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RESEARCH ARTICLE

A note on multidegrees of automorphisms of the form $(\exp D)_\star$

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Abstract. Let k be a field of characteristic zero. For any polynomial mapping $F = (F_1, \dots, F_n) : k^n \to k^n$ by multidegree of F we mean the following n-tuple of natural numbers $\operatorname{mdeg} F = (\deg F_1, \dots, \deg F_n)$.

Let us denote by $k[x] = k[x_1, \dots, x_n]$ a ring of polynomials in n variables $x_1, \dots, x_n$ over k. If $D : k[x] \to k[x]$ is a locally nilpotent k-derivation, then one can define the automorphism $\exp D$ of k-algebra $k[x]$ and then the polynomial automorphism $(\exp D)_\star$ of k^n. In this note we present a general upper bound of $\operatorname{mdeg}(\exp D)_\star$ in the case of a triangular derivation D, and also show that this estimation is exact.

Introduction

Let k be a field of characteristic zero, and let $k[x] = k[x_1, \dots, x_n]$ be a ring of polynomials in n variables $x_1, \dots, x_n$ over k. Let us recall that a mapping $D : k[x] \to k[x]$ is called *k-derivation* of $k[x]$ when it is k-linear and satisfies the Leibniz rule:

$$D(fg) = D(f)g + fD(g) \qquad \text{for all } f, g \in k[x]. \tag{1}$$

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The set of all k-derivations of $k[x]$ we will denote by $\mathrm{Der}_k(k[x])$. For any $D \in \mathrm{Der}_k(k[x])$ we define the *kernel* of D as the following subset of $k[x]$:

$$\ker D = \{\, a \in k[x] \,:\, D(a) = 0 \,\}. \tag{2}$$

If $D \in \mathrm{Der}_k(k[x])$, then $\ker D$ is a k-subalgebra of $k[x]$. In particular, if $D_1, D_2 \in \mathrm{Der}_k(k[x])$ are such that $D_1(x_i) = D_2(x_i)$ for $i = 1, \dots, n$, then $\ker(D_1 - D_2) = k[x]$, and so $D_1 = D_2$. This means that for any $D \in \mathrm{Der}_k(k[x])$ we have the following equality

$$D = D(x_1)\frac{\partial}{\partial x_1} + \cdots + D(x_n)\frac{\partial}{\partial x_n}, \tag{3}$$

where $\frac{\partial}{\partial x_i} : k[x] \to k[x]$ is the usually defined partial derivative with respect to the variable x_i.

Let us also recall that a derivation $D \in \mathrm{Der}_k(k[x])$ is called *locally nilpotent* if for any $f \in k[x]$ there is a number $m \in \mathbb{N}$ such that $D^m(f) = 0$, where $D^0 = \mathrm{id}_{k[x]}$ and $D^{l+1} = D \circ D^l$ for any $l \in \mathbb{N}$. The set of all locally nilpotent derivations of $k[x]$ will be denoted by $\mathrm{LND}_k(k[x])$.

Assume that we are given an arbitrary derivation $D \in \mathrm{LND}_k(k[x])$. Then, one can define the following map

$$\exp D : k[x] \ni f \mapsto \sum_{i=0}^{\infty} \frac{1}{i!} D^i(f) \in k[x], \tag{4}$$

which is a homomorphism of k-algebras. If $D_1, D_2 \in \mathrm{LND}_k(k[x])$ are such that $D_1 \circ D_2 = D_2 \circ D_1$, then

$$\exp D_1 \circ \exp D_2 = \exp(D_1 + D_2) = \exp D_2 \circ \exp D_1. \tag{5}$$

In particular

$$\exp D \circ \exp(-D) = \exp(-D) \circ \exp D = \exp 0 = \mathrm{id}_{k[x]} \tag{6}$$

for any $D \in \mathrm{LND}_k(k[x])$. This means that for any $D \in \mathrm{LND}_k(k[x])$ the mapping $\exp D$ is an automorphism of the k-algebra $k[x]$. For more information about derivations and polynomial automorphisms we refer to [1, 3].

For the convenience of the reader let us recall that for any polynomial mapping $F = (F_1, \dots, F_n) : k^n \to k^n$ the mapping $F^\star : k[x] \ni h \mapsto h \circ F = h(F_1, \dots, F_n) \in k[x]$ is a k-algebra homomorphism and for any

k-algebra homomorphism $\varphi : k[x] \to k[x]$ the mapping $\varphi_\star = (F_1, \dots, F_n)$, where $F_i = \varphi(x_i)$ for $i = 1, \dots, n$, is a polynomial mapping of k^n.

The multidegrees of polynomial mappings seem to be a useful tool in studying polynomial automorphisms. For example, the first author and J. Zygadło proved in [6], using multidegrees, that for the following slight modification of the Nagata automorphism $\tilde{\sigma} : \mathbb{C}^3 \ni (x, y, z) \mapsto (z, y - z(zx + y^2), x + 2y(zx + y^2) - z(zx + y^2)^2) \in \mathbb{C}^3$ and any $n \in \mathbb{N} \setminus \{0\}$, the automorphism $\tilde{\sigma}^n : \mathbb{C}^3 \to \mathbb{C}^3$ is wild (i.e. it is not a composition of triangular and affine automorphisms). The question about wildness of the Nagata automorphism $\sigma : \mathbb{C}^3 \ni (x, y, z) \mapsto (x + 2y(zx + y^2) - z(zx + y^2)^2, y - z(zx + y^2), z) \in \mathbb{C}^3$ was open since 1972 up to 2003 [9, 10]. It is known that the Nagata automorphism can be obtained in the form $(\exp D)_\star$ for some locally nilpotent derivation (see e.g. [8]). In this context it seems to be interesting to know something about $\operatorname{mdeg}(\exp D)_\star$, and in this note we establish an upper bound of $\operatorname{mdeg}(\exp D)_\star$ in the case of a triangular derivation D, and show that this estimation cannot be improved. For the first result about multidegrees of polynomial automorphisms see [4], and for more information about multidegrees we refer to [2, 5, 7].

1. Weighted degree and general estimation of multidegree for triangular derivation

Consider a k-derivation $D = f_1 \frac{\partial}{\partial x_1} + \dots + f_n \frac{\partial}{\partial x_n}$ of $k[x]$, where $f_1, \dots, f_n \in k[x]$. We say that D is *triangular* if $f_1 \in k$ and $f_i \in k[x_1, \dots, x_{i-1}]$ for $i = 2, \dots, n$. One can check that if $D \in \operatorname{Der}_k(k[x])$ is triangular, then $D \in \operatorname{LND}_k(k[x])$.

Now, we define an useful weighted degree on $k[x]$ associated with a given triangular derivation $D = f_1 \frac{\partial}{\partial x_1} + \dots + f_n \frac{\partial}{\partial x_n} \in \operatorname{Der}_k(k[x])$. In order to define $w = (w_1, \dots, w_n) \in \mathbb{N}_+^n$, we put

$$w_1 = 1 \qquad \text{and} \qquad w_i = \max\{1, \deg_{(w_1,\dots,w_{i-1})} f_i\} \text{ for } i = 2, \dots, n. \quad (7)$$

In the above formula for $w_2, \dots, w_n$ we use the fact that $f_i \in k[x_1, \dots, x_{i-1}]$ for $i = 2, \dots, n$, and so $\deg_{(w_1,\dots,w_{i-1})} f_i$ means the weighted degree of f_i considered as an element of $k[x_1, \dots, x_{i-1}]$, where the weighted degree function $\deg_{(w_1,\dots,w_{i-1})} : k[x_1, \dots, x_{i-1}] \to \mathbb{N} \cup \{-\infty\}$ is defined by $\deg_{(w_1,\ldots,w_{i-1})} x_l = w_l$ for $l = 1,\ldots,i-1$. One can notice that, in the case $f_2,\ldots,f_n \notin k$, the above defined $w = (w_1,\ldots,w_n) \in \mathbb{N}_+^n$ is the unique element of $\mathbb{N}_+^n$ such that $w_1 = 1$ and $\deg_w f_i = w_i$ for $i = 2,\ldots,n$, where $\deg_w f_i$ means, of course, the w-degree of $f_2,\ldots,f_n$ considered as elements of $k[x_1,\ldots,x_n]$.

Now, we are in a position to prove the following theorem.

Theorem 1. *Let $D = f_1\frac{\partial}{\partial x_1} + f_2\frac{\partial}{\partial x_2} + \cdots + f_n\frac{\partial}{\partial x_n}$ be a triangular k-derivation of $k[x]$ with $f_1 \in k$ and $f_i \in k[x_1,\ldots,x_{i-1}]$ for $i = 2,\ldots,n$.*

If $w = (w_1,\ldots,w_n) \in \mathbb{N}_+^n$ is defined as above and $m = (m_1,\ldots,m_n) =$ $\operatorname{mdeg}(\exp D)_\star$, then we have

$$m_1 = w_1, \quad m_2 = w_2 \qquad \textit{and} \qquad m_i \le w_i \quad \textit{for } i = 3,\ldots,n. \tag{8}$$

Proof. First, notice that

$$(\exp D)(x_1) = x_1 + f_1 \tag{9}$$

and

$$(\exp D)(x_2) = \begin{cases} x_2 + f_2 & \text{if } f_2 \in k \\ x_2 + f_2 + \sum\limits_{l=1}^{d} \frac{1}{(l+1)!} f_1^l \left(\frac{\partial}{\partial x_1}\right)^l (f_2), & \text{if } f_2 \in k[x_1] \setminus k \end{cases} \tag{10}$$

where $d = \deg_{x_1} f_2$.

By (9) and $f_1 \in k$, we obtain $\deg\left((\exp D)(x_1)\right) = 1$. In the case $f_2 \in k$, by (10), we also obtain $\deg\left((\exp D)(x_2)\right) = 1$. On the other hand, in the case $f_2 \in k[x_1] \setminus k$ (in which $d \ge 1$), we have $\deg f_2 >$ $\deg\left(\frac{\partial}{\partial x_1}(f_2)\right) > \ldots > \deg\left(\left(\frac{\partial}{\partial x_1}\right)^d (f_2)\right)$, and so

$$\deg\left((\exp D)(x_2)\right) = \deg\left(x_2 + f_2\right) = \deg f_2 = \deg_w f_2.$$

In both cases, we have $\deg\left((\exp D)(x_2)\right) = w_2$.

Now, take any $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n \setminus \{0\}$. By the chain rule for the derivation D and properties of degree function, we have

$$\begin{aligned}
&\deg_w\left(D(x_1^{\alpha_1}\cdots x_n^{\alpha_n})\right) \qquad (11)\\
=&\ \deg_w\left(\sum_{i=1}^{n}\alpha_i x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}x_i^{\alpha_i-1}x_{i+1}^{\alpha_{i+1}}\cdots x_n^{\alpha_n}D(x_i)\right)\\
=&\ \deg_w\left(\sum_{i=1}^{n}\alpha_i x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}x_i^{\alpha_i-1}x_{i+1}^{\alpha_{i+1}}\cdots x_n^{\alpha_n}f_i\right)\\
\leq&\ \max\left\{\deg_w\left(\alpha_i x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}x_i^{\alpha_i-1}x_{i+1}^{\alpha_{i+1}}\cdots x_n^{\alpha_n}f_i\right)\ :\ i=1,\ldots,n\right\}.
\end{aligned}$$

Let us notice that, by definition of $w=(w_1,\ldots,w_n)$, for $\alpha_i\neq 0$, we have

$$\begin{aligned}
&\deg_w\left(\alpha_i x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}x_i^{\alpha_i-1}x_{i+1}^{\alpha_{i+1}}\cdots x_n^{\alpha_n}f_i\right) \qquad (12)\\
&=\alpha_1w_1+\cdots+\alpha_nw_n-w_i+\deg_w f_i\\
&\leq \alpha_1w_1+\cdots+\alpha_nw_n=\deg_w\left(x_1^{\alpha_1}\cdots x_n^{\alpha_n}\right).
\end{aligned}$$

By (11) and (12), we obtain

$$\deg_w\left(D(x_1^{\alpha_1}\cdots x_n^{\alpha_n})\right)\leq \deg_w(x_1^{\alpha_1}\cdots x_n^{\alpha_n}). \qquad (13)$$

Now, we check that the above inequality is also valid for any polynomial $h\in k[x]$. The inequality is obviously true if $h=0$, so we can assume that $h\neq 0$. Then, $h=\sum_{\alpha\in\operatorname{supp}h}a_\alpha x^\alpha$, where for $\alpha=(\alpha_1,\ldots,\alpha_n)$ we write x^α instead of $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. By (13), k-linearity of D and properties of degree function, we obtain

$$\begin{aligned}
\deg_w\left(D(h)\right) &= \deg_w\left(\sum_{\alpha\in\operatorname{supp}h}a_\alpha D(x^\alpha)\right) \qquad (14)\\
&\leq \max_{\alpha\in\operatorname{supp}h}\deg_w\left(D(x^\alpha)\right)\leq \max_{\alpha\in\operatorname{supp}h}\deg_w(x^\alpha)=\deg_w h.
\end{aligned}$$

Now, take any $h\in k[x]$ and choose $d\in\mathbb{N}_+$ such that $D^{d+1}(h)=0$. Then, by (14), we get

$$\begin{aligned}
\deg_w\left((\exp D)(h)\right) &= \deg_w\left(h+\sum_{i=1}^{d}\frac{1}{i!}D^i(h)\right) \qquad (15)\\
&\leq \max\left\{\deg_w h,\deg_w\left(D(h)\right),\ldots,\deg_w\left(D^d(h)\right)\right\}\\
&= \deg_w h.
\end{aligned}$$

Since $w_1 \geq 1, \ldots, w_n \geq 1$, it follows that for any polynomial $P \in k[x]$ we have $\deg P \leq \deg_w P$. Thus, for any $h \in k[x]$, we get

$$\deg\left((\exp D)(h)\right) \leq \deg_w\left((\exp D)(h)\right) \leq \deg_w h. \tag{16}$$

In particular, we obtain $\deg\left((\exp D)(x_i)\right) \leq \deg_w x_i = w_i$ for $i = 3, \ldots, n$. □

2. Exactness of the estimation in Theorem 1

In this section, we give a large family of triangular derivations for which, in Theorem 1 we obtain the equality. Nonemptiness of this family shows that the estimation given in Theorem 1 cannot be improved.

First, notice that since k is of characteristic zero, we can assume that $\mathbb{Q} \subset k$, where $\mathbb{Q}$ denotes the field of rational numbers. By $\mathbb{Q}_{\geq 0}$ and $\mathbb{Q}_{\geq 0}[x_1, \ldots, x_n]$ we will denote, respectively, the set of all nonnegative rational numbers and the set of all polynomials with coefficients in $\mathbb{Q}_{\geq 0}$.

In order to prove the nonemptiness of the above mentioned family we will use the following fact.

Lemma 1. *Let* $w = (w_1, \ldots, w_n) \in \mathbb{N}_+^n$ *be arbitrary and* $D_1 = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \cdots + a_n \frac{\partial}{\partial x_n}$, $D_2 = b_2 \frac{\partial}{\partial x_2} + \cdots + b_n \frac{\partial}{\partial x_n}$ *be two triangular k-derivations such that* $\deg_w a_i < w_i$, $\deg_w b_i = w_i$ *and* $b_i \in \mathbb{Q}_{\geq 0}[x_1, \ldots, x_{i-1}]$ *for* $i = 2, \ldots, n$.

Then, the following hold:

(1) *For any* $h \in k[x] \setminus \{0\}$ *we have* $\deg_w D_1(h) < \deg_w h$.

(2) *For any* $h \in k[x]$ *we have* $\deg_w D_2(h) \leq \deg_w h$.

(3) *If* $h \in \mathbb{Q}_{\geq 0}[x_1, \ldots, x_n]$, *then* $D_2(h) \in \mathbb{Q}_{\geq 0}[x_1, \ldots, x_n]$.

(4) *If* $h \in \mathbb{Q}_{\geq 0}[x_1, \ldots, x_n] \cap \ker D_2$, *then* $h \in \mathbb{Q}_{\geq 0}[x_1]$.

(5) *If* $h \in \mathbb{Q}_{\geq 0}[x_1, \ldots, x_n] \setminus \ker D_2$ *is w-homogeneous, then* $\deg_w D_2(h) = \deg_w h$.

(6) *If* $b_2, \ldots, b_n$ *are w-homogeneous, then for each w-homogeneous* $h \in \mathbb{Q}_{\geq 0}[x_1, \ldots, x_n] \setminus \ker D_2$, $D_2(h)$ *is w-homogeneous with* $\deg_w D_2(h) = \deg_w h$.

Proof. To obtain (1) and (2) one can use similar arguments as in the proof of Theorem 1 (see the second and third paragraphs of the proof).

The statement (3) is a consequence of the straightforward calculation.

To prove (4) take any $h = \sum_{\alpha\in\operatorname{supp} h} a_\alpha x^\alpha \in \mathbb{Q}_{\geq 0}[x]$. Since $D_2(h) = \sum_{\alpha\in\operatorname{supp} h} D_2(a_\alpha x^\alpha)$ and, by (3), for each $\alpha \in \operatorname{supp} h$ we have $D_2(a_\alpha x^\alpha) \in \mathbb{Q}_{\geq 0}[x]$, it follows that monomials occurring in $D_2\left(a_\beta x^\beta\right)$ for a fixed $\beta \in \operatorname{supp} h$ cannot be vanished by monomials occurring in the sum $\sum_{\alpha\in\operatorname{supp} h\setminus\{\beta\}} D_2(a_\alpha x^\alpha)$.

Thus, we obtain that

$$\operatorname{supp} D_2(h) = \bigcup_{\alpha\in\operatorname{supp} h} \operatorname{supp} D_2(a_\alpha x^\alpha) = \bigcup_{\alpha\in\operatorname{supp} h} \operatorname{supp} D_2(x^\alpha) \qquad (17)$$

and

$$D_2(h) = 0 \qquad \Leftrightarrow \qquad D_2(x^\alpha) = 0 \quad \text{for each } \alpha \in \operatorname{supp} h, \qquad (18)$$

because $f = 0$ iff $\operatorname{supp} f = \emptyset$. By definition of D_2 one can easily check that if $\alpha \in \operatorname{supp} h \setminus \mathbb{N} \times \{(0, \ldots, 0)\}$, then $D_2(x^\alpha) \neq 0$. This completes the proof of (4).

To obtain (5) and (6) one can repeat carefully, for each $\alpha \in \operatorname{supp} h$, similar calculations as in (12). Indeed, if $h = \sum_{\alpha\in\operatorname{supp} h} a_\alpha x^\alpha \in \mathbb{Q}_{\geq 0}[x]$, then

$$\begin{aligned} D_2(h) &= \sum_{\alpha\in\operatorname{supp} h} a_\alpha D_2(x^\alpha) \\ &= \sum_{\alpha\in\operatorname{supp} h} \sum_{i=2}^{n} a_\alpha \left(\alpha_i x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_i^{\alpha_i - 1} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} b_i\right), \end{aligned}$$

and by calculations as in (12), we have

$$\deg_w \left(\alpha_i x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_i^{\alpha_i - 1} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} b_i\right) = \deg_w h.$$

If b_i are w-homogeneous, then all summands of (19) are w-homogeneous of w-degree equal to $\deg_w h$. This proves (6).

When not all of $b_2, \ldots, b_n$ are w-homogeneous, observe that $\bar{b}_i \in \mathbb{Q}_{\geq 0}[x]$, where $\bar{f}$ denotes the w-homogeneous component of f with maximal w-degree. This implies that

$$a_\alpha \left(\alpha_i x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_i^{\alpha_i - 1} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} \bar{b}_i\right) \in \mathbb{Q}_{\geq 0}[x],$$

and so

$$\sum_{\alpha\in\operatorname{supp} h} \sum_{i=2}^{n} a_\alpha \left(\alpha_i x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_i^{\alpha_i - 1} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} \bar{b}_i\right) \neq 0. \qquad (19)$$

Since all summands in the above sum are w-homogeneous of w-degree equal to $\deg_w h$ and any other summands of (19) have strictly lower w-degree than $\deg_w h$ we obtain (5). □

Now, we are in a position to prove the following theorem which gives us the above mentioned family of triangular derivations.

Theorem 2. *Let $D = f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n}$ be a triangular k-derivation of $k[x]$ with $f_1 \in k$ and $f_i \in k[x_1, \ldots, x_{i-1}] \setminus k$ for $i = 2, \ldots, n$.*

Assume that $w = (w_1, \ldots, w_n) \in \mathbb{N}_+^n$ is defined as in the previous section, and that $\bar{f}_i \in \mathbb{Q}_{\geq 0}[x]$ for $i = 2, \ldots, n$, where $\bar{h}$ denotes the w-homogeneous component of h with maximal w-degree. Then

$$\operatorname{mdeg}(\exp D)_\star = (w_1, \ldots, w_n). \tag{20}$$

Proof. First, notice that by assumption that $f_2, \ldots, f_n \notin k$, we have $\deg_w f_i = w_i$ for $i = 2, \ldots, n$.

Consider the following k-derivations:

$$D_1 = f_1\frac{\partial}{\partial x_1} + (f_2 - \bar{f}_2)\frac{\partial}{\partial x_2} + \cdots + (f_n - \bar{f}_n)\frac{\partial}{\partial x_n} \tag{21}$$

and

$$D_2 = \bar{f}_2\frac{\partial}{\partial x_2} + \cdots + \bar{f}_n\frac{\partial}{\partial x_n}. \tag{22}$$

Then, for any $l = 1, 2, \ldots,$ we have

$$D^l = (D_1 + D_2)^l = \sum_{(\varepsilon_1, \ldots, \varepsilon_l) \in \{1,2\}^l} D_{\varepsilon_l} \circ \cdots \circ D_{\varepsilon_1}. \tag{23}$$

Using Lemma 1(1) and (2), we obtain that for any $l = 1, 2, \ldots,$ $h \in k[x] \setminus \{0\}$ and $(\varepsilon_1, \ldots, \varepsilon_l) \in \{1, 2\}^l \setminus \{(2, \ldots, 2)\}$ the following holds

$$\deg_w (D_{\varepsilon_l} \circ \cdots \circ D_{\varepsilon_1})(h) < \deg_w h. \tag{24}$$

Using Lemma 1(3), (5) and (6), we obtain that if $h \in \mathbb{Q}_{\geq 0}[x]$ is w-homogeneous then

$$(D_2)^l(h) = 0 \qquad \text{or} \qquad \deg_w (D_2)^l(h) = \deg_w h \qquad \text{for } l = 1, 2 \ldots \tag{25}$$

Now, we claim that for any $h \in k[x] \setminus k$ such that $\bar{h} \in \mathbb{Q}_{\geq 0}[x]$ we have:

$$\deg((\exp D)(h)) = \deg_w h. \tag{26}$$

If this is the case, then in particular we obtain $\deg((\exp D)(x_i)) = \deg_w x_i = w_i$ for $i = 2, \ldots, n$, Thus, to complete the proof it is enough to show the claim.

First, notice using (16) that

$$\deg\big((\exp D)(h - \bar{h})\big) \leq \deg_w(h - \bar{h}) < \deg_w h. \tag{27}$$

Since $(\exp D)(h) = (\exp D)(h - \bar{h}) + (\exp D)(\bar{h})$, it follows that we only need to show that

$$\deg\big((\exp D)(\bar{h})\big) = \deg_w \bar{h} = \deg_w h. \tag{28}$$

Take $d_1 \in \mathbb{N}_+$ such that $D^{d_1+1}(\bar{h}) = 0$. Then, by (23), we obtain

$$\begin{aligned}(\exp D)(\bar{h}) &= \bar{h} + \sum_{l=1}^{d_1} \frac{1}{l!} D^l(\bar{h}) \qquad (29)\\ &= \bar{h} + \sum_{l=1}^{d_1} \frac{1}{l!} \sum_{(\varepsilon_1,\ldots,\varepsilon_l)\in\{1,2\}^l\setminus\{(2,\ldots,2)\}} \left(D_{\varepsilon_l} \circ \cdots \circ D_{\varepsilon_1}\right)(\bar{h})\\ &\quad + \sum_{l=1}^{d_1} \frac{1}{l!} (D_2)^l(\bar{h}).\end{aligned}$$

By (24), properties of degree function and the fact that $w_1 \geq 1, \ldots, w_n \geq 1$, we obtain

$$\begin{aligned}&\deg\left(\sum_{l=1}^{d_1} \frac{1}{l!} \sum_{(\varepsilon_1,\ldots,\varepsilon_l)\in\{1,2\}^l\setminus\{(2,\ldots,2)\}} \left(D_{\varepsilon_l} \circ \cdots \circ D_{\varepsilon_1}\right)(\bar{h})\right) \qquad (30)\\ \leq\ &\deg_w\left(\sum_{l=1}^{d_1} \frac{1}{l!} \sum_{(\varepsilon_1,\ldots,\varepsilon_l)\in\{1,2\}^l\setminus\{(2,\ldots,2)\}} \left(D_{\varepsilon_l} \circ \cdots \circ D_{\varepsilon_1}\right)(\bar{h})\right) < \deg_w \bar{h}.\end{aligned}$$

Thus, now it is enough to show that $\deg\left(\bar{h} + \sum_{l=1}^{d_1} \frac{1}{l!}(D_2)^l(\bar{h})\right) = \deg_w \bar{h}$.

By Lemma 1(3), we obtain that if $x^\alpha \in \operatorname{supp}\big((D_2)^l(\bar{h})\big)$ for some l (we identify the monomial x^α with α), then

$$x^\alpha \in \operatorname{supp}\left(\bar{h} + \sum_{l=1}^{d_1} \frac{1}{l!}(D_2)^l(\bar{h})\right).$$

Take $d_2 \in \mathbb{N}$ such that $(D_2)^{d_2}(\bar{h}) \neq 0$ and $(D_2)^{d_2+1}(\bar{h}) = 0$. Of course, we have $d_2 \in \{0, 1, \ldots, d_1\}$. Using Lemma 1(3), (5) and (6), we obtain that $(D_2)^{d_2}(\bar{h}) \in \mathbb{Q}_{\geq 0}[x]$ is w-homogeneous of w-degree equal to $\deg_w h$. Since $(D_2)^{d_2}(\bar{h}) \in \mathbb{Q}_{\geq 0}[x] \cap \ker D_2$ is w-homogeneous of w-degree $\deg_w h$, it follows by Lemma 1(4)-(6) that $(D_2)^{d_2}(\bar{h}) = cx_1^{\deg_w h}$ for some $c \in \mathbb{Q}_{\geq 0}$, $c \neq 0$. Hence $x_1^{\deg_w h} \in \operatorname{supp}\left((D_2)^{d_2}(\bar{h})\right)$ and so $x_1^{\deg_w h} \in \operatorname{supp}\left(\bar{h} + \sum_{l=1}^{d_1} \frac{1}{l!}(D_2)^l(\bar{h})\right)$. This means that $\deg\left(\bar{h} + \sum_{l=1}^{d_1} \frac{1}{l!}(D_2)^l(\bar{h})\right)$ $\geq \deg_w \bar{h} = \deg_w h$. Thus the claim is proved, because $\bar{h} + \sum_{l=1}^{d_1} \frac{1}{l!}(D_2)^l(\bar{h})$ $= (\exp D_2)(\bar{h})$ and so one can use (16).

□

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