Lie}}$. This is an analogue on Leibniz algebras of the class defined in [14] on Lie algebras. We provide a characterization of these algebras by using properties of their two-sided ideals of codimension one. We prove that Lie-solvable Leibniz algebras of type ($\mathcal{T}_{\mathsf{Lie}}$) are pseudo-$\mathsf{Lie}$-abelian. Also, we prove that the Lie algebra of Lie-derivations of Leibniz algebras in the class $\mathcal{T}_{\mathsf{Lie}}$ is centerless. Finally, we prove that under certain conditions, Leibniz algebras in the class $\mathcal{T}_{\mathsf{Lie}}$ have direct summands that are Lie-nilpotent Leibniz subalgebras, and admit semi-simple Lie-derivations.

2. Preliminaries on Leibniz algebras

Let $\mathbb{K}$ be a fixed ground field such that $\frac{1}{2} \in \mathbb{K}$. Throughout the paper, all vector spaces and tensor products are considered over $\mathbb{K}$.

A *Leibniz algebra* [12] is a vector space $\mathfrak{g}$ equipped with a bilinear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, usually called the *Leibniz bracket* of $\mathfrak{g}$, satisfying the *Leibniz identity*:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad x,y,z \in \mathfrak{g}.$$

A subalgebra $\mathfrak{h}$ of a Leibniz algebra $\mathfrak{g}$ is said to be *left (resp. right) ideal* of $\mathfrak{g}$ if $[h,g] \in \mathfrak{h}$ (resp. $[g,h] \in \mathfrak{h}$), for all $h \in \mathfrak{h}$, $g \in \mathfrak{g}$. If $\mathfrak{h}$ is both left and right ideal, then $\mathfrak{h}$ is called *two-sided ideal* of $\mathfrak{g}$. In this case $\mathfrak{g}/\mathfrak{h}$ naturally inherits a Leibniz algebra structure.

Given a Leibniz algebra $\mathfrak{g}$, we denote by $\mathfrak{g}^{\rm ann}$ the subspace of $\mathfrak{g}$ spanned by all elements of the form $[x,x]$, $x \in \mathfrak{g}$. It is clear that the quotient $\mathfrak{g}_{_{\rm Lie}} = \mathfrak{g}/\mathfrak{g}^{\rm ann}$ is a Lie algebra. This defines the so-called *Liezation functor* $(-)_{\mathsf{Lie}} : \mathsf{Leib} \to \mathsf{Lie}$, which assigns to a Leibniz algebra $\mathfrak{g}$ the

Lie algebra $\mathfrak{g}_{\mathrm{Lie}}$. Moreover, the canonical epimorphism $\mathfrak{g} \twoheadrightarrow \mathfrak{g}_{\mathrm{Lie}}$ is universal among all homomorphisms from $\mathfrak{g}$ to a Lie algebra, implying that the Liezation functor is left adjoint to the inclusion functor $\mathsf{Lie} \hookrightarrow \mathsf{Leib}$.

Given a Leibniz algebra $\mathfrak{g}$, we define the bracket

$$[-,-]_{lie} : \mathfrak{g} \to \mathfrak{g}, \quad \text{by} \quad [x,y]_{lie} = [x,y] + [y,x], \quad \text{for} \quad x, y \in \mathfrak{g}.$$

Let $\mathfrak{m}$, $\mathfrak{n}$ be two-sided ideals of a Leibniz algebra $\mathfrak{g}$. The following notions come from [9], which were derived from [10].

The Lie-*commutator* of $\mathfrak{m}$ and $\mathfrak{n}$ is the two-sided ideal of $\mathfrak{g}$

$$[\mathfrak{m}, \mathfrak{n}]_{\mathsf{Lie}} = \langle \{ [m,n]_{lie}, m \in \mathfrak{m}, n \in \mathfrak{n} \} \rangle.$$

The Lie-*center* of the Leibniz algebra $\mathfrak{g}$ is the two-sided ideal

$$Z_{\mathsf{Lie}}(\mathfrak{g}) = \{ z \in \mathfrak{g} \mid [g,z]_{lie} = 0 \text{ for all } g \in \mathfrak{g} \}.$$

The Lie-*centralizer* of $\mathfrak{m}$ and $\mathfrak{n}$ over $\mathfrak{g}$ is

$$C_{\mathfrak{g}}^{\mathsf{Lie}}(\mathfrak{m}, \mathfrak{n}) = \{ g \in \mathfrak{g} \mid \ [g,m]_{lie} \in \mathfrak{n}, \text{ for all } m \in \mathfrak{m} \} .$$

Obviously, $C_{\mathfrak{g}}^{\mathsf{Lie}}(\mathfrak{g}, 0) = Z_{\mathsf{Lie}}(\mathfrak{g})$.

Remark 2.1. $[[a,b]_{\mathsf{Lie}}, c]_{\mathsf{Lie}} = [a,[b,c]]_{\mathsf{Lie}} + [b,[a,c]]_{\mathsf{Lie}}$ for all $a, b, c \in \mathfrak{g}$.

Definition 2.2 ([9]). *The lower* Lie-*central series of a Leibniz algebra* $\mathfrak{g}$ *is the sequence*

$$\cdots \trianglelefteq \gamma_i^{\mathsf{Lie}}(\mathfrak{g}) \trianglelefteq \cdots \trianglelefteq \gamma_2^{\mathsf{Lie}}(\mathfrak{g}) \trianglelefteq \gamma_1^{\mathsf{Lie}}(\mathfrak{g})$$

of two-sided ideals of $\mathfrak{g}$ *defined inductively by*

$$\gamma_1^{\mathsf{Lie}}(\mathfrak{g}) = \mathfrak{g} \quad \textit{and} \quad \gamma_i^{\mathsf{Lie}}(\mathfrak{g}) = [\gamma_{i-1}^{\mathsf{Lie}}(\mathfrak{g}), \mathfrak{g}]_{\mathsf{Lie}}, \quad i \geq 2.$$

Recall that in the absolute case, the lower central series of $\mathfrak{g}$ is the sequence of two-sided ideals of $\mathfrak{g}$ defined inductively by

$$\cdots \trianglelefteq \gamma_i(\mathfrak{g}) \trianglelefteq \cdots \trianglelefteq \gamma_2(\mathfrak{g}) \trianglelefteq \gamma_1(\mathfrak{g}) = \mathfrak{g} \quad \text{and} \quad \gamma_i(\mathfrak{g}) = [\gamma_{i-1}(\mathfrak{g}), \mathfrak{g}], \quad i \geq 2.$$

Definition 2.3 ([9]). *The Leibniz algebra* $\mathfrak{g}$ *is said to be* Lie-*nilpotent of class* c *if* $\gamma_{c+1}^{\mathsf{Lie}}(\mathfrak{g}) = 0$ *and* $\gamma_c^{\mathsf{Lie}}(\mathfrak{g}) \neq 0$.

Definition 2.4 ([9, Proposition 1]). *An exact sequence of Leibniz algebras* $0 \to \mathfrak{n} \to \mathfrak{g} \xrightarrow{\pi} \mathfrak{q} \to 0$ *is said to be a* Lie-central *extension if* $[\mathfrak{g}, \mathfrak{n}]_{\mathsf{Lie}} = 0$, *equivalently* $\mathfrak{n} \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$.

Definition 2.5. *A linear map $d : \mathfrak{g} \to \mathfrak{g}$ of a Leibniz algebra $(\mathfrak{g}, [-,-])$ is said to be a* Lie*-derivation if for all $x, y \in \mathfrak{g}$, the following condition holds:*

$$d([x, y]_{lie}) = [d(x), y]_{lie} + [x, d(y)]_{lie}$$

We denote by $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ the set of all Lie-derivations of a Leibniz algebra $\mathfrak{g}$, which can be equipped with a structure of Lie algebra by means of the usual bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$, for all $d_1, d_2 \in \mathsf{Der}(\mathfrak{g})$.

Example 2.6. The absolute derivations, that is linear maps $d \colon \mathfrak{g} \to \mathfrak{g}$ such that $d([x, y]) = [d(x), y] + [x, d(y)]$, are also Lie-derivations, since:

$$d([x, y]_{lie}) = d([x, y] + [y, x]) = [d(x), y]_{lie} + [x, d(y)]_{lie}, \text{ for all } x, y \in \mathfrak{g}. \quad (1)$$

In particular, for a fixed $x \in \mathfrak{g}$, the inner derivation $R_x \colon \mathfrak{g} \to \mathfrak{g}, R_x(y) = [y, x]$, for all $y \in \mathfrak{g}$, is a Lie-derivation, so it gives rise to the following identity:

$$[[y, z]_{lie}, x] = [[y, x], z]_{lie} + [y, [z, x]]_{lie}, \quad \text{for all } x, y \in \mathfrak{g}.$$

Definition 2.7. *A* Lie*-derivation $d : \mathfrak{g} \to \mathfrak{g}$ of a Leibniz algebra $\mathfrak{g}$ is said to be* Lie*-central derivation if its image is contained in the* Lie*-center of $\mathfrak{g}$.*

Remark 2.8. The absolute notion corresponding to Definition 2.7 is the so called central derivations, that is derivations $d : \mathfrak{g} \to \mathfrak{g}$ such that its image is contained in the center of $\mathfrak{g}$. Obviously, every central derivation is a Lie-central derivation. However the converse is not true as the following example shows: let $\mathfrak{g}$ be the two-dimensional Leibniz algebra with basis $\{e, f\}$ and bracket operation given by $[e, f] = -[f, e] = = e$ [11]. The inner derivation R_e is a Lie-central derivation, but it is not central in general.

We denote the set of all Lie-central derivations of a Leibniz algebra $\mathfrak{g}$ by $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. Obviously $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ is a subalgebra of $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ and every element of $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ annihilates $\gamma_2^{\mathsf{Lie}}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$.

3. The main Results

Denote by $\mathfrak{D}_{\mathsf{Lie}}$, the class of Leibniz algebras $\mathfrak{g}$ satisfying the conditions $[\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}} \neq \mathfrak{g}$ and $Z_{\mathsf{Lie}}(\mathfrak{g}) \neq 0$. In this section, we study properties of a subclass of $\mathfrak{D}_{\mathsf{Lie}}$, call it (T_{Lie}).

Definition 3.1. *A Leibniz algebra $\mathfrak{g}$ is of type (T_{Lie}) if $\mathfrak{g} \in \mathfrak{D}_{\mathsf{Lie}}$ and $\mathfrak{g}$ has a nonzero subspace $\mathcal{T}$ satisfying the following conditions:*

a) $\mathfrak{g} = \mathcal{T} + [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}, \quad \mathcal{T} \cap [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} = 0$;

b) $\left[\mathcal{T}, [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}\right]_{\mathsf{Lie}} = 0$;

c) $[\mathcal{T},\mathcal{T}]_{\mathsf{Lie}} = \langle z \rangle$ *for some* $z \in Z_{\mathsf{Lie}}(\mathfrak{g})$;

d) The mapping $\theta : \mathcal{T} \times \mathcal{T} \to [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ *defined by* $\theta(x,y) = \alpha z$ *such that* $[x,y]_{\mathsf{Lie}} = \alpha z$, *is a non-degenerate alternate form on* $\mathcal{T}$.

Definition 3.2 ([3, Definition 4.1])**.** *A* Lie*-stem Leibniz algebra is a Leibniz algebra* $\mathfrak{g}$ *such that* $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$.

Example 3.3. Consider the $\mathbb{K}$-vector space $\mathfrak{g}$ spanned by the nonzero vectors x, y, z and define a bracket on $\mathfrak{g}$ as follows: $[x,x] = 2z$, $[y,y] = z$ and $[x,y] = \frac{1}{2}z = [y,x]$. $\mathfrak{g}$ is a non-Lie Leibniz algebra of dimension 3 with Lie-center $Z_{\mathsf{Lie}}(\mathfrak{g}) = \langle z \rangle$. Moreover one has $\mathfrak{g} = \mathcal{T} + [\mathfrak{g},\mathfrak{g}]_{Lie}$ with $\mathcal{T} = \langle \{x,y\} \rangle$ and $[\mathfrak{g},\mathfrak{g}]_{Lie} = \langle z \rangle = Z_{\mathsf{Lie}}(\mathfrak{g})$. Thus $\mathcal{T} \cap [\mathfrak{g},\mathfrak{g}]_{Lie} = 0$ and the bilinear form $\theta : \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{K}$ defined by $\theta(x,x) = 4$, $\theta(y,y) = 2$, and $\theta(x,y) = 1$, is nondegenerate. Therefore $\mathfrak{g}$ is a non-Lie Leibniz algebra of type (T_{Lie}).

Example 3.4. Consider the Leibniz algebra $\mathfrak{g}$ spanned by $\{x,y,z\}$ with nonzero bracket $[x,x] = z$; $[y,y] = z$; $[x,y] = z$; $[y,x] = \alpha z$, $\alpha \in \mathbb{F} \setminus \{1;-1\}$. Then, $Z_{\mathsf{Lie}}(\mathfrak{g}) = span\{z\}$ and $[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} = span\{z\}$. So $(0) \neq Z_{\mathsf{Lie}}(\mathfrak{g}) \subset [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \neq \mathfrak{g}$. This means that $\mathfrak{g}$ is Lie-stem Leibniz algebra and $\mathfrak{g} \in \mathfrak{D}_{\mathsf{Lie}}$. Consider the subspace $\mathcal{T} =: span\{x,y\}$ of $\mathfrak{g}$. It is easy to check that $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} + \mathcal{T}$, $[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \cap \mathcal{T} = 0$, $[[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}, \mathcal{T}]_{\mathsf{Lie}} = 0$ and $[\mathcal{T},\mathcal{T}]_{\mathsf{Lie}} = span\{z\}$. Moreover, the bilinear form $\theta : \mathcal{T} \times \mathcal{T} \to \mathbb{F}$ defined by $\theta(x,x) = 2 = \theta(y,y); \theta(x,y) = 1 + \alpha, \alpha \in \mathbb{F} \setminus \{1;-1\}$, is nondegenerate. Therefore $\mathfrak{g}$ is a non-Lie Leibniz algebra of type (T_{Lie}).
Now let $\mathfrak{M} = \mathrm{span}\{y,z\}$. $\mathfrak{M}$ is an ideal of $\mathfrak{g}$ of codimension one, and $Z_{\mathsf{Lie}}(\mathfrak{M}) = span\{z\} = [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$.

Remark 3.5. If $\mathfrak{g}$ is a non-Lie Leibniz algebra of type (T_{Lie}) then $Z_{\mathsf{Lie}}(\mathfrak{g}) = Z_{\mathsf{Lie}}(\mathfrak{g}^{(1)}_{\mathsf{Lie}})$.

Proof. In fact, for any $z \in Z_{\mathsf{Lie}}(\mathfrak{g})$ and any $x \in \mathcal{T}$ one has $[z,x]_{\mathsf{Lie}} = 0$. But $z = z_1 + z_2$ with $z_1 \in \mathcal{T}$ and $z_2 \in [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ and $[x,z_2]_{\mathsf{Lie}} = 0$; hence $[x,z_1]_{\mathsf{Lie}} = 0, \ \forall x \in \mathcal{T}$. Therefore $z_1 = 0$ since the map

$$\begin{aligned} \theta : \mathcal{T} \times \mathcal{T} &\longrightarrow \mathbb{K} \\ (x,y) &\longmapsto \theta(x,y) = \alpha; [x,y]_{\mathsf{Lie}} = \alpha z \end{aligned}$$

is a nondegenerate bilinear form. The converse inclusion comes from the definition of a (T_{Lie})-Leibniz algebra. □

Lemma 3.6. *Let $\mathfrak{g}$ be a* Lie-*stem Leibniz algebra in class $\mathfrak{D}_{\mathsf{Lie}}$ such that $Z_{\mathsf{Lie}}(\mathfrak{M}) \not\subset [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ for every two-sided ideal $\mathfrak{M}$ of $\mathfrak{g}$ of codimension 1. Then for every two-sided ideal $\mathfrak{M}$ of $\mathfrak{g}$ of codimension 1, there exists a two-sided ideal $\mathfrak{M}'$ of codimension 1 satisfying the following conditions:*

a) $\mathfrak{g} = \langle e \rangle + \mathfrak{M} = \langle e' \rangle + \mathfrak{M}'$ *for some* $e \in Z_{\mathsf{Lie}}(\mathfrak{M}') - [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ *and* $e' \in Z_{\mathsf{Lie}}(\mathfrak{M}) - [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$;

b) $[e,e']_{\mathsf{Lie}} \neq 0, \quad [e,\mathfrak{g}]_{\mathsf{Lie}} = [e',\mathfrak{g}]_{\mathsf{Lie}} = \langle [e,e']_{\mathsf{Lie}} \rangle \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$;

c) $[\mathfrak{g}, Z_{\mathsf{Lie}}(\mathfrak{M})]_{\mathsf{Lie}} \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$ *and* $\dim([\mathfrak{g}, Z_{\mathsf{Lie}}(\mathfrak{M})]_{\mathsf{Lie}}) = 1$;

d) $dim(Z_{\mathsf{Lie}}(\mathfrak{M})) = 1 + dim(Z_{\mathsf{Lie}}(\mathfrak{g}))$.

Proof. Assume that $\mathfrak{g}$ satisfies the hypotheses of the Lemma and let $\mathfrak{M}$ be an arbitrary two-sided ideal of $\mathfrak{g}$ of codimension 1. We need to find a two-sided ideal $\mathfrak{M}'$ of codimension 1 satisfying the conditions $a), b), c)$ and $d)$. Since $Z_{\mathsf{Lie}}(\mathfrak{M}) \not\subset [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$, there exists a nonzero vector $e' \in Z_{\mathsf{Lie}}(\mathfrak{M}) - [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ and a subspace $\mathfrak{M}'$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathbb{K}e' \oplus \mathfrak{M}'$ with $[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \subset \mathfrak{M}'$. Clearly, $\mathfrak{M}'$ is a two-sided ideal of $\mathfrak{g}$ of codimension 1. This implies that $Z_{\mathsf{Lie}}(\mathfrak{M}') \nsubseteq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$, thus there exists a nonzero vector $e \in Z_{\mathsf{Lie}}(\mathfrak{M}') - [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$. Now, $e \notin \mathfrak{M}$, otherwise $e \in Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$, which contradicts the choice of e. Therefore $\mathfrak{g} = \mathbb{K}e + \mathfrak{M} = \mathbb{K}e' + \mathfrak{M}'$ with $e \in Z_{\mathsf{Lie}}(\mathfrak{M}') - [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ and $e' \in Z_{\mathsf{Lie}}(\mathfrak{M}) - [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$. This proves a). To prove b), notice from the construction above that since $e, e' \notin Z_{\mathsf{Lie}}(\mathfrak{g})$, then $[e,e']_{\mathsf{Lie}} \neq 0$. Moreover, we have $[e,\mathfrak{g}]_{\mathsf{Lie}} = [e, \mathbb{K}e' + \mathfrak{M}']_{\mathsf{Lie}} = [e, \mathbb{K}e']_{\mathsf{Lie}} + [e, \mathfrak{M}']_{\mathsf{Lie}} =< [e,e']_{\mathsf{Lie}} >$, since $e \in Z_{\mathsf{Lie}}(\mathfrak{M}')$. Similarly, $[e',\mathfrak{g}]_{\mathsf{Lie}} =< [e,e']_{\mathsf{Lie}} >$. Furthermore, for every $g \in \mathfrak{g}$, we have by Remark 2.1 $[[e,e']_{\mathsf{Lie}}, g]_{\mathsf{Lie}} = [e,[e',g]]_{\mathsf{Lie}} + [e',[e,g]]_{\mathsf{Lie}} = 0$ since $e' \in Z_{\mathsf{Lie}}(\mathfrak{M})$, $e \in Z_{\mathsf{Lie}}(\mathfrak{M})$ and $[e',g]_{\mathsf{Lie}}, [e,g]_{\mathsf{Lie}} \in \mathfrak{M} \cap \mathfrak{M}'$. Therefore $[e,e']_{\mathsf{Lie}} \in Z_{\mathsf{Lie}}(\mathfrak{g})$. To prove c), notice that $0 \neq [\mathfrak{g}, Z_{\mathsf{Lie}}(\mathfrak{M})]_{\mathsf{Lie}} = [< e >, Z_{\mathsf{Lie}}(\mathfrak{M})]_{\mathsf{Lie}} \subseteq [< e >, \mathfrak{g}]_{\mathsf{Lie}} =< [e,e']_{\mathsf{Lie}} > \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$, thanks to b). To show d), let $\mathfrak{M}$ be an arbitrary two-sided ideal of $\mathfrak{g}$ with codimension one. Then from a), we obtain that for any other two-sided ideal $\mathfrak{M}'$ of $\mathfrak{g}$ of codimension one, $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq \mathfrak{M} \cap \mathfrak{M}'$. Moreover, if $B_{Z_{\mathsf{Lie}}(\mathfrak{g})}$ denotes the basis of $\mathfrak{g}$, then $B_{Z_{\mathsf{Lie}}(\mathfrak{M})} = B_{Z_{\mathsf{Lie}}(\mathfrak{g})} \cup \{e'\}$. Therefore $\dim(Z_{\mathsf{Lie}}(\mathfrak{M})) = 1 + \dim(Z_{\mathsf{Lie}}(\mathfrak{g}))$. □

Lemma 3.7. *If a Leibniz algebra $\mathfrak{g}$ is of type (T_{Lie}), then $\mathfrak{g}$ is a* Lie-*stem Leibniz algebra such that $\mathfrak{g} \in \mathfrak{D}_{\mathsf{Lie}}$ and for every two-sided ideal $\mathfrak{M}$ of $\mathfrak{g}$ of codimension 1 such that $Z_{\mathsf{Lie}}(\mathfrak{g}) \neq Z_{\mathsf{Lie}}(\mathfrak{M})$, $Z_{\mathsf{Lie}}(\mathfrak{M}) \not\subset [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$.*

Proof. Assume that $\mathfrak{g}$ is a Leibniz algebra of type (T_{Lie}). Then $\mathfrak{g} \in \mathfrak{D}_{\mathsf{Lie}}$ by definition. Let $\mathcal{T}$, z and θ be as in definition 3.1. For any $z := z_1 + z_2 \in Z_{\mathsf{Lie}}(\mathfrak{g})$, with $z_1 \in \mathcal{T}$ and $z_2 \in [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$, we have $[\mathcal{T}, z_1]_{\mathsf{Lie}} + [\mathcal{T}, z_2]_{\mathsf{Lie}} = [\mathcal{T}, z]_{\mathsf{Lie}} = 0$. So $\theta(x, z_1) = [x, z_1]_{\mathsf{Lie}} = 0$ for all $x \in \mathcal{T}$, since $[x, z_2]_{\mathsf{Lie}} = 0$ by Definition 3.1(b). It follows that $z_1 = 0$ because θ is a non-degenerate bilinear form on $\mathcal{T}$. Therefore $z = z_2 \in [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. This implies that $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$ i.e. $\mathfrak{g}$ is a Lie-stem Leibniz algebra. Now consider $\mathfrak{M}$ be an arbitrary two-sided ideal of $\mathfrak{g}$ of codimension 1. Clearly $\mathfrak{M}$ contains $[\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$ and $\mathfrak{g} = \mathcal{T} + [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}} = \mathbb{K}e + \mathfrak{M}$, for some $e \in \mathfrak{g} - [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. But since $Z_{\mathsf{Lie}}(\mathfrak{g}) \subsetneqq Z_{\mathsf{Lie}}(\mathfrak{M})$, there exists $e' \in Z_{\mathsf{Lie}}(\mathfrak{M})$ such that $[e', e]_{\mathsf{Lie}} \neq 0$. Consider the subspace $\mathfrak{M}'$ of $\mathfrak{g}$ complementary of $< \{e'\} >$. $\mathfrak{M}'$ is a two sided ideal of $\mathfrak{g}$ of codimension 1 containing $[\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$ and $\mathfrak{g} = \mathfrak{M}' \oplus \mathbb{K}e' = \mathcal{T} \oplus [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. Hence $e' \in \mathcal{T} \cap Z_{\mathsf{Lie}}(\mathfrak{M})$, consequently $Z_{\mathsf{Lie}}(\mathfrak{M}) \nsubseteq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. □

Theorem 3.8. *Let $\mathfrak{g}$ be a* Lie*-stem Leibniz algebra such that $\mathfrak{g} \in \mathfrak{D}_{\mathsf{Lie}}$ and $Z_{\mathsf{Lie}}(\mathfrak{M}) \not\subset [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$ for every two-sided ideal $\mathfrak{M}$ of $\mathfrak{g}$ of codimension 1. Then $\mathfrak{g}$ is of type (T_{Lie}).*

Proof. Assume that $\mathfrak{g}$ is a Lie-stem Leibniz algebra such that $\mathfrak{g} \in \mathfrak{D}_{\mathsf{Lie}}$ and for any two-sided ideal of $\mathfrak{g}$ of codimension 1, $Z_{\mathsf{Lie}}(\mathfrak{M}) \nsubseteq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. We construct a subspace $\mathcal{T}$ of $\mathfrak{g}$ satisfying the properties of Definition 3.1 above. Let $\mathfrak{M}$ be any subspace of $\mathfrak{g}$ of codimension 1 containing $[\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. Clearly, $\mathfrak{M}$ is a two-sided ideal of $\mathfrak{g}$. Since $Z_{\mathsf{Lie}}(\mathfrak{M}) \nsubseteq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$, then by Lemma 3.6, there exists a two-sided ideal $\mathfrak{M}'$ of $\mathfrak{g}$ such that

$$\begin{cases} \mathfrak{g} & = \mathbb{K}e \oplus \mathfrak{M} \qquad \text{with} \qquad e \in Z_{\mathsf{Lie}}(\mathfrak{M})' - [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}, \\ \mathfrak{g} & = \mathbb{K}e' \oplus \mathfrak{M}' \qquad \text{with} \qquad e' \in Z_{\mathsf{Lie}}(\mathfrak{M}) - [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}. \end{cases}$$

Since $\mathfrak{M}$ and $\mathfrak{M}'$ contain $[\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$, It follows that

$$\begin{cases} \mathfrak{M} & = & \mathbb{K}e' \oplus (\oplus_{i=1}^{s} \mathbb{K}a_i) \oplus [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}, \\ \mathfrak{M}' & = & \mathbb{K}e \oplus (\oplus_{i=1}^{s} \mathbb{K}a_i) \oplus [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}. \end{cases}$$

We construct the subspace $\mathcal{T}$ by induction. If $s = 1$ then $\mathfrak{g} = \mathbb{K}e \oplus \mathbb{K}e' \oplus [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. Take $\mathcal{T} = \langle \{e, e'\} \rangle$. Clearly, for some vectors $a_i \in \mathfrak{g}, i = 1, \ldots, s$, $\mathcal{T}$ satisfies all the assertions of Definition 3.1, therefore $\mathfrak{g}$ is a (T_{Lie}) Leibniz algebra. Now, assume that $s \geq 2$ and set $\mathfrak{M}_1 = \mathbb{K}e \oplus \mathbb{K}e' \oplus (\oplus_{i=2}^{s} \mathbb{K}a_i) \oplus [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. $\mathfrak{M}_1$ is also a two-sided ideal of $\mathfrak{g}$ of codimension 1. Since $[e, e']_{\mathsf{Lie}} \neq 0$, then $Z_{\mathsf{Lie}}(\mathfrak{M}_1) \subseteq (\oplus_{i=2}^{s} \mathbb{K}a_i) \oplus [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. Moreover, we have by hypothesis $Z_{\mathsf{Lie}}(\mathfrak{M}_1) \nsubseteq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. Again, by Lemma 3.6, one has

$$\begin{cases} \mathfrak{g} & = \mathbb{K}e_1 \oplus \mathfrak{M}_1 \qquad \text{with} \qquad e_1 \in Z_{\mathsf{Lie}}(\mathfrak{M}_1'), \\ \mathfrak{g} & = \mathbb{K}e_1' \oplus \mathfrak{M}_1' \qquad \text{with} \qquad e_1' \in Z_{\mathsf{Lie}}(\mathfrak{M}_1), \end{cases}$$

and $[e_1,e'_1]_{\mathsf{Lie}} \neq 0, [e_1,\mathfrak{g}]_{\mathsf{Lie}} = [\mathfrak{g},e'_1]_{\mathsf{Lie}} = \langle [e_1,e'_1]_{\mathsf{Lie}}\rangle \subset Z_{\mathsf{Lie}}(\mathfrak{g})$; $e_1, e'_1 \notin \{e,e'\}$, $[e,e_1]_{\mathsf{Lie}} = [e',e_1]_{\mathsf{Lie}} = [e,e'_1]_{\mathsf{Lie}} = [e',e'_1]_{\mathsf{Lie}} = 0$.
Continuing this construction and using appropriate notations, we obtain

$$\mathfrak{g} = \mathbb{K}e_1 \oplus \mathbb{K}e'_1 \oplus \mathbb{K}e_2 \oplus \mathbb{K}e'_2 \cdots \oplus \mathbb{K}e_n \oplus \mathbb{K}e'_n \oplus [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$$

with $(0 \neq [e_i,e'_i]_{\mathsf{Lie}} = z_i \in Z_{\mathsf{Lie}}(\mathfrak{g}))_{1\le i\le n}$, and for all $i,j \in \{1;2;\cdots n\}$, $i \neq j$, $[e_i,e_j]_{\mathsf{Lie}} = [e_i,e'_j]_{\mathsf{Lie}} = [e'_i,e_j]_{\mathsf{Lie}} = [e'_i,e'_j]_{\mathsf{Lie}} = 0$. Set $\mathcal{T} = \mathbb{K}e_1 \oplus \mathbb{K}e'_1 \oplus \mathbb{K}e_2 \oplus \mathbb{K}e'_2 \cdots \oplus \mathbb{K}e_n \oplus \mathbb{K}e'_n$. We show that $\mathcal{T}$ satisfies the assertions of Definition 3.1. From the construction, we have $\mathfrak{g} = \mathcal{T} \oplus [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ and $[\mathcal{T},[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}]_{\mathsf{Lie}} = 0$. It remains to show that $[\mathcal{T},\mathcal{T}]_{\mathsf{Lie}}$ is a one dimensional vector subspace of $Z_{\mathsf{Lie}}(\mathfrak{g})$. We show that any set $\{z_i,z_j\}_{\{1\le\ i\neq j\le n\}}$ of two vectors in $Z_{\mathsf{Lie}}(\mathfrak{g})$ is linearly dependent. Indeed, assume that $\{z_{i_0},z_{j_0}\}$ is independent for some $1 \le\ i_0 \neq j_0 \le n$, and denote by $\widetilde{\mathfrak{M}}$ the two sided-ideal of $\mathfrak{g}$ of codimension 1 defined by

$$\widetilde{\mathfrak{M}} = \mathbb{K}(e_{i_0}-e_{j_0}) \oplus \mathbb{K}e'_{i_0} \oplus \mathbb{K}e'_{j_0} \oplus \sum_{1\leqq s\leqq n,\ s\neq i_0,j_0} (\mathbb{K}e'_s \oplus \mathbb{K}e_s) \oplus [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}.$$

Then for any element, x in $Z_{\mathsf{Lie}}(\widetilde{\mathfrak{M}})$, we have $[x,e_s]_{\mathsf{Lie}} = [x,e'_s]_{\mathsf{Lie}} = 0$ for all $1 \le s \le n,\ s \neq i_0,j_0$. Moreover $[x,e_{i_0}-e_{j_0}]_{\mathsf{Lie}} = 0 = \beta_{i_0}z_{i_0} - \beta_{j_0}z_{j_0}$ for some $\beta_{i_0},\beta_{j_0} \in \mathbb{K}$. Thus $\beta_{i_0} = \beta_{j_0} = 0$ since the set $\{z_{i_0},z_{j_0}\}_{\{1\le\ i\neq j\le n\}}$ is linearly independent. Hence $x \in [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$. This contradicts the fact that $Z_{\mathsf{Lie}}(\widetilde{\mathfrak{M}}) \nsubseteq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$. Therefore $[\mathcal{T},\mathcal{T}]_{\mathsf{Lie}}$ is a one dimensional vector subspace of $Z_{\mathsf{Lie}}(\mathfrak{g})$. □

Definition 3.9. *A* Lie-*nilpotent Leibniz algebra is said to be pseudo*-Lie-*abelian if* $[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{g}) = \langle z\rangle$ *for some* $z \in \mathfrak{g}$.

Remark 3.10. Every Lie-solvable Leibniz algebra of type (T_{Lie}) is pseudo-Lie-abelian.

Proof. Let $\mathfrak{g}$ be a Lie-solvable Leibniz algebra of type (T_{Lie}). Then $\mathfrak{g}$ is a Lie-stem Leibniz algebra i.e. $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$. Moreover, there exists a subalgebra $\mathcal{T}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathcal{T} + \mathfrak{g}^{(1)}_{\mathsf{Lie}}$ with $[\mathcal{T},\mathfrak{g}^{(1)}_{\mathsf{Lie}}]_{\mathsf{Lie}} = 0$, and $[\mathcal{T},\mathcal{T}]_{\mathsf{Lie}} = \langle z\rangle$ for some $z \in Z_{\mathsf{Lie}}(\mathfrak{g})$, where $\mathfrak{g}^{(i+1)}_{\mathsf{Lie}} = [\mathfrak{g}^{(i)}_{\mathsf{Lie}},\mathfrak{g}^{(i)}_{\mathsf{Lie}}]_{\mathsf{Lie}}$ and $\mathfrak{g}^{(0)}_{\mathsf{Lie}} = \mathfrak{g}$. So

$$\mathfrak{g}^{(1)}_{\mathsf{Lie}} = [\mathcal{T}+\mathfrak{g}^{(1)}_{\mathsf{Lie}},\mathcal{T}+\mathfrak{g}^{(1)}_{\mathsf{Lie}}]_{\mathsf{Lie}} \subseteq \langle z\rangle + \mathfrak{g}^{(2)}.$$

Inductively, we have that

$$\mathfrak{g}^{(2)}_{\mathsf{Lie}} = [\mathfrak{g}^{(1)}_{\mathsf{Lie}},\mathfrak{g}^{(1)}_{\mathsf{Lie}}]_{\mathsf{Lie}} \subseteq [\langle z\rangle+\mathfrak{g}^{(2)}_{\mathsf{Lie}},\langle z\rangle+\mathfrak{g}^{(2)}_{\mathsf{Lie}}]_{\mathsf{Lie}} \subseteq \mathfrak{g}^{(3)}_{\mathsf{Lie}} \subseteq \ldots \subseteq \mathfrak{g}^{(n)}_{\mathsf{Lie}} \subseteq \ldots \subseteq 0$$

since $\mathfrak{g}$ is Lie-solvable. So, $[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \subseteq \langle z\rangle \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$. This completes the proof. □

The following theorem characterizes Leibniz algebras of type (T_{Lie}) with 1-dimensional Lie-center.

Theorem 3.11. *A Leibniz algebra is of type* $(\mathcal{T}_{\mathsf{Lie}})$ *such that* $\dim(Z_{\mathsf{Lie}}(\mathfrak{g})) = 1$ *if and only if* $\mathfrak{g}$ *is a* Lie*-stem Leibniz algebra such that* $\mathfrak{g} \in \mathfrak{D}_{\mathsf{Lie}}$, *and* $[Z_{\mathsf{Lie}}(\mathfrak{M}),\mathfrak{g}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{M}) \cap [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ *for every two-sided ideal* $\mathfrak{M}$ *of* $\mathfrak{g}$ *of codimension 1 such that* $Z_{\mathsf{Lie}}(\mathfrak{M}) \neq Z_{\mathsf{Lie}}(\mathfrak{g})$.

Proof. Assume that $\mathfrak{g}$ is a Leibniz algebra of type $(\mathcal{T}_{\mathsf{Lie}})$ with 1-dimensional Lie-center. Then by Lemma 3.7, $\mathfrak{g} \in \mathfrak{D}_{\mathsf{Lie}}$ and $\mathfrak{g}$ is a Lie-stem Leibniz algebra. Let $\mathfrak{M}$ be a two-sided ideal of $\mathfrak{g}$ of codimension 1. Then by Lemma 3.6(c), $[Z_{\mathsf{Lie}}(\mathfrak{M}),\mathfrak{g}]_{\mathsf{Lie}} \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$. Since $dim(Z_{\mathsf{Lie}}(\mathfrak{g})) = 1$, it follows that $[Z_{\mathsf{Lie}}(\mathfrak{M}),\mathfrak{g}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{g})$. Also, as $Z_{\mathsf{Lie}}(\mathfrak{M})$ is a two-sided ideal of $\mathfrak{g}$, we have $[Z_{\mathsf{Lie}}(\mathfrak{M}),\mathfrak{g}]_{\mathsf{Lie}} \subseteq Z_{\mathsf{Lie}}(\mathfrak{M}) \cap [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \subseteq Z_{\mathsf{Lie}}([\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}) = Z_{\mathsf{Lie}}(\mathfrak{g})$. Therefore $[Z_{\mathsf{Lie}}(\mathfrak{M}),\mathfrak{g}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{M}) \cap [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$. Conversely, assume that $\mathfrak{g}$ is a Lie-stem Leibniz algebra such that $\mathfrak{g} \in \mathfrak{D}_{\mathsf{Lie}}$, and $[Z_{\mathsf{Lie}}(\mathfrak{M}),\mathfrak{g}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{M}) \cap [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ for every two-sided ideal $\mathfrak{M}$ of $\mathfrak{g}$ of codimension 1, and let $e \in \mathfrak{g}$ such that $\mathfrak{g} = \mathbb{K}e+\mathfrak{M}$. If $Z_{\mathsf{Lie}}(\mathfrak{M}) \subseteq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$, then we have on one hand $[Z_{\mathsf{Lie}}(\mathfrak{M}),e]_{\mathsf{Lie}} = [Z_{\mathsf{Lie}}(\mathfrak{M}),\mathfrak{g}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{M})$. On the other hand, the map $\alpha : Z_{\mathsf{Lie}}(\mathfrak{M}) \to [Z_{\mathsf{Lie}}(\mathfrak{M}),e]_{\mathsf{Lie}}$ defined by $\alpha(m) = [m,e]_{\mathsf{Lie}}$, is a surjective linear map with $\ker(\alpha) = \{m \in Z_{\mathsf{Lie}}(\mathfrak{M}) \mid [m,e]_{\mathsf{Lie}} = 0\} = Z_{\mathsf{Lie}}(\mathfrak{g}) \neq 0$. This implies that $dim(Z_{\mathsf{Lie}}(\mathfrak{M})) > dim([Z_{\mathsf{Lie}}(\mathfrak{M}),e]_{\mathsf{Lie}})$. A contradiction. Hence $Z_{\mathsf{Lie}}(\mathfrak{M}) \nsubseteq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ for every two-sided ideal $\mathfrak{M}$ of $\mathfrak{g}$ of codimension 1. We conclude by Theorem 3.8 that $\mathfrak{g}$ is of type $(\mathcal{T}_{\mathsf{Lie}})$. To show that $dim(Z_{\mathsf{Lie}}(\mathfrak{g})) = 1$, notice that since $\mathfrak{g}$ is a Lie-stem Leibniz algebra, $Z_{\mathsf{Lie}}(\mathfrak{g}) = Z_{\mathsf{Lie}}([\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{M}) \cap [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} = [Z_{\mathsf{Lie}}(\mathfrak{M}),\mathfrak{g}]_{\mathsf{Lie}}$. The result follows by Lemma 3.6 since $Z_{\mathsf{Lie}}(\mathfrak{g}) \neq 0$ and $dim([\mathfrak{g}, Z_{\mathsf{Lie}}(\mathfrak{M})]_{\mathsf{Lie}}) = 1$. □

Proposition 3.12. *Let* $\mathfrak{g}$ *be a Leibniz algebra. Then the following assertions are equivalents:*

a) $\mathfrak{g}$ *is* Lie*-pseudo-abelian;*

b) $[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{g})$ *and* $dim(Z_{\mathsf{Lie}}(\mathfrak{g})) = 1$;

c) $[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{g})$ *and* $dim(Z_{\mathsf{Lie}}(\mathfrak{M})) = 2dim(Z_{\mathsf{Lie}}(\mathfrak{g}))$ *for every two-sided ideal* $\mathfrak{M}$ *of* $\mathfrak{g}$ *of codimension 1;*

d) $[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{g})$ *and* $dim(Z_{\mathsf{Lie}}(\mathfrak{M}) \cap [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}) = 1$ *for every two-sided ideal* $\mathfrak{M}$ *of* $\mathfrak{g}$ *of codimension 1.*

Proof. *a*) $\Rightarrow$ *b*) Straightforward.

b) $\Rightarrow$ *c*) Assume that b) holds. If $Z_{\mathsf{Lie}}(\mathfrak{M}) \subseteq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$, then $Z_{\mathsf{Lie}}(\mathfrak{M}) = [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$, and the result follows. Otherwise, $\mathfrak{g} \in (\mathcal{T}_{\mathsf{Lie}})$, and thus $dim(Z_{\mathsf{Lie}}(\mathfrak{M})) = dim(Z_{\mathsf{Lie}}(\mathfrak{g})) + 1 = 2dim(Z_{\mathsf{Lie}}(\mathfrak{g}))$.

c) $\Rightarrow$ *a*) Assume that c) holds. Then, $Z_{\mathsf{Lie}}(\mathfrak{M}) \nsubseteq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$. By Theorem 3.8, $\mathfrak{g}$ is of type $(\mathcal{T}_{\mathsf{Lie}})$. Since $[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{g})$, $\mathfrak{g}$ is Lie-solvable. Therefore, $\mathfrak{g}$ is Lie-solvable of type $(\mathcal{T}_{\mathsf{Lie}})$. It follows by Remark 3.10 that $\mathfrak{g}$ is pseudo-abelian.

b) $\Rightarrow$ *d*) Assume that b) holds and let $\mathfrak{M}$ be a two-sided ideal of $\mathfrak{g}$ of codimension 1. Then, $Z_{\mathsf{Lie}}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \subseteq \mathfrak{M}$. So, $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{M})$ and $Z_{\mathsf{Lie}}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \cap Z_{\mathsf{Lie}}(\mathfrak{M})$. Hence $dim([\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \cap Z_{\mathsf{Lie}}(\mathfrak{M})) = 1$.

d) $\Rightarrow$ *b*) Assume that d) holds. For every two-sided ideal $\mathfrak{M}$ of $\mathfrak{g}$ of codimension 1, $Z_{\mathsf{Lie}}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \subseteq \mathfrak{M}$. So, $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{M})$ and thus $Z_{\mathsf{Lie}}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \cap Z_{\mathsf{Lie}}(\mathfrak{M})$. Hence $dim(Z_{\mathsf{Lie}}(\mathfrak{g})) = 1$.

This completes the proof. □

Proposition 3.13. *Let $\mathfrak{g}$ be a non-Lie Leibniz algebra. If $\mathfrak{g}$ is* Lie*-pseudo-abelian, then $dim(Z_{\mathsf{Lie}}(\mathfrak{M})) = 2$ and $Z_{\mathsf{Lie}}(\mathfrak{g}) = Z_{\mathsf{Lie}}(\mathfrak{M}) \cap [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ for every two-sided ideal $\mathfrak{M}$ of $\mathfrak{g}$ of codimension one.*

Proof. For every two-sided ideal $\mathfrak{M}$ of $\mathfrak{g}$ of codimension 1, $Z_{\mathsf{Lie}}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \subseteq \mathfrak{M}$ since $\mathfrak{g}$ is a Lie-pseudo-abelian. So, $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{M})$ and thus $Z_{\mathsf{Lie}}(\mathfrak{g}) = Z_{\mathsf{Lie}}(\mathfrak{M}) \cap [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$. Finally, we have by the assertions b) and c) of Proposition 3.12 that $dim(Z_{\mathsf{Lie}}(\mathfrak{M})) = 2$. □

Proposition 3.14. *Let $\mathfrak{g}$ be a Leibniz algebra of type $(\mathcal{T}_{\mathsf{Lie}})$. Then*

$$Z(\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})) = 0.$$

Proof. Let $\mathcal{T}$, and θ be as in Definition 3.1 and let $d \in Z(\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g}))$. For any $a, b \in \mathfrak{g}$, $d([a,b]_{\mathsf{Lie}}) = d([a,b]) + d([b,a]) = d(R_b)(a) + d(R_a)(b) = 0$, since the inner derivations R_a and R_b are Lie-derivations (see Example 2.6). Now let $x \in \mathcal{T}$, since θ is nondegenerate, there exists a non zero vector $y \in \mathcal{T}$ such that $\theta(x,y) \neq 0$ for some $y \in \mathcal{T}$. Set $T_1 = \{u \in \mathcal{T} \mid \theta(x,u) = 0\}$. It is easy to show that $\mathcal{T}_1$ is a subspace of $\mathcal{T}$ of codimension 1. Set $\mathfrak{M}_1 := [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} + \mathcal{T}_1$. Clearly, $\mathfrak{M}_1$ is an ideal of $\mathfrak{g}$ of codimension 1, $x \in Z_{\mathsf{Lie}}(\mathfrak{M}_1)$ by Definition 3.1(b), and $\mathfrak{g} = \mathbb{K}y + \mathfrak{M}_1$. Now, let $d' : \mathfrak{g} \to \mathfrak{g}$ defined by $d'(\lambda y + m_1) := \lambda x$. It is easy to show that d' is a Lie-derivation of $\mathfrak{g}$. So $d(x) = d(d'(y)) - d'(d(y)) = 0$. Therefore $d(\mathfrak{g}) = 0$, and hence $d = 0$. □

Proposition 3.15. *Let $\mathfrak{g}$ be a non-Lie Leibniz algebra in the class $\mathfrak{D}_{\mathsf{Lie}}$ such that $\mathfrak{g}$ is not of type (T_{Lie}) and has no nonzero abelian direct summands. Then $\mathfrak{g}$ has a Lie-derivation in $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$.*

Proof. Let $\mathfrak{g}$ be a non-Lie Leibniz algebra satisfying the hypothesis of the proposition. Then $\mathfrak{g}$ is a Lie-stem Leibniz algebra. Otherwise, there exists $x \in Z_{\mathsf{Lie}}(\mathfrak{g})$ such that $0 \neq x \notin [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. Consider the one-dimensional subspace of $\mathfrak{g}$ spanned by $\langle x \rangle$ and let M_x be its complementary in $\mathfrak{g}$. Then M_x is a two-sided ideal of $\mathfrak{g}$ and $\mathfrak{g} = \langle x \rangle \oplus M_x$. Thus $\langle x \rangle$ is a nonzero abelian direct summand of $\mathfrak{g}$. This contradicts the hypothesis. Now, since $\mathfrak{g}$ is a Lie-stem Leibniz algebra and is not of type (T_{Lie}), then by Theorem 3.8, there exists a two-sided ideal M_e of $\mathfrak{g}$ of codimension one such that $Z_{\mathsf{Lie}}(M_e) \subset [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. So we have $\mathfrak{g} = \langle e \rangle \oplus M_e$ with $0 \neq Z_{\mathsf{Lie}}(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}} \subset M_e$, $Z_{\mathsf{Lie}}(\mathfrak{g}) \neq M_e$ and $e \in \mathfrak{g} - M_e$. Choose $z_0 \in Z_{\mathsf{Lie}}(\mathfrak{g})$ and define the linear map $d : \mathfrak{g} \to \mathfrak{g}$ by $d(x) = \alpha_x z_0$ for any $x = \alpha_x e + m_x$. Clearly, d is an derivation of the Leibniz algebra $\mathfrak{g}$ in $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. □

Definition 3.16. *A Lie-derivation $d : \mathfrak{g} \to \mathfrak{g}$ is said to be semi-simple if $Im(d) = \oplus_{\alpha \in \wedge} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is a subspace of $\mathfrak{g}$ defined by $\mathfrak{g}_\alpha = \{a \in \mathfrak{g} : d(a) = \alpha a\}$and $\wedge$ is a set of indexes.*

Proposition 3.17. *Let $\mathfrak{g}$ be a non-Lie Leibniz algebra of type (T_{Lie}) such that $\mathfrak{g}^{(1)}_{\mathsf{Lie}} \neq \mathfrak{g}^{(2)}_{\mathsf{Lie}}$. Then $\mathfrak{g}$ is a direct sum of a* Lie*-nilpotent Leibniz subalgebra of $\mathfrak{g}$ and $\mathfrak{g}$ has a semi-simple* Lie*-derivation.*

Proof. Since $\mathfrak{g}$ is of type (T_{Lie}) with the condition $\mathfrak{g}^{(1)}_{\mathsf{Lie}} \neq \mathfrak{g}^{(2)}_{\mathsf{Lie}}$, then there exists $z \in Z_{\mathsf{Lie}}(\mathfrak{g})$ such that $\mathfrak{g}^{(1)}_{\mathsf{Lie}} = \langle z \rangle \oplus \mathfrak{g}^{(2)}_{\mathsf{Lie}}$. Set $\mathfrak{g}_1 = \mathcal{T} \oplus \langle z \rangle$ and $\mathfrak{g}_2 = \mathfrak{g}^{(2)}_{\mathsf{Lie}}$. We have $[\mathfrak{g}_1, \mathfrak{g}_1]_{\mathsf{Lie}} = [\mathcal{T}, \mathcal{T}]_{\mathsf{Lie}} = \langle z \rangle = Z_{\mathsf{Lie}}(\mathfrak{g}_1)$. Thus $\mathfrak{g}_1$ is a pseudo-abelian Leibniz subalgebra of $\mathfrak{g}$, and therefore it is Lie-nilpotent Leibniz subalgebra of $\mathfrak{g}$ with nilpotency class 2. So $\mathfrak{g} = \mathcal{T} \oplus \langle z \rangle \oplus \mathfrak{g}^{(2)} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Assume now that $\mathbb{K}$ is a field of characteristic different from 2 and consider the linear map defined as follows:

$$\begin{aligned} d_0 : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x = x_1 + \alpha_x z + x_2 &\longmapsto d_0(x) = x_1 + 2\alpha_x z \end{aligned} .$$

We show that d_0 is a derivation. Indeed,

$$\begin{gathered} d_0([x_1 + \alpha_x z + x_2, y_1 + \alpha_y z + y_2]_{\mathsf{Lie}}) = \\ = d_0(\alpha_1 z) = 2\alpha_1 z = \alpha_1 z + \alpha_1 z = [x_1, y_1]_{\mathsf{Lie}} + [x_1, y_1]_{\mathsf{Lie}} = \\ = [x_1 + 2\alpha_x z, y_1 + \alpha_y z + y_2]_{\mathsf{Lie}} + [x_1 + \alpha_x z + x_2, y_1 + 2\alpha_y z]_{\mathsf{Lie}} = \\ = [d_0(x_1+\alpha_x z+x_2), y_1+\alpha_y z+y_2]_{\mathsf{Lie}}+[x_1+\alpha_x z+x_2, d_0(y_1+2\alpha_y z+y_2)]_{\mathsf{Lie}}. \end{gathered}$$

In addition $Im(d_0) = \mathcal{T} \oplus [\mathcal{T},\mathcal{T}]_{\mathsf{Lie}}$, $\mathcal{T} = \{x \in \mathfrak{g} : d_0(x) = x\}$ and $[\mathcal{T},\mathcal{T}]_{\mathsf{Lie}} = \{y \in \mathfrak{g} : d_0(y) = 2y\}$. Thus d_0 is a semi-simple Lie-derivation □

Remark 3.18. If $\mathfrak{g}$ is a non-Lie Leibniz algebra, then $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ is an ideal of $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$.

Proof. Let $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$, $d' \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ and $x, y \in \mathfrak{g}$. Then

$$[[d', d](x), y]_{\mathsf{Lie}} = [d'(d(x)), y]_{\mathsf{Lie}} - [d(d'(x)), y]_{\mathsf{Lie}} = [d'(d(x)), y]_{\mathsf{Lie}}.$$

In addition

$$d'([d(x), y]_{\mathsf{Lie}}) = 0 = [d'(d(x)), y]_{\mathsf{Lie}} + [d(x), d'(y)]_{\mathsf{Lie}} = [d'(d(x)), y]_{\mathsf{Lie}}.$$

Hence $[d', d] \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. Therefore $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ is an ideal of $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$. □

Remark 3.19. Let $\mathfrak{g}$ be a non-Lie Leibniz algebra of type (T_{Lie}). Then $\mathfrak{g}$ has a central Lie-derivation.

Proof. Since $\mathfrak{g}$ is of type (T_{Lie}), then $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}} \subsetneq \mathfrak{g}$ by the proof of Lemma 3.7. Let $\mathfrak{M}$ be a two-sided ideal of $\mathfrak{g}$ of codimension one, and set $\mathfrak{g} = \langle e \rangle \oplus \mathfrak{M}$. Choose $z \in Z_{\mathsf{Lie}}(\mathfrak{g})$ and define the linear map

$$\begin{aligned} d : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x = \alpha_x e + m &\longmapsto d(x) = \alpha_x z. \end{aligned}$$

It is easy to show that d is a central Lie-derivation. □

Proposition 3.20. *Let $\mathfrak{g}$ be a Lie-solvable Leibniz algebra with nonzero Lie-center. If $\mathfrak{g}$ is not pseudo-abelian and if $\mathfrak{g}$ has no nonzero abelian direct summands, then $\mathfrak{g}$ has a Lie-derivation in $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$.*

Proof. Let $\mathfrak{g}$ be a Lie-solvable Leibniz algebra such that $Z_{\mathsf{Lie}}(\mathfrak{g}) \neq \{0\}$, and assume that $\mathfrak{g}$ satisfies the hypotheses of the proposition. First, we prove that $\mathfrak{g} \in \mathfrak{D}_{\mathsf{Lie}}$. Indeed, assume the contrary, then since $Z_{\mathsf{Lie}}(\mathfrak{g}) \neq \{0\}$, it follows that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. So the sequence of ideals $(\mathfrak{g}_{\mathsf{Lie}}^{(k)})_{k \geq 0}$ defined inductively by: $\mathfrak{g}_{\mathsf{Lie}}^{(0)} = \mathfrak{g}$ and for all $k \geq 1$, $\mathfrak{g}_{\mathsf{Lie}}^{(k)} = [\mathfrak{g}_{\mathsf{Lie}}^{(k-1)}, \mathfrak{g}_{\mathsf{Lie}}^{(k-1)}]_{\mathsf{Lie}}$, is a constant sequence. It follows by the Lie-solvability of $\mathfrak{g}$ that there exists $r > 0$ such that $\{0\} \neq Z_{\mathsf{Lie}}(\mathfrak{g}) \subset \mathfrak{g} = \mathfrak{g}_{\mathsf{Lie}}^{(0)} = \mathfrak{g}_{\mathsf{Lie}}^{(r)} = \{0\}$. This is absurd. Thus $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$ and $\mathfrak{g} \in \mathfrak{D}_{\mathsf{Lie}}$. Moreover $\mathfrak{g}_{\mathsf{Lie}}^{(2)} \subsetneq \mathfrak{g}$. Now assume that $\mathfrak{g}$ is of type (T_{Lie}). Then $\mathfrak{g} = \mathcal{T} + [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$ for some nonzero subspace $\mathcal{T}$ of $\mathfrak{g}$. This implies that $[\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}} = [\mathcal{T}, \mathcal{T}]_{\mathsf{Lie}} + [[\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}, [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}]_{\mathsf{Lie}} =$

$\mathbb{K}z + \mathfrak{g}^{(2)}_{\mathsf{Lie}}$. Thus $\mathfrak{g}^{(2)}_{\mathsf{Lie}} = \mathfrak{g}^{(3)}_{\mathsf{Lie}}$. So the sequence $(\mathfrak{g}^{(k)}_{\mathsf{Lie}})_{k\geq 0}$, as defined above, is stationary. It follows that $\mathfrak{g}^{(2)}_{\mathsf{Lie}} = 0$ since $\mathfrak{g}$ is Lie-solvable. Therefore $[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} = \mathbb{K}z = [\mathcal{T},\mathcal{T}]_{\mathsf{Lie}}$ with $z \in Z_{\mathsf{Lie}}(\mathfrak{g})$. Moreover, by the proof of Theorem 3.8, $\mathfrak{g}$ is a Lie-stem Leibniz algebra. Therefore, $[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{g}) = \langle z \rangle$, that is, $\mathfrak{g}$ is a pseudo-abelian Leibniz algebra satisfying $\mathfrak{g}^{(2)}_{\mathsf{Lie}} \neq \mathfrak{g}^{(1)}_{\mathsf{Lie}}$. This contradicts the hypothesis. So $\mathfrak{g}$ is not of type (T_{Lie}). We now conclude by Proposition 3.15 that $\mathfrak{g}$ has a Lie-derivation in $\mathsf{Der}^{\mathsf{Lie}}_z(\mathfrak{g})$. □

References

[1] G.R. Biyogmam and J.M. Casas, *On Lie-isoclinic Leibniz algebras*, J. Algebra, **499**, 2018, pp. 337–357.

[2] G.R. Biyogmam and J.M. Casas, *The c-nilpotent Schur Lie-multiplier of Leibniz algebras*, J. Geom. Phys., **138**, 2019, pp. 55–69.

[3] G.R. Biyogmam and J.M. Casas, *A Study of n-Lie-isoclinic Leibniz Algebras*, J. Algebra Appl., **19** (1), 2019, 2050013.

[4] G.R. Biyogmam and J.M. Casas, *Lie-central derivations, Lie-centroids and Lie-stem Leibniz algebras*, Publ. Math. Debrecen, **97**, 1-2, 2020, pp. 217–239.

[5] G.R. Biyogmam, C. Tcheka and D.A. Kamgam, *Some Properties of ID_*-n-Lie-derivations of Leibniz algebras*, Asian-Eur. J. Math., 2021, 2250054.

[6] G.R. Biyogmam and C. Tcheka, *A note on outer derivations of Leibniz algebras*, Communication in Algebra, **49** (5), 2021, pp. 2190–2198.

[7] A. Bloh, *A generalization of the concept of a Lie algebra,* Dokl. Akad. Nauk SSSR, **165** (3), 1965, pp. 471–473.

[8] J.M. Casas and M.A. Insua, *The Schur Lie-multiplier of Leibniz algebras*, Quaest. Math., **41** (7), 2018, pp. 917–936.

[9] J.M. Casas and E. Khmaladze, *On Lie-central extensions of Leibniz algebras*, RACSAM, **111** (1), 2017, pp. 39–56.

[10] J.M. Casas and T. Van der Linden, *Universal central extensions in semi-abelian categories*, Appl. Categor. Struct., **22** (1), 2014, pp. 253–268.

[11] C. Cuvier, *Algèbres de Leibnitz: définitions, propriétés*, Ann. Sci. École. Norm. Sup., **27** (4), 1994, pp. 1–45.

[12] J.-L. Loday, *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Enseign. Math., **39**, 1993, pp. 269–292.

[13] I.S. Rakhimov, K.K. Masutova, and B.A. Omirov, *On Derivations of Semisimple Leibniz Algebras*, Bull. Malays. Math. Sci. Soc., **40**, 2017, pp. 295–306.

[14] S. Tôgô, *Outer derivations of Lie algebras*, Trans. Amer. Math. Soc., **128**, 1967, pp. 264–276.

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