© Algebra and Discrete Mathematics
Volume **34** (2022). Number 1, pp. 68–104
DOI:10.12958/adm2036 RESEARCH ARTICLE

On the structure of low-dimensional Leibniz algebras: some revision

L. A. Kurdachenko, O. O. Pypka, and I. Ya. Subbotin

Dedicated to Professor M. M. Semko on the occasion of his 65th birthday

Abstract. Let L be an algebra over a field F with the binary operations $+$ and $[\cdot,\cdot]$. Then L is called a *left Leibniz algebra* if $[[a,b],c]=[a,[b,c]]-[b,[a,c]]$ for all $a,b,c\in L$. We describe the inner structure of left Leibniz algebras having dimension 3.

1. Introduction

Let L be an algebra over a field F with the binary operations $+$ and $[\cdot,\cdot]$. Then, L is called a *Leibniz algebra* (more precisely, a *left Leibniz algebra*) if, for all elements $a,b,c\in L$, it satisfies the Leibniz identity:

$$[[a,b],c]=[a,[b,c]]-[b,[a,c]].$$

We will also use another form of this identity:

$$[a,[b,c]]=[[a,b],c]+[b,[a,c]].$$

Leibniz algebras first appeared in the paper of A. Blokh [2] where they were called *D-algebras*. However, at that time, these works were not in demand. Only after two decades was there a real rise in interest towards Leibniz algebras. It happened thanks to the rediscovery of these algebras

The first author is grateful for the support Isaac Newton Institute for Mathematical Sciences and University of Edinburgh provided in the frame of LMS Solidarity Supplementary Grant Program.

2020 MSC: 17A32, 17A60, 17A99.

Key words and phrases: Leibniz algebra, nilpotent Leibniz algebra, dimension.

by J.-L. Loday [9] (see also [8, Section 10.6]), who used the term "Leibniz algebras" since it was Leibniz who discovered and proved the Leibniz rule for differentiation of the product of functions. The main motivation for the introduction of Leibniz algebras was the study of periodicity phenomena in algebraic K-theory. The Leibniz algebras appeared to be naturally related to several areas such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, non-commutative geometry, physics and so on. Nowadays, the theory of Leibniz algebras is one of actively developing areas of modern algebra.

Note that Lie algebras are a partial case of Leibniz algebras. Conversely, if L is a Leibniz algebra in which $[a, a] = 0$ for every element $a \in L$, then it is a Lie algebra. Thus, Leibniz algebras can be seen as a non-commutative generalization of Lie algebras.

The theory of Leibniz algebras has been intensely developing in many different directions. Some results of this theory were presented in the recent book [1].

One of the first steps in the theory of Leibniz algebras is the description of algebras with small dimensions. Unlike Lie algebras, the situation with Leibniz algebras of dimension 3 is very diverse. Leibniz algebras of dimension 3 are mostly described. The description of Leibniz algebras of dimensions 4 and 5 is quite complex. The list of papers devoted to these studies is quite large and we will not give it here in full. We only note that the Section 3.1 of book [1] is devoted to study of right Leibniz algebras having dimension 3. The investigation of Leibniz algebras having dimension 3 was carried out in articles [3–5, 10–12]. Some of these papers use the language of the right Leibniz algebras, whilst others use the language of left Leibniz algebras. Basically, the description is reduced to determining the structural constants of these algebras. However, the structural constants do not always give an idea of the internal structure of these algebras. Elucidation of the structure requires some additional analysis. The overall picture seems fragmented. Thus, the articles dealt with Leibniz algebras over concrete fields of real, complex, p-adic numbers, etc. Therefore, we cannot decide if the internal structure of these algebras really contains a complete description. For example, when passing from the field of rational numbers to the field of real numbers, some types of algebras disappear altogether. Furthermore, some sections describe right Leibniz algebras, while others describe left Leibniz algebras. Moreover, only structural constants were found. None of the articles considered the internal structure. So, under these circumstances, in order to observe the entire scope, there are two options that arise: analyze these articles and really make sure everything is done there or do it yourself all over again. The second option is preferable since it is better to complete the list within your own framework if it turns out that not all types of algebras have been considered. More importantly, we are interested in the description of the inner structure, not just structural constants. Of course, you can extract some information about the structure from structural constants, but it is more logical to obtain the structural constants in the process of describing the inner structure.

Therefore, in the current article, we present a description of left Leibniz algebras having dimension 3, focusing on clarifying their structure and obtaining structural constants by passing. This consideration of the structure of Leibniz algebras of dimension 3 is carried out over an arbitrary field F, and when studying these specific types of algebras, additional natural restrictions on the field F appear. These restrictions are very significant in some cases. Some types of algebras can exist only if sufficiently strict restrictions are imposed. Our goal was the most detailed description of these algebras, reflecting all the nuances of their structure.

2. Main results

Let L be a Leibniz algebra over a field F. Then L is called *abelian* if $[a, b] = 0$ for every elements $a, b \in L$. In particular, an abelian Leibniz algebra is a Lie algebra.

If A, B are subspaces of L, then $[A, B]$ will denote a subspace generated by all elements $[a, b]$ where $a \in A$, $b \in B$. As usual, a subspace A of L is called a *subalgebra* of L, if $[a, b] \in A$ for every $a, b \in A$. It follows that $[A, A] \leqslant A$. A subalgebra A of L is called a *left* (respectively *right*) *ideal* of L, if $[b, a] \in A$ (respectively $[a, b] \in A$) for every $a \in A$, $b \in L$. In other words, if A is a left (respectively right) ideal, then $[L, A] \leqslant A$ (respectively $[A, L] \leqslant A$). A subalgebra A of L is called an *ideal* of L (more precisely, *two-sided ideal*) if it is both a left ideal and a right ideal.

Every Leibniz algebra L possesses the following specific ideal. Denote by $\mathrm{Leib}(L)$ the subspace generated by the elements $[a, a]$, $a \in L$. It is not hard to prove that $\mathrm{Leib}(L)$ is an ideal of L. The ideal $\mathrm{Leib}(L)$ is called the *Leibniz kernel* of algebra L.

We note the following important property of the Leibniz kernel:

$$[[a, a], x] = [a, [a, x]] - [a, [a, x]] = 0.$$

The *left* (respectively *right*) *center* $\zeta^{\text{left}}(L)$ (respectively $\zeta^{\text{right}}(L)$) of a Leibniz algebra L is defined by the rule:

$$\zeta^{\text{left}}(L) = \{x \in L | \ [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively,

$$\zeta^{\text{right}}(L) = \{x \in L | \ [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but that is not true for the right center. Moreover, $\text{Leib}(L) \leqslant \zeta^{\text{left}}(L)$ so that $L/\zeta^{\text{left}}(L)$ is a Lie algebra. The right center is a subalgebra of L and, in general, the left and right centers are distinct (see, for example, [7]).

The *center* $\zeta(L)$ of L is defined by the rule:

$$\zeta(L) = \{x \in L | \ [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L.

Now we define the *upper central series*

$$\langle 0 \rangle = \zeta_0(L) \leqslant \zeta_1(L) \leqslant \dots \zeta_\alpha(L) \leqslant \zeta_{\alpha+1}(L) \leqslant \dots \zeta_\eta(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) = \bigcup_{\mu<\lambda} \zeta_\mu(L)$ for the limit ordinals λ. By definition, each term of this series is an ideal of L.

Define the *lower central series* of L

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \dots \gamma_\alpha(L) \geqslant \gamma_{\alpha+1}(L) \geqslant \dots \gamma_\tau(L)$$

by the rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ.

As usual, we say that a Leibniz algebra L is *nilpotent*, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class* c if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$. We denote the nilpotency class of L by $\text{ncl}(L)$.

Define the *lower derived series* of L

$$L = \delta_0(L) \geqslant \delta_1(L) \geqslant \dots \delta_\alpha(L) \geqslant \delta_{\alpha+1}(L) \geqslant \dots \delta_\nu(L)$$

by the rule: $\delta_0(L) = L$, $\delta_1(L) = [L, L]$, and recursively $\delta_{\alpha+1}(L) = [\delta_\alpha(L), \delta_\alpha(L)]$ for all ordinals α and $\delta_\lambda(L) = \bigcap_{\mu<\lambda} \delta_\mu(L)$ for the limit ordinals λ. If $\delta_n(L) = \langle 0 \rangle$ for some positive integer n, then we say that L is a *soluble* Leibniz algebra.

As usual, we say that a Leibniz algebra L is *finite dimensional* if the dimension of L as a vector space over F is finite.

If $\dim_F(L) = 1$, then L is abelian.

If $\dim_F(L) = 2$, then we obtain the following types of Leibniz algebras:

$$\mathrm{Lei}_1(2, F) = Fa \oplus Fb \quad \text{where } [a, a] = b, [a, b] = [b, a] = [b, b] = 0;$$
$$\mathrm{Lei}_2(2, F) = Fc \oplus Fd \quad \text{where } [c, c] = [c, d] = d, [d, c] = [d, d] = 0$$

(see, for example, [6]).

Moving on to Leibniz algebras of dimension 3, we immediately note that we will consider Leibniz algebras, which are not Lie algebras. This means that their Leibniz kernel is non-zero. Then, the factor-algebra over Leibniz kernel has dimension, at most, 2. Note that Lie algebras having dimension at most 2 are soluble. Thus, we obtain the following

Proposition 1. *Let L be a Leibniz algebra over a field F. Suppose that L is not a Lie algebra. If L has dimension* 3*, then L is soluble.*

For the Leibniz kernel $\mathrm{Leib}(L)$ of a Leibniz but not a Lie algebra, L having dimension 3 will give us only two possibilities: $\dim_F(\mathrm{Leib}(L)) = 1$ and $\dim_F(\mathrm{Leib}(L)) = 2$.

First, we will consider the situation when $\dim_F(\mathrm{Leib}(L)) = 1$. Immediately, we obtain the following two subcases:

(IA) the center of L includes $\mathrm{Leib}(L)$;

(IB) the Leibniz kernel of L is not central.

For each of these subcases, we have the following two possibilities:

(IA1) the factor-algebra $L/\mathrm{Leib}(L)$ is abelian;

(IA2) the factor-algebra $L/\mathrm{Leib}(L)$ is not abelian;

and

(IB1) the factor-algebra $L/\mathrm{Leib}(L)$ is abelian;

(IB2) the factor-algebra $L/\mathrm{Leib}(L)$ is not abelian.

Consider these cases.

Theorem 1. *Let L be a Leibniz algebra over a field F having dimension* 3*. Suppose that L is not a Lie algebra. If the center of L includes the Leibniz kernel,* $\dim_F(\mathrm{Leib}(L)) = 1$ *and the factor-algebra* $L/\mathrm{Leib}(L)$ *is abelian, then L is an algebra of one of the following types.*

(i) $\mathrm{Lei}_3(3, F) = L_3$ *is a direct sum of two ideals* $A = Fa_1 \oplus Fa_3$ *and* $B = Fa_2$*. Moreover, A is a nilpotent cyclic Leibniz algebra of dimension* 2*,*

$[A, B] = [B, A] = \langle 0 \rangle$, *so that* $L_3 = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_3$, $[a_1, a_2] = [a_1, a_3] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_3) = [L_3, L_3] = Fa_3$, $\zeta^{\mathrm{left}}(L_3) = \zeta^{\mathrm{right}}(L_3) = \zeta(L_3) = Fa_2 \oplus Fa_3$, L_3 *is nilpotent and* $\mathrm{ncl}(L_3) = 2$.

(ii) $\mathrm{Lei}_4(3, F) = L_4$ *is a direct sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2$. *Moreover, A is a nilpotent cyclic Leibniz algebra of dimension* 2, $[A, B] = Fa_3$, $[B, A] = \langle 0 \rangle$, *so that* $L_4 = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_1, a_2] = a_3$, $[a_1, a_3] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_4) = [L_4, L_4] = \zeta^{\mathrm{right}}(L_4) = \zeta(L_4) = Fa_3$, $\zeta^{\mathrm{left}}(L_4) = Fa_2 \oplus Fa_3$, L_4 *is nilpotent and* $\mathrm{ncl}(L_4) = 2$.

(iii) $\mathrm{Lei}_5(3, F) = L_5$ *is a direct sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2$. *Moreover, A is a nilpotent cyclic Leibniz algebra of dimension* 2, $[A, B] = \langle 0 \rangle$, $[B, A] = Fa_3$, *so that* $L_5 = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_2, a_1] = a_3$, $[a_1, a_2] = [a_1, a_3] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_5) = [L_5, L_5] = \zeta^{\mathrm{left}}(L_5) = \zeta(L_5) = Fa_3$, $\zeta^{\mathrm{right}}(L_5) = Fa_2 \oplus Fa_3$, L_5 *is nilpotent and* $\mathrm{ncl}(L_5) = 2$.

(iv) $\mathrm{Lei}_6(3, F) = L_6$ *is a direct sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2$. *Moreover, A is a nilpotent cyclic Leibniz algebra of dimension* 2, $[A, B] = [B, A] = Fa_3$, *so that* $L_6 = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_2, a_1] = a_3$, $[a_1, a_2] = \alpha a_3$ $(\alpha \neq 0)$, $[a_1, a_3] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_6) = [L_6, L_6] = \zeta^{\mathrm{right}}(L_6) = \zeta^{\mathrm{left}}(L_6) = \zeta(L_6) = Fa_3$, L_6 *is nilpotent and* $\mathrm{ncl}(L_6) = 2$.

(v) $\mathrm{Lei}_7(3, F) = L_7$ *is a sum of two nilpotent cyclic ideals* $A = Fa_1 \oplus Fa_3$ *and* $C = Fa_2 \oplus Fa_3$, $[A, C] = [C, A] = \langle 0 \rangle$, *so that* $L_7 = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_3$, $[a_2, a_2] = \beta a_3$ $(\beta \neq 0)$, $[a_1, a_2] = [a_1, a_3] = [a_2, a_1] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_7) = [L_7, L_7] = \zeta^{\mathrm{right}}(L_7) = \zeta^{\mathrm{left}}(L_7) = \zeta(L_7) = Fa_3$. *Moreover, polynomial* $X^2 + \beta$ *has no root in field* F, L_7 *is nilpotent and* $\mathrm{ncl}(L_7) = 2$.

(vi) $\mathrm{Lei}_8(3, F) = L_8$ *is a sum of two nilpotent cyclic ideals* $A = Fa_1 \oplus Fa_3$ *and* $C = Fa_2 \oplus Fa_3$, $[A, C] = Fa_3$, $[C, A] = \langle 0 \rangle$, *so that* $L_8 = Fa_1 \oplus Fa_2 \oplus Fa_3$, $[a_1, a_1] = a_3$, $[a_1, a_2] = \alpha a_3$, $[a_2, a_2] = \beta a_3$ $(\alpha, \beta \neq 0)$, $[a_1, a_3] = [a_2, a_1] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_8) = [L_8, L_8] = \zeta^{\mathrm{right}}(L_8) = \zeta^{\mathrm{left}}(L_8) = \zeta(L_8) = Fa_3$. *Moreover, polynomial* $X^2 + \alpha X + \beta$ *has no root in field* F, L_8 *is nilpotent and* $\mathrm{ncl}(L_8) = 2$.

(vii) $\mathrm{Lei}_9(3, F) = L_9$ *is a sum of two nilpotent cyclic ideals* $A = Fa_1 \oplus Fa_3$ *and* $B = Fa_2 \oplus Fa_3$ *such that* $[A, B] = \langle 0 \rangle$, $[B, A] = Fa_3$, *so that* $L_9 = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_3$, $[a_2, a_1] = a_3$, $[a_2, a_2] = \sigma a_3$ $(\sigma \neq 0)$, $[a_1, a_2] = [a_1, a_3] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_9) =$

$[L_9, L_9] = \zeta^{\text{right}}(L_9) = \zeta(L_9) = \zeta^{\text{left}}(L_9) = Fa_3$. *Moreover, polynomial* $X^2 + X + \sigma$ *has no root in field* F, L_9 *is nilpotent and* $\text{ncl}(L_9) = 2$.

(viii) $\text{Lei}_{10}(3, F) = L_{10}$ *is a sum of two nilpotent cyclic ideals* $A = Fa_1 \oplus Fa_3$ *and* $B = Fa_2 \oplus Fa_3$ *such that* $[A, B] = [B, A] = Fa_3$, *so that* $L_{10} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_3$, $[a_2, a_1] = a_3$, $[a_1, a_2] = \tau a_3$ $(\tau \neq 0)$, $[a_2, a_2] = \sigma a_3$ $(\sigma \neq 0)$, $[a_1, a_3] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\text{Leib}(L_{10}) = [L_{10}, L_{10}] = \zeta^{\text{right}}(L_{10}) = \zeta^{\text{left}}(L_{10}) = \zeta(L_{10}) = Fa_3$. *Moreover, polynomial* $X^2 + (\tau + 1)X + \sigma$ *has no root in field* F, L_{10} *is nilpotent and* $\text{ncl}(L_{10}) = 2$.

Proof. We note that the center $\zeta(L)$ has dimension at most 2. Suppose first that $\dim_F(\zeta(L)) = 2$. Since L is not a Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. We note that $a_3 \in \zeta(L)$. It follows that $[a_1, a_3] = [a_3, a_1] = [a_3, a_3] = 0$. Being an abelian algebra of dimension 2, $\zeta(L)$ has a direct decomposition $\zeta(L) = Fa_2 \oplus Fa_3$ for some element a_2. Put $A = Fa_1 \oplus Fa_3$ and $B = Fa_2$, then $B \leqslant \zeta(L)$, so that B is an ideal of L. Clearly, $L = Fa_1 \oplus Fa_2 \oplus Fa_3 = A \oplus B$ and A is also an ideal of L. Moreover, A is a nilpotent cyclic Leibniz algebra of dimension 2. Thus, we come to the following type of nilpotent Leibniz algebras:

$$\begin{gathered} L_3 = Fa_1 \oplus Fa_2 \oplus Fa_3 \quad \text{where } [a_1, a_1] = a_3, \\ [a_1, a_2] = [a_1, a_3] = [a_2, a_1] = [a_2, a_2] \\ = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0. \end{gathered}$$

Note also that $\text{Leib}(L_3) = [L_3, L_3] = Fa_3$, $\zeta^{\text{left}}(L_3) = \zeta^{\text{right}}(L_3) = \zeta(L_3) = Fa_2 \oplus Fa_3$, $\text{ncl}(L_3) = 2$.

Suppose now that the center of L has dimension 1. In this case, $\zeta(L) = \text{Leib}(L)$. Since L is not a Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. We note that $a_3 \in \zeta(L)$. It follows that $[a_1, a_3] = [a_3, a_1] = [a_3, a_3] = 0$. Then $\zeta(L) = Fa_3$. Since $L/\text{Leib}(L)$ is abelian, for every element $x \in L$ we have $[a_1, x], [x, a_1] \in \zeta(L) \leqslant \langle a_1 \rangle = Fa_1 \oplus Fa_3$. It follows that subalgebra $\langle a_1 \rangle$ is an ideal of L. Since $\dim_F(\langle a_1 \rangle) = 2$, $\langle a_1 \rangle \neq L$.

Suppose first that there exists an element b such that $b \notin \langle a_1 \rangle$ and $[b, b] = 0$. We have $[b, a_1] = \gamma a_3$ for some $\gamma \in F$. The following two cases appear here: $\gamma = 0$ and $\gamma \neq 0$. Let $\gamma = 0$. Then $[a_1, b] = \alpha a_3$ for some $\alpha \in F$. If we suppose that $\alpha = 0$, then $b \in \zeta(L)$. But in this case, $\dim_F(\zeta(L)) = 2$, and we obtain a contradiction, which shows that $\alpha \neq 0$. Put $a_2 = \alpha^{-1}b$, then $[a_2, a_2] = [a_2, a_1] = 0$, $[a_1, a_2] = a_3$, and we come to the following nilpotent Leibniz algebra:

$$L_4 = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = [a_1, a_2] = a_3,$$
$$[a_1, a_3] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_4) = [L_4, L_4] = Fa_3$, $\zeta^{\mathrm{left}}(L_4) = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_4) = \zeta(L_4) = Fa_3$, $\mathrm{ncl}(L_4) = 2$.

Let $\gamma \neq 0$. Put $a_2 = \gamma^{-1}b$, then $[a_2, a_2] = 0$, $[a_2, a_1] = a_3$. We have $[a_1, a_2] = \alpha a_3$ for some element $\alpha \in F$. If $\alpha = 0$, then $a_2 \in \zeta^{\mathrm{right}}(L)$, and we come to the following nilpotent Leibniz algebra:

$$L_5 = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = [a_2, a_1] = a_3,$$
$$[a_1, a_2] = [a_1, a_3] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_5) = [L_5, L_5] = Fa_3$, $\zeta^{\mathrm{right}}(L_5) = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{left}}(L_5) = \zeta(L_5) = Fa_3$, $\mathrm{ncl}(L_5) = 2$.

Suppose that $\alpha \neq 0$. Then, we come to the following type of nilpotent Leibniz algebras:

$$L_6 = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = [a_2, a_1] = a_3,$$
$$[a_1, a_2] = \alpha a_3 \ (\alpha \neq 0),$$
$$[a_1, a_3] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_6) = [L_6, L_6] = \zeta^{\mathrm{right}}(L_6) = \zeta^{\mathrm{left}}(L_6) = \zeta(L_6) = Fa_3$, $\mathrm{ncl}(L_6) = 2$.

Suppose now that $[b, b] \neq 0$ for every element $b \notin \langle a_1 \rangle$. In particular, it follows that $b \notin \zeta(L)$. Put $[b, b] = \beta a_3$ where $\beta \in F$. We have $[b, a_1] = \gamma a_3$ and $[a_1, b] = \alpha a_3$ for some elements $\alpha, \gamma \in F$. If $\alpha = \gamma = 0$, then put $a_2 = b$ and denote by C the subalgebra generates by a_2. Then C is a cyclic nilpotent ideal such that $[A, C] = [C, A] = \langle 0 \rangle$. Furthermore, let $u = \lambda a_1 + \mu a_2 + \nu a_3$ be the arbitrary element of L. Then,

$$[\lambda a_1 + \mu a_2 + \nu a_3, \lambda a_1 + \mu a_2 + \nu a_3] = \lambda^2 [a_1, a_1] + \mu^2 [a_2, a_2] =$$
$$\lambda^2 a_3 + \beta \mu^2 a_3 = (\lambda^2 + \beta \mu^2) a_3.$$

If $u \notin \langle a_1 \rangle$, then $(\lambda, \mu) \neq (0, 0)$. It follows that polynomial $X^2 + \beta$ has no root in field F. Thus, we come to the following type of nilpotent Leibniz algebras:

$$L_7 = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3, [a_2, a_2] = \beta a_3 \ (\beta \neq 0),$$
$$[a_1, a_2] = [a_1, a_3] = [a_2, a_1] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\text{Leib}(L_7) = [L_7, L_7] = \zeta^{\text{right}}(L_7) = \zeta^{\text{left}}(L_7) = \zeta(L_7) = Fa_3$. Moreover, polynomial $X^2 + \beta$ has no root in field F, $\text{ncl}(L_7) = 2$.

Suppose now that $\gamma = 0$ and $\alpha \neq 0$. Put $a_2 = b$. Then, $[a_2, a_1] = 0$, $[a_1, a_2] = \alpha a_3$, $[a_2, a_2] = \beta a_3$. Let $\lambda a_1 + \mu a_2 + \nu a_3$ be an arbitrary element of L. Then,

$$[\lambda a_1 + \mu a_2 + \nu a_3, \lambda a_1 + \mu a_2 + \nu a_3] =$$
$$\lambda^2[a_1, a_1] + \lambda\mu[a_1, a_2] + \lambda\mu[a_2, a_1] + \mu^2[a_2, a_2] =$$
$$\lambda^2 a_3 + \lambda\mu\alpha a_3 + \mu^2\beta a_3 = (\lambda^2 + \lambda\mu\alpha + \mu^2\beta)a_3.$$

As above, $\lambda \neq 0$, $\mu \neq 0$. It follows that polynomial $X^2 + \alpha X + \beta$ has no root in field F. Thus, we come to the following type of nilpotent Leibniz algebras:

$$L_8 = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3,$$
$$[a_1, a_2] = \alpha a_3 \ (\alpha \neq 0), [a_2, a_2] = \beta a_3 \ (\beta \neq 0),$$
$$[a_1, a_3] = [a_2, a_1] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\text{Leib}(L_8) = [L_8, L_8] = \zeta^{\text{right}}(L_8) = \zeta^{\text{left}}(L_8) = \zeta(L_8) = Fa_3$. Moreover, polynomial $X^2 + \alpha X + \beta$ has no root in field F, $\text{ncl}(L_8) = 2$.

Suppose now that $\gamma \neq 0$ and $\alpha = 0$. Put $a_2 = \gamma^{-1}b$. Then, $[a_2, a_1] = a_3$, $[a_1, a_2] = 0$, $[a_2, a_2] = \gamma^{-2}\beta a_3 = \sigma a_3$. Let $\lambda a_1 + \mu a_2 + \nu a_3$ be an arbitrary element of L. Then,

$$[\lambda a_1 + \mu a_2 + \nu a_3, \lambda a_1 + \mu a_2 + \nu a_3] =$$
$$\lambda^2[a_1, a_1] + \lambda\mu[a_2, a_1] + \mu^2[a_2, a_2] =$$
$$\lambda^2 a_3 + \lambda\mu a_3 + \mu^2\sigma a_3 = (\lambda^2 + \lambda\mu + \mu^2\sigma)a_3.$$

As above, $\lambda \neq 0$, $\mu \neq 0$. It follows that polynomial $X^2 + X + \sigma$ has no root in field F. Thus, we come to the following type of nilpotent Leibniz algebras:

$$L_9 = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3, [a_2, a_1] = a_3,$$
$$[a_2, a_2] = \sigma a_3 \ (\sigma \neq 0),$$
$$[a_1, a_2] = [a_1, a_3] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\text{Leib}(L_9) = [L_9, L_9] = \zeta^{\text{right}}(L_9) = \zeta^{\text{left}}(L_9) = \zeta(L_9) = Fa_3$. Moreover, polynomial $X^2 + X + \sigma$ has no root in field F, $\text{ncl}(L_9) = 2$.

Suppose now that $\gamma \neq 0$ and $\alpha \neq 0$. Put $a_2 = \gamma^{-1}b$. Then, $[a_2, a_1] = a_3$, $[a_1, a_2] = \gamma^{-1}\alpha a_3 = \tau a_3$, $[a_2, a_2] = \gamma^{-2}\beta a_3 = \sigma a_3$. Let $\lambda a_1 + \mu a_2 + \nu a_3$ be an arbitrary element of L. Then,

$$[\lambda a_1 + \mu a_2 + \nu a_3, \lambda a_1 + \mu a_2 + \nu a_3] =$$
$$\lambda^2[a_1, a_1] + \lambda\mu[a_1, a_2] + \lambda\mu[a_2, a_1] + \mu^2[a_2, a_2] =$$
$$\lambda^2 a_3 + \lambda\mu\tau a_3 + \lambda\mu a_3 + \mu^2\sigma a_3 = (\lambda^2 + \lambda\mu(\tau + 1) + \mu^2\sigma)a_3.$$

As above, $\lambda \neq 0$, $\mu \neq 0$. It follows that polynomial $X^2 + (\tau + 1)X + \sigma$ has no root in field F. Thus, we come to the following type of nilpotent Leibniz algebras:

$$L_{10} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3, [a_2, a_1] = a_3,$$
$$[a_1, a_2] = \tau a_3\ (\tau \neq 0), [a_2, a_2] = \sigma a_3\ (\sigma \neq 0),$$
$$[a_1, a_3] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{10}) = [L_{10}, L_{10}] = \zeta^{\mathrm{right}}(L_{10}) = \zeta^{\mathrm{left}}(L_{10}) = \zeta(L_{10}) = Fa_3$. Moreover, polynomial $X^2 + (\tau + 1)X + \sigma$ has no root in field F, $\mathrm{ncl}(L_{10}) = 2$. □

Theorem 2. *Let L be a Leibniz algebra of dimension 3 over a field F. Suppose that L is not a Lie algebra. If the center of L includes the Leibniz kernel, $\dim_F(\mathrm{Leib}(L)) = 1$ and factor-algebra $L/\mathrm{Leib}(L)$ is non-abelian, then L is an algebra of one of the following types.*

(i) $\mathrm{Lei}_{11}(3, F) = L_{11}$ *is a direct sum of ideal $A = Fa_1 \oplus Fa_3$ and subalgebra $B = Fa_2$. Moreover, A is a nilpotent cyclic Leibniz algebra of dimension 2, $[A, B] = [B, A] = Fa_1$, so that $L_{11} = Fa_1 \oplus Fa_2 \oplus Fa_3$ where $[a_1, a_1] = a_3$, $[a_1, a_2] = -a_1$, $[a_2, a_1] = a_1$, $[a_1, a_3] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{11}) = \zeta^{\mathrm{left}}(L_{11}) = \zeta^{\mathrm{right}}(L_{11}) = \zeta(L_{11}) = Fa_3$, $[L_{11}, L_{11}] = Fa_1 \oplus Fa_3$, L_{11} is non-nilpotent.*

(ii) $\mathrm{Lei}_{12}(3, F) = L_{12}$ *is a direct sum of ideal $A = Fa_1 \oplus Fa_3$ and subalgebra $B = Fa_2$. Moreover, A is a nilpotent cyclic Leibniz algebra of dimension 2, $[A, B] = [B, A] = Fa_1 \oplus Fa_3$, so that $L_{12} = Fa_1 \oplus Fa_2 \oplus Fa_3$ where $[a_1, a_1] = a_3$, $[a_1, a_2] = -a_1 - \alpha a_3$, $[a_2, a_1] = a_1 + \alpha a_3$ $(\alpha \neq 0)$, $[a_1, a_3] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{12}) = \zeta^{\mathrm{left}}(L_{12}) = \zeta^{\mathrm{right}}(L_{12}) = \zeta(L_{12}) = Fa_3$, $[L_{12}, L_{12}] = Fa_1 \oplus Fa_3$, L_{12} is non-nilpotent.*

(iii) $\mathrm{Lei}_{13}(3, F) = L_{13}$ *is a sum of ideal $A = Fa_1 \oplus Fa_3$ and subalgebra $B = Fa_2 \oplus Fa_3$. Moreover, A, B are nilpotent cyclic Leibniz algebras of dimension 2, $[A, B] = [B, A] = Fa_1$, so that $L_{13} = Fa_1 \oplus Fa_2 \oplus Fa_3$*

where $[a_1, a_1] = a_3$, $[a_1, a_2] = -a_1$, $[a_2, a_1] = a_1$, $[a_2, a_2] = \gamma a_3$ $(\gamma \neq 0)$, $[a_1, a_3] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{13}) = \zeta^{\mathrm{left}}(L_{13}) = \zeta^{\mathrm{right}}(L_{13}) = \zeta(L_{13}) = Fa_3$, $[L_{13}, L_{13}] = Fa_1 \oplus Fa_3$. *Moreover, polynomial* $X^2 + \gamma$ *has no root in field* F, L_{13} *is non-nilpotent.*

(iv) $\mathrm{Lei}_{14}(3, F) = L_{14}$ *is a sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2 \oplus Fa_3$. *Moreover,* A, B *are nilpotent cyclic Leibniz algebras of dimension* 2, $[A, B] = [B, A] = Fa_1 \oplus Fa_3$, *so that* $L_{14} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_3$, $[a_1, a_2] = -a_1 - \alpha a_3$, $[a_2, a_1] = a_1 + \alpha a_3$ $(\alpha \neq 0)$, $[a_2, a_2] = \gamma a_3$ $(\gamma \neq 0)$, $[a_1, a_3] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{14}) = \zeta^{\mathrm{left}}(L_{14}) = \zeta^{\mathrm{right}}(L_{14}) = \zeta(L_{14}) = Fa_3$, $[L_{14}, L_{14}] = Fa_1 \oplus Fa_3$. *Moreover, polynomial* $X^2 + \gamma$ *has no root in field* F, L_{14} *is non-nilpotent.*

(v) $\mathrm{Lei}_{15}(3, F) = L_{15}$ *is a direct sum of ideal* $B = Fa_2$ *and a subalgebra* $A = Fa_1 \oplus Fa_3$. *Moreover,* A *is a nilpotent cyclic Leibniz algebra of dimension* 2, $[A, B] = [B, A] = Fa_2$, *so that* $L_{15} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_3$, $[a_1, a_2] = a_2$, $[a_2, a_1] = -a_2$, $[a_1, a_3] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{15}) = \zeta^{\mathrm{left}}(L_{15}) = \zeta^{\mathrm{right}}(L_{15}) = \zeta(L_{15}) = Fa_3$, $[L_{15}, L_{15}] = Fa_2 \oplus Fa_3$, L_{15} *is non-nilpotent.*

(vi) $\mathrm{Lei}_{16}(3, F) = L_{16}$ *is a sum of abelian ideal* $B = Fa_2 \oplus Fa_3$ *and a subalgebra* $A = Fa_1 \oplus Fa_3$. *Moreover,* A *is a nilpotent cyclic Leibniz algebra of dimension* 2, $[A, B] = [B, A] = Fa_2 \oplus Fa_3$, *so that* $L_{16} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_3$, $[a_1, a_2] = a_2 + \alpha a_3$, $[a_2, a_1] = -a_2 - \alpha a_3$ $(\alpha \neq 0)$, $[a_1, a_3] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{16}) = \zeta^{\mathrm{left}}(L_{16}) = \zeta^{\mathrm{right}}(L_{16}) = \zeta(L_{16}) = Fa_3$, $[L_{16}, L_{16}] = Fa_2 \oplus Fa_3$, L_{16} *is non-nilpotent.*

Proof. We note that the center $\zeta(L)$ has dimension at most 2. Suppose first that $\dim_F(\zeta(L)) = 2$. Since L is a not Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. We note that $a_3 \in \zeta(L)$. It follows that $[a_1, a_3] = [a_3, a_1] = [a_3, a_3] = 0$. Being an abelian algebra of dimension 2, $\zeta(L)$ has a direct decomposition $\zeta(L) = Fa_2 \oplus Fa_3$ for some element a_2. We have $[a_1, a_2] = [a_2, a_1] = 0$. But, in this case, the factor-algebra $L/\mathrm{Leib}(L)$ is abelian, and we obtain a contradiction. This contradiction shows that $\zeta(L)$ has dimension 1 and hence, $\zeta(L) = \mathrm{Leib}(L)$.

As noted above, $L/\mathrm{Leib}(L)$ has an ideal $C/\mathrm{Leib}(L)$ of dimension 1 (i.e., $C = Fc \oplus \mathrm{Leib}(L)$ for some element c). If $[c, c] \neq 0$ without loss of generality, we can put $c = a_1$. The ideal $\langle a_1 \rangle = Fa_1 \oplus Fa_3 = A$ is nilpotent and has codimension 1. Let b be an element such that $L = A \oplus Fb$. We have $[b, b] = \gamma a_3$ for some element $\gamma \in F$. As noted above, in this case, $[b, a_1] \in a_1 + Fa_3$ so that $[b, a_1] = a_1 + \alpha a_3$ for some element $\alpha \in F$. We have also $[a_1, b] = -a_1 + \beta a_3$ for some element $\beta \in F$. Using the equality

$$[b, [a_1, b]] = [[b, a_1], b] + [a_1, [b, b]] = [[b, a_1], b],$$

we obtain $[b, -a_1 + \beta a_3] = [a_1 + \alpha a_3, b]$. It follows that $-[b, a_1] = [a_1, b]$, so that $[a_1, b] = -a_1 - \alpha a_3$.

Suppose that $\gamma = \alpha = 0$. Put $a_2 = b$. Then, $[a_2, a_2] = 0$, $[a_1, a_2] = -a_1$, $[a_2, a_1] = a_1$. Thus, we come to the following type of Leibniz algebras:

$$L_{11} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3, [a_1, a_2] = -a_1, [a_2, a_1] = a_1,$$
$$[a_1, a_3] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{11}) = \zeta^{\mathrm{left}}(L_{11}) = \zeta^{\mathrm{right}}(L_{11}) = \zeta(L_{11}) = Fa_3$, $[L_{11}, L_{11}] = Fa_1 \oplus Fa_3$, L_{11} is non-nilpotent.

Suppose now that $\gamma = 0$ and $\alpha \neq 0$. Put again $a_2 = b$. Then, $[a_2, a_2] = 0$, $[a_1, a_2] = -a_1 - \alpha a_3$, $[a_2, a_1] = a_1 + \alpha a_3$. Then, we come to the following type of Leibniz algebras:

$$L_{12} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3,$$
$$[a_1, a_2] = -a_1 - \alpha a_3, [a_2, a_1] = a_1 + \alpha a_3 \ (\alpha \neq 0),$$
$$[a_1, a_3] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{12}) = \zeta^{\mathrm{left}}(L_{12}) = \zeta^{\mathrm{right}}(L_{12}) = \zeta(L_{12}) = Fa_3$, $[L_{12}, L_{12}] = Fa_1 \oplus Fa_3$, L_{12} is non-nilpotent.

Suppose that $\gamma \neq 0$ and $\alpha = 0$. Put again $a_2 = b$. Then, $[a_2, a_2] = \gamma a_3$, $[a_1, a_2] = -a_1$, $[a_2, a_1] = a_1$. Let $\lambda a_1 + \mu a_2 + \nu a_3$ be an arbitrary element of L. Then,

$$[\lambda a_1 + \mu a_2 + \nu a_3, \lambda a_1 + \mu a_2 + \nu a_3] =$$
$$\lambda^2[a_1, a_1] + \lambda\mu[a_1, a_2] + \lambda\mu[a_2, a_1] + \mu^2[a_2, a_2] =$$
$$\lambda^2 a_3 - \lambda\mu a_1 + \lambda\mu a_1 + \mu^2\gamma a_3 = (\lambda^2 + \mu^2\gamma)a_3.$$

As above, $\lambda \neq 0$, $\mu \neq 0$. It follows that polynomial $X^2 + \gamma$ has no root in field F. Then, we come to the following type of Leibniz algebras:

$$L_{13} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3, [a_1, a_2] = -a_1, [a_2, a_1] = a_1,$$
$$[a_2, a_2] = \gamma a_3 \ (\gamma \neq 0),$$
$$[a_1, a_3] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{13}) = \zeta^{\mathrm{left}}(L_{13}) = \zeta^{\mathrm{right}}(L_{13}) = \zeta(L_{13}) = Fa_3$, $[L_{13}, L_{13}] = Fa_1 \oplus Fa_3$. Moreover, polynomial $X^2 + \gamma$ has no root in field F, L_{13} is non-nilpotent.

Suppose now that $\gamma \neq 0$ and $\alpha \neq 0$. Put again $a_2 = b$. Then, $[a_2, a_2] = \gamma a_3$, $[a_1, a_2] = -a_1 - \alpha a_3$, $[a_2, a_1] = a_1 + \alpha a_3$. Let $\lambda a_1 + \mu a_2 + \nu a_3$ be an arbitrary element of L. Then,

$$[\lambda a_1 + \mu a_2 + \nu a_3, \lambda a_1 + \mu a_2 + \nu a_3] =$$
$$\lambda^2 [a_1, a_1] + \lambda\mu[a_1, a_2] + \lambda\mu[a_2, a_1] + \mu^2[a_2, a_2] =$$
$$\lambda^2 a_3 - \lambda\mu(-a_1 - \alpha a_3) + \lambda\mu(a_1 + \alpha a_3) + \mu^2 \gamma a_3 = (\lambda^2 + \mu^2\gamma) a_3.$$

As above, $\lambda \neq 0$, $\mu \neq 0$. It follows that polynomial $X^2 + \gamma$ has no root in field F. Then, we come to the following type of Leibniz algebras:

$$L_{14} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3,$$
$$[a_1, a_2] = -a_1 - \alpha a_3, [a_2, a_1] = a_1 + \alpha a_3 \ (\alpha \neq 0),$$
$$[a_2, a_2] = \gamma a_3 \ (\gamma \neq 0),$$
$$[a_1, a_3] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{14}) = \zeta^{\mathrm{left}}(L_{14}) = \zeta^{\mathrm{right}}(L_{14}) = \zeta(L_{14}) = Fa_3$, $[L_{14}, L_{14}] = Fa_1 \oplus Fa_3$. Moreover, polynomial $X^2 + \gamma$ has no root in field F, L_{14} is non-nilpotent.

Suppose now that $[c, c] = 0$. Put again $\mathrm{Leib}(L) = Fa_3$. Since $\mathrm{Leib}(L) = \zeta(L)$, the ideal $C = Fc \oplus \mathrm{Leib}(L)$ is abelian. Suppose that there exists an element $b \notin C$ such that $[b, b] = 0$. Using the above arguments without loss of generality, we can assume that $[b, c] \in c + Fa_3$, $[c, b] \in -c + Fa_3$, so that $[b, c] = c + \alpha a_3$, $[c, b] = -c + \beta a_3$ for some elements $\alpha, \beta \in F$. Using the equality

$$[[b, c], b] = [b, [c, b]] - [c, [b, b]] = [b, [c, b]],$$

we obtain $[c + \alpha a_3, b] = [b, -c + \beta a_3]$. It follows that $[c, b] = -[b, c]$, so that $[c, b] = -c - \alpha a_3$. Let $u = \lambda c + \mu b + \nu a_3$ be an arbitrary element of L. Then,

$$[\lambda c + \mu b + \nu a_3, \lambda c + \mu b + \nu a_3] =$$
$$\lambda^2[c, c] + \lambda\mu[c, b] + \lambda\mu[b, c] + \mu^2[b, b] = \lambda\mu[c, b] + \lambda\mu[b, c] = 0.$$

Thus, we obtain a contradiction with the fact that L is not a Lie algebra. This contradiction shows that $[b, b] \neq 0$ for every element $b \notin C$. Hence, $[b, b] = \gamma a_3$ where γ is a non-zero element of field F. Without loss of generality, we can assume that $[b, b] = a_3$. Since $[b, b] \in \mathrm{Leib}(L) = \zeta(L)$, we obtain that $[c, b] = -[b, c]$, so that $[c, b] = -c - \alpha a_3$.

If $\alpha = 0$, then $[b, c] = c$, $[c, b] = -c$. Put $b = a_1$, $c = a_2$, then Fa_2 is an ideal of L, and we come to the following type of Leibniz algebras:

$$L_{15} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3, [a_1, a_2] = a_2, [a_2, a_1] = -a_2,$$
$$[a_1, a_3] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{15}) = \zeta^{\mathrm{left}}(L_{15}) = \zeta^{\mathrm{right}}(L_{15}) = \zeta(L_{15}) = Fa_3$, $[L_{15}, L_{15}] = Fa_2 \oplus Fa_3$, L_{15} is non-nilpotent.

If $\alpha \neq 0$, then $[b, c] = c + \alpha a_3$, $[c, b] = -c - \alpha a_3$, and we come to the following type of Leibniz algebras:

$$L_{16} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3,$$
$$[a_1, a_2] = a_2 + \alpha a_3, [a_2, a_1] = -a_2 - \alpha a_3 \ (\alpha \neq 0),$$
$$[a_1, a_3] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{16}) = \zeta^{\mathrm{left}}(L_{16}) = \zeta^{\mathrm{right}}(L_{16}) = \zeta(L_{16}) = Fa_3$, $[L_{16}, L_{16}] = Fa_2 \oplus Fa_3$, L_{16} is non-nilpotent. □

Theorem 3. *Let L be a Leibniz algebra of dimension 3 over a field F. Suppose that L is not a Lie algebra. If the center of L does not include the Leibniz kernel, $\dim_F(\mathrm{Leib}(L)) = 1$ and the factor-algebra $L/\mathrm{Leib}(L)$ is abelian, then L is an algebra of one of the following types.*

(i) $\mathrm{Lei}_{17}(3, F) = L_{17}$ *is a direct sum of two ideals $A = Fa_1 \oplus Fa_3$ and $B = Fa_2$. Moreover, A is a non-nilpotent cyclic Leibniz algebra of dimension 2 and $B = \zeta(L)$, $[A, B] = [B, A] = \langle 0 \rangle$, so that $L_{17} = Fa_1 \oplus Fa_2 \oplus Fa_3$ where $[a_1, a_1] = [a_1, a_3] = a_3$, $[a_1, a_2] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{17}) = [L_{17}, L_{17}] = Fa_3$, $\zeta^{\mathrm{left}}(L_{17}) = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{17}) = \zeta(L_{17}) = Fa_2$, L_{17} is non-nilpotent.*

(ii) $\mathrm{Lei}_{18}(3, F) = L_{18}$ *is a direct sum of ideal $A = Fa_1 \oplus Fa_3$ and a subalgebra $B = Fa_2$. Moreover, A is a non-nilpotent cyclic Leibniz algebra of dimension 2, $[A, B] = Fa_3$, $[B, A] = \langle 0 \rangle$, so that $L_{18} = Fa_1 \oplus Fa_2 \oplus Fa_3$ where $[a_1, a_1] = [a_1, a_2] = [a_1, a_3] = a_3$, $[a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{18}) = [L_{18}, L_{18}] = Fa_3$, $\zeta^{\mathrm{left}}(L_{18}) = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{18}) = \zeta(L_{18}) = \langle 0 \rangle$, L_{18} is non-nilpotent.*

(iii) $\mathrm{Lei}_{19}(3, F) = L_{19}$ *is a direct sum of ideal $A = Fa_1 \oplus Fa_3$ and a subalgebra $B = Fa_2$. Moreover, A is a non-nilpotent cyclic Leibniz algebra of dimension 2, $[A, B] = \langle 0 \rangle$, $[B, A] = Fa_3$, so that $L_{19} = Fa_1 \oplus Fa_2 \oplus Fa_3$ where $[a_1, a_1] = [a_1, a_3] = [a_2, a_1] = [a_2, a_3] = a_3$, $[a_1, a_2] = [a_2, a_2] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{19}) = [L_{19}, L_{19}] = \zeta^{\mathrm{left}}(L_{19}) = Fa_3$, $\zeta^{\mathrm{right}}(L_{19}) = Fa_2$, $\zeta(L_{19}) = \langle 0 \rangle$, L_{19} is non-nilpotent.*

(iv) $\mathrm{Lei}_{20}(3, F) = L_{20}$ *is a sum of two ideals* $A = Fa_1 \oplus Fa_3$ *and* $B = Fa_2 \oplus Fa_3$. *Furthermore, A is a non-nilpotent cyclic Leibniz algebra of dimension* 2, *B is a nilpotent cyclic Leibniz algebra of dimension* 2, $[A, B] = Fa_3$, $[B, A] = \langle 0 \rangle$, *so that* $L_{20} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_1, a_3] = a_3$, $[a_2, a_2] = \sigma a_3$ $(\sigma \neq 0)$, $[a_1, a_2] = [a_2, a_1] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{20}) = [L_{20}, L_{20}] = \zeta^{\mathrm{left}}(L_{20}) = Fa_3$, $\zeta^{\mathrm{right}}(L_{20}) = \zeta(L_{20}) = \langle 0 \rangle$. *Moreover, if* $\alpha a_1 + \beta a_2 + \gamma a_3 \notin A$, *then* $\alpha^2 + \alpha\gamma + \beta^2\sigma \neq 0$, L_{20} *is non-nilpotent.*

(v) $\mathrm{Lei}_{21}(3, F) = L_{21}$ *is a sum of two ideals* $A = Fa_1 \oplus Fa_3$ *and* $B = Fa_2 \oplus Fa_3$. *Furthermore, A is a non-nilpotent cyclic Leibniz algebra of dimension* 2, *B is a nilpotent cyclic Leibniz algebra of dimension* 2, $[A, B] = Fa_3$, $[B, A] = \langle 0 \rangle$, *so that* $L_{21} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_1, a_2] = [a_1, a_3] = a_3$, $[a_2, a_2] = \tau a_3$ $(\tau \neq 0)$, $[a_2, a_1] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{21}) = [L_{21}, L_{21}] = \zeta^{\mathrm{left}}(L_{21}) = Fa_3$, $\zeta^{\mathrm{right}}(L_{21}) = \zeta(L_{21}) = \langle 0 \rangle$. *Moreover, if* $\alpha a_1 + \beta a_2 + \gamma a_3 \notin A$, *then* $\alpha^2 + \alpha\beta + \alpha\gamma + \beta^2\tau \neq 0$, L_{21} *is non-nilpotent.*

(vi) $\mathrm{Lei}_{22}(3, F) = L_{22}$ *is a sum of two ideals* $A = Fa_1 \oplus Fa_3$ *and* $B = Fa_2 \oplus Fa_3$. *Furthermore, A, B are non-nilpotent cyclic Leibniz algebras of dimension* 2, $[A, B] = [B, A] = Fa_3$, *so that* $L_{22} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_1, a_3] = [a_2, a_1] = [a_2, a_3] = a_3$, $[a_2, a_2] = \tau a_3$ $(\tau \neq 0)$, $[a_1, a_2] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{22}) = [L_{22}, L_{22}] = \zeta^{\mathrm{left}}(L_{22}) = Fa_3$, $\zeta^{\mathrm{right}}(L_{22}) = \zeta(L_{22}) = \langle 0 \rangle$. *Moreover, if* $\alpha a_1 + \beta a_2 + \gamma a_3 \notin A$, *then* $\alpha^2 + \alpha\gamma + \alpha\beta + \beta^2\tau + \beta\gamma \neq 0$, L_{22} *is non-nilpotent.*

(vii) $\mathrm{Lei}_{23}(3, F) = L_{23}$ *is a sum of two ideals* $A = Fa_1 \oplus Fa_3$ *and* $B = Fa_2 \oplus Fa_3$. *Furthermore, A, B are non-nilpotent cyclic Leibniz algebras of dimension* 2, $[A, B] = [B, A] = Fa_3$, *so that* $L_{23} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_1, a_3] = [a_2, a_1] = [a_2, a_3] = a_3$, $[a_1, a_2] = \delta a_3$ $(\delta \neq 0)$, $[a_2, a_2] = \tau a_3$ $(\tau \neq 0)$, $[a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{23}) = [L_{23}, L_{23}] = \zeta^{\mathrm{left}}(L_{23}) = Fa_3$, $\zeta^{\mathrm{right}}(L_{23}) = \zeta(L_{23}) = \langle 0 \rangle$. *Moreover, if* $\alpha a_1 + \beta a_2 + \gamma a_3 \notin A$, *then* $\alpha^2 + \alpha\beta\delta + \alpha\gamma + \alpha\beta + \beta^2\tau + \beta\gamma \neq 0$, L_{23} *is non-nilpotent.*

Proof. Since $\dim_F(\mathrm{Leib}(L)) = 1$, $\mathrm{Leib}(L) \cap \zeta(L) = \langle 0 \rangle$. If we suppose that the center $\zeta(L)$ has dimension 2, then $L = \mathrm{Leib}(L) \oplus \zeta(L)$. But, in this case, L is abelian and we obtain a contradiction. Suppose now that $\dim_F(\zeta(L)) = 1$.

Since L is not a Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. Then, $\mathrm{Leib}(L) = Fa_3$. Then, $A = \langle a_1 \rangle = Fa_1 \oplus Fa_3$ is a subalgebra of L. It is obvious that $A \cap \zeta(L) = \langle 0 \rangle$, so that $L = A \oplus \zeta(L)$, A is an ideal of L and $B = \zeta(L) = Fa_2$. If we suppose that A is nilpotent (that is, $[a_1, a_3] = 0$), then $\mathrm{Leib}(L) = \mathrm{Leib}(A) = \zeta(A) \leqslant \zeta(L)$, and we obtain a contradiction. Thus, A is not nilpotent. As we have seen above, we can choose an element a_1 such that $[a_1, a_3] = a_3$. Thus, we come to the following type of Leibniz algebras:

$$L_{17} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = [a_1, a_3] = a_3,$$
$$[a_1, a_2] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{17}) = [L_{17}, L_{17}] = Fa_3$, $\zeta^{\mathrm{left}}(L_{17}) = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{17}) = \zeta(L_{17}) = Fa_2$, L_{17} is non-nilpotent.

Suppose now that $\zeta(L)$ is zero. Since L is not a Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. Then, $\mathrm{Leib}(L) = Fa_3$. Since $L/\mathrm{Leib}(L)$ is abelian, a subalgebra $A = \langle a_1 \rangle = Fa_1 \oplus Fa_3$ is an ideal of L. Suppose that A is nilpotent (that is, $[a_1, a_3] = 0$). Let b be an element of L such that $b \notin A$. We have:

$$[b, a_3] = [b, [a_1, a_1]] = [[b, a_1], a_1] + [a_1, [b, a_1]] = [\lambda a_3, a_1] + [a_1, \lambda a_3] = 0.$$

Since $[a_3, b] = 0$, we obtain that $a_3 \in \zeta(L)$, and we obtain a contradiction. This contradiction shows that subalgebra $A = \langle a_1 \rangle = Fa_1 \oplus Fa_3$ is not nilpotent. As we have seen above, we can choose an element a_1 such that $[a_1, a_3] = a_3$.

Suppose that L contains an element b such that $b \notin A$ and $[b, b] = 0$. Since $L/\mathrm{Leib}(L)$ is abelian, $[b, a_1] = \lambda a_3$, $[a_1, b] = \mu a_3$ for some elements $\lambda, \mu \in F$. As above,

$$[b, a_3] = [a_1, [b, a_1]] = [a_1, \lambda a_3] = \lambda [a_1, a_3] = \lambda a_3.$$

If $\lambda = \mu = 0$, then $b \in \zeta(L)$, and we obtain a contradiction with our assumption concerning the inclusion of $\zeta(L)$.

Suppose, now, that $\lambda = 0$, $\mu \neq 0$. Put $a_2 = \mu^{-1} b$, then $[a_2, a_1] = 0$, $[a_1, a_2] = a_3$, $[a_2, a_2] = 0$. As seen above, we can observe that $[a_2, a_3] = 0$, and we obtain that $[a_2, A] = \langle 0 \rangle$. It follows that $a_2 \in \zeta^{\mathrm{left}}(L)$. Thus, we come to the following type of Leibniz algebras:

$$L_{18} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = [a_1, a_2] = [a_1, a_3] = a_3,$$
$$[a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{18}) = [L_{18}, L_{18}] = Fa_3$, $\zeta^{\mathrm{left}}(L_{18}) = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{18}) = \zeta(L_{14}) = \langle 0 \rangle$, L_{18} is non-nilpotent.

Suppose now that $\lambda \neq 0$, $\mu = 0$. Put $a_2 = \lambda^{-1}b$, then $[a_2, a_1] = a_3$, $[a_1, a_2] = 0$, $[a_2, a_2] = 0$. Since $[a_3, a_2] = 0$, $[A, a_2] = \langle 0 \rangle$. It follows that $a_2 \in \zeta^{\text{right}}(L)$. Thus, we come to the following type of Leibniz algebras:

$$L_{19} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where}$$
$$[a_1, a_1] = [a_1, a_3] = [a_2, a_1] = [a_2, a_3] = a_3,$$
$$[a_1, a_2] = [a_2, a_2] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\text{Leib}(L_{19}) = [L_{19}, L_{19}] = \zeta^{\text{left}}(L_{19}) = Fa_3$, $\zeta^{\text{right}}(L_{19}) = Fa_2$, $\zeta(L_{19}) = \langle 0 \rangle$, L_{19} is non-nilpotent.

Suppose now that $\mu \neq 0$, $\lambda \neq 0$. We have:

$$0 = [a_1, 0] = [a_1, [b, b]] = [[a_1, b], b] + [b, [a_1, b]] =$$
$$[b, [a_1, b]] = [b, \mu a_3] = \mu[b, a_3] = \mu\lambda a_3.$$

It follows that $\mu\lambda = 0$, and we obtain a contradiction.

Suppose that $[b, b] \neq 0$ for every element b such that $b \notin A$. Since $L/\text{Leib}(L)$ is abelian, $[b, a_1] = \lambda a_3$, $[a_1, b] = \mu a_3$ for some elements $\lambda, \mu \in F$, and $[b, b] = \sigma a_3$ for some non-zero element $\sigma \in F$.

As above,

$$[b, a_3] = [a_1, [b, a_1]] = [a_1, \lambda a_3] = \lambda[a_1, a_3] = \lambda a_3.$$

If $\lambda = \mu = 0$, then $[b, a_1] = [a_1, b] = [b, a_3] = 0$. Put $a_2 = b$. Let $u = \alpha a_1 + \beta a_2 + \gamma a_3$ be an arbitrary element of L. Then,

$$[u, u] = [\alpha a_1 + \beta a_2 + \gamma a_3, \alpha a_1 + \beta a_2 + \gamma a_3] =$$
$$\alpha^2[a_1, a_1] + \alpha\gamma[a_1, a_3] + \beta^2[a_2, a_2] =$$
$$\alpha^2 a_3 + \alpha\gamma a_3 + \beta^2 a_3 = (\alpha^2 + \alpha\gamma + \beta^2\sigma)a_3.$$

If $u \notin A$, then $[u, u] \neq 0$. It follows that $\alpha^2 + \alpha\gamma + \beta^2\sigma \neq 0$. Thus, we come to the following type of Leibniz algebras:

$$L_{20} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = [a_1, a_3] = a_3,$$
$$[a_2, a_2] = \sigma a_3 \ (\sigma \neq 0),$$
$$[a_1, a_2] = [a_2, a_1] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\text{Leib}(L_{20}) = [L_{20}, L_{20}] = \zeta^{\text{left}}(L_{20}) = Fa_3$, $\zeta^{\text{right}}(L_{20}) = \zeta(L_{20}) = \langle 0 \rangle$. Moreover, if $\alpha a_1 + \beta a_2 + \gamma a_3 \notin A$, then $\alpha^2 + \alpha\gamma + \beta^2\sigma \neq 0$, L_{20} is non-nilpotent.

Suppose now that $\lambda = 0$, $\mu \neq 0$. Put $a_2 = \mu^{-1}b$. Then $[a_2, a_1] = [a_2, a_3] = 0$, $[a_1, a_2] = a_3$, $[a_2, a_2] = \mu^{-2}[b, b] = \mu^{-2}\sigma a_3 = \tau a_3$. Let $\alpha a_1 + \beta a_2 + \gamma a_3$ be an arbitrary element of L. Then,

$$[\alpha a_1 + \beta a_2 + \gamma a_3, \alpha a_1 + \beta a_2 + \gamma a_3] =$$
$$\alpha^2[a_1, a_1] + \alpha\beta[a_1, a_2] + \alpha\gamma[a_1, a_3] + \beta^2[a_2, a_2] =$$
$$\alpha^2 a_3 + \alpha\beta a_3 + \alpha\gamma a_3 + \beta^2\tau a_3 = (\alpha^2 + \alpha\beta + \alpha\gamma + \beta^2\tau)a_3.$$

As above, $\alpha^2 + \alpha\beta + \alpha\gamma + \beta^2\tau \neq 0$. Thus, we come to the following type of Leibniz algebras:

$$L_{21} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = [a_1, a_2] = [a_1, a_3] = a_3,$$
$$[a_2, a_2] = \tau a_3 \ (\tau \neq 0),$$
$$[a_2, a_1] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{21}) = [L_{21}, L_{21}] = \zeta^{\mathrm{left}}(L_{21}) = Fa_3$, $\zeta^{\mathrm{right}}(L_{21}) = \zeta(L_{21}) = \langle 0\rangle$. Moreover, if $\alpha a_1 + \beta a_2 + \gamma a_3 \notin A$, then $\alpha^2 + \alpha\beta + \alpha\gamma + \beta^2\tau \neq 0$, L_{21} is non-nilpotent.

Suppose now that $\lambda \neq 0$, $\mu = 0$. Put $a_2 = \lambda^{-1}b$. Then $[a_2, a_1] = [a_2, a_3] = a_3$, $[a_1, a_2] = 0$, $[a_2, a_2] = \lambda^{-2}[b, b] = \lambda^{-2}\sigma a_3 = \tau a_3$. Let $\alpha a_1 + \beta a_2 + \gamma a_3$ be an arbitrary element of L. Then,

$$[\alpha a_1 + \beta a_2 + \gamma a_3, \alpha a_1 + \beta a_2 + \gamma a_3] =$$
$$\alpha^2[a_1, a_1] + \alpha\gamma[a_1, a_3] + \alpha\beta[a_2, a_1] + \beta^2[a_2, a_2] + \beta\gamma[a_2, a_3] =$$
$$\alpha^2 a_3 + \alpha\gamma a_3 + \alpha\beta a_3 + \beta^2\tau a_3 + \beta\gamma a_3 =$$
$$(\alpha^2 + \alpha\gamma + \alpha\beta + \beta^2\tau + \beta\gamma)a_3.$$

As above, $\alpha^2 + \alpha\gamma + \alpha\beta + \beta^2\tau + \beta\gamma \neq 0$. Thus, we come to the following type of Leibniz algebras:

$$L_{22} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where}$$
$$[a_1, a_1] = [a_1, a_3] = [a_2, a_1] = [a_2, a_3] = a_3,$$
$$[a_2, a_2] = \tau a_3 \ (\tau \neq 0),$$
$$[a_1, a_2] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{22}) = [L_{22}, L_{22}] = \zeta^{\mathrm{left}}(L_{22}) = Fa_3$, $\zeta^{\mathrm{right}}(L_{22}) = \zeta(L_{22}) = \langle 0\rangle$. Moreover, if $\alpha a_1 + \beta a_2 + \gamma a_3 \not{\!\!\;} inA$, then $\alpha^2 + \alpha\gamma + \alpha\beta + \beta^2\tau + \beta\gamma \neq 0$, L_{22} is non-nilpotent.

Suppose now that $\lambda \neq 0$, $\mu \neq 0$. Put $a_2 = \lambda^{-1}b$. Then $[a_2, a_1] = [a_2, a_3] = a_3$, $[a_1, a_2] = \lambda^{-1}[a_1, b] = \lambda^{-1}\mu a_3 = \delta a_3$, $[a_2, a_2] = \lambda^{-2}[b, b] = \lambda^{-2}\sigma a_3 = \tau a_3$. Let $\alpha a_1 + \beta a_2 + \gamma a_3$ be an arbitrary element of L. Then,

$$[\alpha a_1 + \beta a_2 + \gamma a_3, \alpha a_1 + \beta a_2 + \gamma a_3] =$$
$$\alpha^2[a_1, a_1] + \alpha\beta[a_1, a_2] + \alpha\gamma[a_1, a_3]+$$
$$\alpha\beta[a_2, a_1] + \beta^2[a_2, a_2] + \beta\gamma[a_2, a_3] =$$
$$\alpha^2 a_3 + \alpha\beta\delta a_3 + \alpha\gamma a_3 + \alpha\beta a_3 + \beta^2\tau a_3 + \beta\gamma a_3 =$$
$$(\alpha^2 + \alpha\beta\delta + \alpha\gamma + \alpha\beta + \beta^2\tau + \beta\gamma)a_3.$$

As above, $\alpha^2 + \alpha\beta\delta + \alpha\gamma + \alpha\beta + \beta^2\tau + \beta\gamma \neq 0$. Thus, we come to the following type of Leibniz algebras:

$$L_{23} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where}$$
$$[a_1, a_1] = [a_1, a_3] = [a_2, a_1] = [a_2, a_3] = a_3,$$
$$[a_1, a_2] = \delta a_3 \ (\delta \neq 0), [a_2, a_2] = \tau a_3 \ (\tau \neq 0),$$
$$[a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{23}) = [L_{23}, L_{23}] = \zeta^{\mathrm{left}}(L_{23}) = Fa_3$, $\zeta^{\mathrm{right}}(L_{23}) = \zeta(L_{23}) = \langle 0 \rangle$. Moreover, if $\alpha a_1 + \beta a_2 + \gamma a_3 \notin A$, then $\alpha^2 + \alpha\beta\delta + \alpha\gamma + \alpha\beta + \beta^2\tau + \beta\gamma \neq 0$, L_{23} is non-nilpotent. □

Let L be a Leibniz algebra over a field F, M be non-empty subset of L and H be a subalgebra of L. Put

$$\mathrm{Ann}_H^{\mathrm{left}}(M) = \{a \in H \mid [a, M] = \langle 0 \rangle\},$$
$$\mathrm{Ann}_H^{\mathrm{right}}(M) = \{a \in H \mid [M, a] = \langle 0 \rangle\}.$$

The subset $\mathrm{Ann}_H^{\mathrm{left}}(M)$ is called the *left annihilator* of M in subalgebra H. The subset $\mathrm{Ann}_H^{\mathrm{right}}(M)$ is called the *right annihilator* of M in subalgebra H. The intersection

$$\mathrm{Ann}_H(M) = \mathrm{Ann}_H^{\mathrm{left}}(M) \cap \mathrm{Ann}_H^{\mathrm{right}}(M)$$

is called the *annihilator* of M in subalgebra H.

It is not hard to see that all of these subsets are subalgebras of L. Moreover, if M is an ideal of L, then $\mathrm{Ann}_H(M)$ is an ideal of L (see, for example, [7]).

Theorem 4. *Let L be a Leibniz algebra over a field F having dimension 3. Suppose that L is not a Lie algebra. Suppose that the center of L does not include the Leibniz kernel,* $\dim_F(\mathrm{Leib}(L)) = 1$, *and the factor-algebra* $L/\mathrm{Leib}(L)$ *is non-abelian. Then, L is an algebra of one of the following types.*

(i) $\mathrm{Lei}_{24}(3, F) = L_{24}$ *is a direct sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2$. *Moreover, A is a nilpotent cyclic Leibniz algebra of dimension* 2, $\mathrm{char}(F) \neq 2$, $[A, B] = Fa_1$, $[B, A] = Fa_1 \oplus Fa_3$, *so that* $L_{24} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_3$, $[a_1, a_2] = -a_1$, $[a_2, a_1] = a_1$, $[a_2, a_3] = 2a_3$, $[a_1, a_3] = [a_2, a_2] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{24}) = \zeta^{\mathrm{left}}(L_{24}) = Fa_3$, $\zeta^{\mathrm{right}}(L_{24}) = \zeta(L_{24}) = \langle 0 \rangle$, $[L_{24}, L_{24}] = Fa_1 \oplus Fa_3$, *$L_{24}$ is non-nilpotent.*

(ii) $\mathrm{Lei}_{25}(3, F) = L_{25}$ *is a direct sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2$. *Moreover, A is a nilpotent cyclic Leibniz algebra of dimension* 2, $\mathrm{char}(F) \neq 2$, $[A, B] = [B, A] = Fa_1 \oplus Fa_3$, *so that* $L_{25} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_3$, $[a_1, a_2] = -a_1 + \alpha a_3$, $[a_2, a_1] = a_1 + \alpha a_3$ $(\alpha \neq 0)$, $[a_2, a_3] = 2a_3$, $[a_1, a_3] = [a_2, a_2] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{25}) = \zeta^{\mathrm{left}}(L_{25}) = Fa_3$, $\zeta^{\mathrm{right}}(L_{25}) = \zeta(L_{25}) = \langle 0 \rangle$, $[L_{25}, L_{25}] = Fa_1 \oplus Fa_3$, *$L_{25}$ is non-nilpotent.*

(iii) $\mathrm{Lei}_{26}(3, F) = L_{26}$ *is a direct sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2$. *Furthermore, A is a non-nilpotent cyclic Leibniz algebra of dimension* 2, $[A, B] = Fa_1$, $[B, A] = Fa_1 \oplus Fa_3$, *so that* $L_{26} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_1, a_3] = a_3$, $[a_1, a_2] = -a_1$, $[a_2, a_1] = a_1$, $[a_2, a_3] = 2a_3$, $[a_2, a_2] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{26}) = \zeta^{\mathrm{left}}(L_{26}) = Fa_3$, $[L_{26}, L_{26}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{26}) = \zeta(L_{26}) = \langle 0 \rangle$. *Moreover,* $\mathrm{char}(F) \neq 2$, *L_{26} is non-nilpotent.*

(iv) $\mathrm{Lei}_{27}(3, F) = L_{27}$ *is a direct sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2$. *Moreover, A is a non-nilpotent cyclic Leibniz algebra of dimension* 2, $[A, B] = [B, A] = Fa_1 \oplus Fa_3$, *so that* $L_{27} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_1, a_3] = a_3$, $[a_1, a_2] = -a_1 + \beta a_3$ $(\beta = \alpha(1 + \alpha)^{-1})$, $[a_2, a_1] = a_1 + \alpha a_3$, $[a_2, a_3] = (2 + \alpha)a_3$ $(\alpha \neq 0, \alpha \neq -1, \alpha \neq -2)$, $[a_2, a_2] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{27}) = \zeta^{\mathrm{left}}(L_{27}) = Fa_3$, $[L_{27}, L_{27}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{27}) = \zeta(L_{27}) = \langle 0 \rangle$, *$L_{27}$ is non-nilpotent.*

(v) $\mathrm{Lei}_{28}(3, F) = L_{28}$ *is a sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2 \oplus Fa_3$. *Furthermore, A is a nilpotent cyclic Leibniz algebra of dimension* 2, *B is a non-nilpotent cyclic Leibniz algebra of dimension* 2, $\mathrm{char}(F) \neq 2$, $[A, B] = Fa_1$, $[B, A] = Fa_1 \oplus Fa_3$, *so that* $L_{28} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_3$, $[a_1, a_2] = -a_1$, $[a_2, a_1] = a_1$, $[a_2, a_2] = \gamma a_3$ $(\gamma \neq 0)$, $[a_2, a_3] = 2a_3$, $[a_1, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{28}) = \zeta^{\mathrm{left}}(L_{28}) = Fa_3$, $[L_{28}, L_{28}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{28}) =$

$\zeta(L_{28}) = \langle 0 \rangle$. *Moreover, if* $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, *then* $\lambda^2 + \mu^2\gamma + 2\mu\nu \neq 0$, L_{28} *is non-nilpotent.*

(vi) $\mathrm{Lei}_{29}(3, F) = L_{29}$ *is a sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2 \oplus Fa_3$. *Furthermore,* A *is a nilpotent cyclic Leibniz algebra of dimension* 2, B *is a non-nilpotent cyclic Leibniz algebra of dimension* 2, $\mathrm{char}(F) \neq 2$, $[A, B] = [B, A] = Fa_1 \oplus Fa_3$, *so that* $L_{29} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_3$, $[a_1, a_2] = -a_1 + \alpha a_3$, $[a_2, a_1] = a_1 + \alpha a_3$ $(\alpha \neq 0)$, $[a_2, a_2] = \gamma a_3$ $(\gamma \neq 0)$, $[a_2, a_3] = 2a_3$, $[a_1, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{29}) = \zeta^{\mathrm{left}}(L_{29}) = Fa_3$, $[L_{29}, L_{29}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{29}) = \zeta(L_{29}) = \langle 0 \rangle$. *Moreover, if* $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, *then* $\lambda^2 + 2\lambda\mu\alpha + \mu^2\gamma + 2\mu\nu \neq 0$, L_{29} *is non-nilpotent.*

(vii) $\mathrm{Lei}_{30}(3, F) = L_{30}$ *is a sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2 \oplus Fa_3$. *Moreover,* A, B *are non-nilpotent cyclic Leibniz algebras of dimension* 2, $\mathrm{char}(F) \neq 2$, $[A, B] = [B, A] = Fa_1 \oplus Fa_3$, *so that* $L_{30} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_1, a_3] = a_3$, $[a_1, a_2] = -a_1 + \gamma a_3$ $(\gamma \neq 0)$, $[a_2, a_1] = a_1$, $[a_2, a_2] = \gamma a_3$, $[a_2, a_3] = 2a_3$, $[a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{30}) = \zeta^{\mathrm{left}}(L_{30}) = Fa_3$, $[L_{30}, L_{30}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{30}) = \zeta(L_{30}) = \langle 0 \rangle$. *Moreover, if* $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, *then* $\lambda^2 + \lambda\mu\gamma + \lambda\nu + \mu^2\gamma + 2\mu\nu \neq 0$, L_{30} *is non-nilpotent.*

(viii) $\mathrm{Lei}_{31}(3, F) = L_{31}$ *is a sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2 \oplus Fa_3$. *Moreover,* A, B *are non-nilpotent cyclic Leibniz algebras of dimension* 2, $[A, B] = [B, A] = Fa_1 \oplus Fa_3$, *so that* $L_{31} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_1, a_3] = [a_2, a_2] = [a_2, a_3] = a_3$, $[a_1, a_2] = -a_1$, $[a_2, a_1] = a_1 - a_3$, $[a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{31}) = \zeta^{\mathrm{left}}(L_{31}) = Fa_3$, $[L_{31}, L_{31}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{31}) = \zeta(L_{31}) = \langle 0 \rangle$. *Moreover, if* $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, *then* $\lambda^2 + \lambda\nu - \lambda\mu + \mu^2 + \mu\nu \neq 0$, L_{31} *is non-nilpotent.*

(ix) $\mathrm{Lei}_{32}(3, F) = L_{31}$ *is a sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2 \oplus Fa_3$. *Moreover,* A, B *are non-nilpotent cyclic Leibniz algebras of dimension* 2, $[A, B] = [B, A] = Fa_1 \oplus Fa_3$, *so that* $L_{32} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_1, a_3] = [a_2, a_2] = [a_2, a_3] = a_3$, $[a_1, a_2] = -a_1 + \beta a_3$ $(\beta \neq 0)$, $[a_2, a_1] = a_1 - a_3$, $[a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{32}) = \zeta^{\mathrm{left}}(L_{32}) = Fa_3$, $[L_{32}, L_{32}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{32}) = \zeta(L_{32}) = \langle 0 \rangle$. *Moreover, if* $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, *then* $\lambda^2 + \lambda\mu\beta + \lambda\nu - \lambda\mu + \mu^2 + \mu\nu \neq 0$, L_{32} *is non-nilpotent.*

(x) $\mathrm{Lei}_{33}(3, F) = L_{31}$ *is a sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2 \oplus Fa_3$. *Moreover,* A, B *are non-nilpotent cyclic Leibniz algebras of dimension* 2, $[A, B] = [B, A] = Fa_1 \oplus Fa_3$, $L_{33} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = [a_1, a_3] = a_3$, $[a_1, a_2] = -a_1$, $[a_2, a_1] = a_1 - \gamma a_3$ $(\gamma \neq 0, \gamma \neq 1, \gamma \neq 2)$, $[a_2, a_2] = \gamma a_3$, $[a_2, a_3] = (2 - \gamma)a_3$, $[a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{33}) = \zeta^{\mathrm{left}}(L_{33}) = Fa_3$, $[L_{33}, L_{33}] = Fa_1 \oplus Fa_3$,

$\zeta^{\text{right}}(L_{33}) = \zeta(L_{33}) = \langle 0\rangle$. *Moreover, if* $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, *then* $\lambda^2 + \lambda\nu - \lambda\mu\gamma + \mu^2\gamma + \mu\nu(2-\gamma) \neq 0$, L_{33} *is non-nilpotent.*

(xi) $\mathrm{Lei}_{34}(3,F) = L_{34}$ *is a sum of ideal* $A = Fa_1 \oplus Fa_3$ *and subalgebra* $B = Fa_2 \oplus Fa_3$. *Moreover,* A, B *are non-nilpotent cyclic Leibniz algebras of dimension* 2, $[A,B] = [B,A] = Fa_1 \oplus Fa_3$, $L_{34} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1,a_1] = [a_1,a_3] = a_3$, $[a_1,a_2] = -a_1 + \beta a_3$ $(\beta = (\alpha+\gamma)(1+\alpha)^{-1})$, $[a_2,a_1] = a_1 + \alpha a_3$ $(\alpha \neq 0, \alpha \neq -1, \alpha \neq -2)$, $[a_2,a_2] = \gamma a_3$ $(\gamma \neq 0)$, $[a_2,a_3] = (2+\alpha)a_3$, $[a_3,a_1] = [a_3,a_2] = [a_3,a_3] = 0$, $\mathrm{Leib}(L_{34}) = \zeta^{\text{left}}(L_{34}) = Fa_3$, $[L_{34}, L_{34}] = Fa_1 \oplus Fa_3$, $\zeta^{\text{right}}(L_{34}) = \zeta(L_{34}) = \langle 0\rangle$. *Moreover, if* $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, *then* $\lambda^2 + \lambda\mu\beta + \lambda\nu + \lambda\mu\alpha + \mu^2\gamma + \mu\nu(2+\alpha) \neq 0$, L_{34} *is non-nilpotent.*

(xii) $\mathrm{Lei}_{35}(3,F) = L_{35}$ *is a direct sum of ideal* $B = Fa_2$ *and a cyclic non-nilpotent subalgebra* $A = Fa_1 \oplus Fa_3$ *of dimension* 2, $[A,B] = [B,A] = Fa_2$, *so that* $L_{35} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1,a_1] = \gamma a_3$ $(\gamma \neq 0)$, $[a_1,a_2] = a_2$, $[a_1,a_3] = a_3$, $[a_2,a_1] = -a_2$, $[a_2,a_2] = [a_2,a_3] = [a_3,a_1] = [a_3,a_2] = [a_3,a_3] = 0$, $\mathrm{Leib}(L_{35}) = \zeta^{\text{left}}(L_{35}) = Fa_3$, $[L_{35}, L_{35}] = Fa_2 \oplus Fa_3$, $\zeta^{\text{right}}(L_{35}) = \zeta(L_{35}) = \langle 0\rangle$, L_{35} *is non-nilpotent.*

(xiii) $\mathrm{Lei}_{36}(3,F) = L_{36}$ *is a sum of abelian ideal* $B = Fa_2 \oplus Fa_3$ *and a cyclic non-nilpotent subalgebra* $A = Fa_1 \oplus Fa_3$ *of dimension* 2, $[A,B] = [B,A] = Fa_2 \oplus Fa_3$, *so that* $L_{36} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1,a_1] = \gamma a_3$ $(\gamma \neq 0)$, $[a_1,a_2] = a_2$, $[a_1,a_3] = a_3$, $[a_2,a_1] = -a_2 + \beta a_3$ $(\beta \neq 0)$, $[a_2,a_2] = [a_2,a_3] = [a_3,a_1] = [a_3,a_2] = [a_3,a_3] = 0$, $\mathrm{Leib}(L_{36}) = \zeta^{\text{left}}(L_{36}) = Fa_3$, $[L_{36}, L_{36}] = Fa_2 \oplus Fa_3$, $\zeta^{\text{right}}(L_{36}) = \zeta(L_{36}) = \langle 0\rangle$, L_{36} *is non-nilpotent.*

Proof. As in the previous theorem, we can see that $\dim_F(\zeta(L)) \leqslant 1$. Suppose that $\dim_F(\zeta(L)) = 1$. Since L is not a Lie algebra, there is an element a_1 such that $[a_1,a_1] = a_3 \neq 0$. Then, $\mathrm{Leib}(L) = Fa_3$. Let $A = \langle a_1\rangle = Fa_1 \oplus Fa_3$. An equality $\mathrm{Leib}(L) \cap \zeta(L) = \langle 0\rangle$ implies that $A \cap \zeta(L) = \langle 0\rangle$, so that $L = A \oplus \zeta(L)$ and A is an ideal of L. But, in this case, the factor-algebra $L/\mathrm{Leib}(L)$ is abelian, and we obtain a contradiction. This contradiction shows that $\zeta(L) = \langle 0\rangle$.

By what is noted above, $L/\mathrm{Leib}(L)$ has an ideal $C/\mathrm{Leib}(L)$ of dimension 1 (i.e., $C = Fc \oplus \mathrm{Leib}(L)$ for some element c). If $[c,c] \neq 0$ without loss of generality, we can put $c = a_1$. Then, subalgebra $\langle a_1\rangle = Fa_1 \oplus Fa_3 = A$ is an ideal which has codimension 1. Then, for every element b such that $b \notin A$, we have $L = A \oplus Fb$. By what is noted above, in this case, $[b,a_1] \in a_1 + Fa_3$, so that $[b,a_1] = a_1 + \alpha a_3$ for some element $\alpha \in F$. We have also $[a_1,b] = -a_1 + \beta a_3$ for some element $\beta \in F$. Since $[b,b] \in \mathrm{Leib}(L)$, $[b,b] = \gamma a_3$ for some element $\gamma \in F$.

Suppose first that $\gamma = 0$. In other words, we suppose that there exists an element b such that $b \notin A$ and $[b, b] = 0$. Consider first the case when A is nilpotent. We have:

$$[b, a_3] = [b, [a_1, a_1]] = [[b, a_1], a_1] + [a_1, [b, a_1]] =$$
$$[a_1 + \alpha a_3, a_1] + [a_1, a_1 + \alpha a_3] = [a_1, a_1] + [a_1, a_1] = 2a_3.$$

In particular, if we suppose that $\mathrm{char}(F) = 2$, then $[b, a_3] = 0$. Since $[a_1, a_3] = [a_3, a_1] = [a_3, b] = 0$, $\mathrm{Leib}(L) = Fa_3 \leqslant \zeta(L)$, and we obtain a contradiction. This contradiction shows that $\mathrm{char}(F) \neq 2$. Further,

$$[b, [a_1, b]] = [[b, a_1], b] + [a_1, [b, b]] = [a_1 + \alpha a_3, b] = [a_1, b] = -a_1 + \beta a_3.$$

On the other hand,

$$[b, [a_1, b]] = [b, -a_1 + \beta a_3] = -[b, a_1] + \beta[b, a_3] =$$
$$-(a_1 + \alpha a_3) + 2\beta a_3 = -a_1 + (2\beta - \alpha)a_3.$$

It follows that $2\beta - \alpha = \beta$ or $\beta = \alpha$ and $[a_1, b] = -a_1 + \alpha a_3$.

Suppose that $\alpha = 0$. Put $a_2 = b$. Then $[a_1, a_2] = -a_1$, $[a_2, a_1] = a_1$, $[a_2, a_2] = 0$, $[a_2, a_3] = 2a_3$. Thus, we come to the following type of Leibniz algebras:

$$L_{24} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3,$$
$$[a_1, a_2] = -a_1, [a_2, a_1] = a_1, [a_2, a_3] = 2a_3,$$
$$[a_1, a_3] = [a_2, a_2] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{24}) = \zeta^{\mathrm{left}}(L_{24}) = Fa_3$, $\zeta^{\mathrm{right}}(L_{24}) = \zeta(L_{24}) = \langle 0 \rangle$, $[L_{24}, L_{24}] = Fa_1 \oplus Fa_3$, $\mathrm{char}(F) \neq 2$, L_{24} is non-nilpotent.

Suppose now that $\alpha \neq 0$. Put again $a_2 = b$. Thus, we come to the following type of Leibniz algebras:

$$L_{25} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3,$$
$$[a_1, a_2] = -a_1 + \alpha a_3, [a_2, a_1] = a_1 + \alpha a_3 \ (\alpha \neq 0),$$
$$[a_2, a_3] = 2a_3, [a_1, a_3] = [a_2, a_2] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{25}) = \zeta^{\mathrm{left}}(L_{25}) = Fa_3$, $\zeta^{\mathrm{right}}(L_{25}) = \zeta(L_{25}) = \langle 0 \rangle$, $[L_{25}, L_{25}] = Fa_1 \oplus Fa_3$, $\mathrm{char}(F) \neq 2$, L_{25} is non-nilpotent.

Consider now the case when A is not nilpotent. As we have seen above, we can choose an element a_1 such that $[a_1, a_3] = a_3$. Then,

$$[b, a_3] = [b, [a_1, a_1]] = [[b, a_1], a_1] + [a_1, [b, a_1]] =$$
$$[a_1 + \alpha a_3, a_1] + [a_1, a_1 + \alpha a_3] =$$
$$[a_1, a_1] + [a_1, a_1] + \alpha[a_1, a_3] = (2 + \alpha)a_3.$$

Further,

$$[b,[a_1,b]] = [[b,a_1],b] + [a_1,[b,b]] =$$
$$[a_1 + \alpha a_3, b] = [a_1, b] = -a_1 + \beta a_3.$$

On the other hand,

$$[b,[a_1,b]] = [b, -a_1 + \beta a_3] = -[b,a_1] + \beta[b,a_3] =$$
$$-(a_1 + \alpha a_3) + \beta(2+\alpha)a_3 = -a_1 + (2\beta + \beta\alpha - \alpha)a_3.$$

It follows that $2\beta + \beta\alpha - \alpha = \beta$ or $\beta + \beta\alpha - \alpha = 0$. It follows that $\beta(1+\alpha) = \alpha$. We consider separately the case when $\alpha = 0$. Then, $\beta = 0$. If we suppose that $\mathrm{char}(F) = 2$, then $[b, a_3] = 0$. Since $[a_3, b] = 0$, $b \in \mathrm{Ann}_L(Fa_3) = \mathrm{Ann}_L(\mathrm{Leib}(L))$. But, $\mathrm{Ann}_L(\mathrm{Leib}(L))$ is an ideal of L. Then, $[b, a_1] = a_1 \in Fb \oplus Fa_3$, and we obtain a contradiction. This contradiction shows that $\mathrm{char}(F) \neq 2$. Put $a_2 = b$. We come to the following type of Leibniz algebras:

$$L_{26} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1,a_1] = [a_1,a_3] = a_3, [a_1,a_2] = -a_1,$$
$$[a_2,a_1] = a_1, [a_2,a_3] = 2a_3, [a_2,a_2] = [a_3,a_1] = [a_3,a_2] = [a_3,a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{26}) = \zeta^{\mathrm{left}}(L_{26}) = Fa_3$, $[L_{26}, L_{26}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{26}) = \zeta(L_{26}) = \langle 0\rangle$, $\mathrm{char}(F) \neq 2$, L_{26} is non-nilpotent.

Suppose that $\alpha \neq 0$. The equality $\beta(1+\alpha) = \alpha$ shows that $\alpha \neq -1$. In this case, $\beta = \alpha(1+\alpha)^{-1}$. If we suppose that $\alpha = -2$, then $[b, a_3] = 0$. Using the above arguments, we obtain a contradiction. This contradiction shows that $\alpha \neq -2$. Put $a_2 = b$. We come to the following type of Leibniz algebras:

$$L_{27} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1,a_1] = [a_1,a_3] = a_3,$$
$$[a_1,a_2] = -a_1 + \beta a_3 \; (\beta = \alpha(1+\alpha)^{-1}), [a_2,a_1] = a_1 + \alpha a_3,$$
$$[a_2,a_3] = (2+\alpha)a_3 \; (\alpha \neq 0, \alpha \neq -1, \alpha \neq -2),$$
$$[a_2,a_2] = [a_3,a_1] = [a_3,a_2] = [a_3,a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{27}) = \zeta^{\mathrm{left}}(L_{27}) = Fa_3$, $[L_{27}, L_{27}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{27}) = \zeta(L_{27}) = \langle 0\rangle$, L_{27} is non-nilpotent.

Suppose now that $[b, b] \neq 0$ for every element b such that $b \notin A$, so that $[b, b] = \gamma a_3$ and $\gamma \neq 0$. Consider first the case when A is nilpotent. We have:

$$[b,a_3] = [b,[a_1,a_1]] = [[b,a_1],a_1] + [a_1,[b,a_1]] =$$
$$[a_1 + \alpha a_3, a_1] + [a_1, a_1 + \alpha a_3] = [a_1,a_1] + [a_1,a_1] = 2a_3.$$

Again, we obtain that $\mathrm{char}(F) \neq 2$. Using the arguments above, we can get that $[a_1, b] = -a_1 + \alpha a_3$ again. We consider separately the case when $\alpha = 0$ (that is, $[a_1, b] = -a_1$, $[b, a_1] = a_1$). Let $\lambda a_1 + \mu b + \nu a_3$ be the arbitrary element of L. We have:

$$\begin{gathered}[\lambda a_1 + \mu b + \nu a_3, \lambda a_1 + \mu b + \nu a_3] = \\ \lambda^2[a_1, a_1] + \lambda\mu[a_1, b] + \lambda\mu[b, a_1] + \mu^2[b, b] + \mu\nu[b, a_3] = \\ \lambda^2 a_3 - \lambda\mu a_1 + \lambda\mu a_1 + \mu^2\gamma a_3 + 2\mu\nu a_3 = \\ (\lambda^2 + \mu^2\gamma + 2\mu\nu)a_3.\end{gathered}$$

Then, a condition $[b, b] \neq 0$ for every element $b \notin A$ yields that $\lambda^2 + \mu^2\gamma + 2\mu\nu \neq 0$. Put $a_2 = b$. Thus, we come to the following type of Leibniz algebras:

$$\begin{gathered}L_{28} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3, [a_1, a_2] = -a_1, [a_2, a_1] = a_1, \\ [a_2, a_2] = \gamma a_3 \ (\gamma \neq 0), [a_2, a_3] = 2a_3, \\ [a_1, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.\end{gathered}$$

Note also that $\mathrm{Leib}(L_{28}) = \zeta^{\mathrm{left}}(L_{28}) = Fa_3$, $[L_{28}, L_{28}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{28}) = \zeta(L_{28}) = \langle 0 \rangle$, $\mathrm{char}(F) \neq 2$. Moreover, if $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, then $\lambda^2 + \mu^2\gamma + 2\mu\nu \neq 0$, L_{28} is non-nilpotent.

Suppose that $\alpha \neq 0$. Let $\lambda a_1 + \mu b + \nu a_3$ be the arbitrary element of L. We have:

$$\begin{gathered}[\lambda a_1 + \mu b + \nu a_3, \lambda a_1 + \mu b + \nu a_3] = \\ \lambda^2[a_1, a_1] + \lambda\mu[a_1, b] + \lambda\mu[b, a_1] + \mu^2[b, b] + \mu\nu[b, a_3] = \\ \lambda^2 a_3 + \lambda\mu(-a_1 + \alpha a_3) + \lambda\mu(a_1 + \alpha a_3) + \mu^2\gamma a_3 + 2\mu\nu a_3 = \\ \lambda^2 a_3 - \lambda\mu a_1 + \lambda\mu\alpha a_3 + \lambda\mu a_1 + \lambda\mu\alpha a_3 + \mu^2\gamma a_3 + 2\mu\nu a_3 = \\ (\lambda^2 + 2\lambda\mu\alpha + \mu^2\gamma + 2\mu\nu)a_3.\end{gathered}$$

As above, $\lambda^2 + 2\lambda\mu\alpha + \mu^2\gamma + 2\mu\nu \neq 0$. Put again $a_2 = b$. Then, we come to the following type of Leibniz algebras:

$$\begin{gathered}L_{29} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3, \\ [a_1, a_2] = -a_1 + \alpha a_3, [a_2, a_1] = a_1 + \alpha a_3 \ (\alpha \neq 0), \\ [a_2, a_2] = \gamma a_3 \ (\gamma \neq 0), [a_2, a_3] = 2a_3, \\ [a_1, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.\end{gathered}$$

Note also that $\text{Leib}(L_{29}) = \zeta^{\text{left}}(L_{29}) = Fa_3$, $[L_{29}, L_{29}] = Fa_1 \oplus Fa_3$, $\zeta^{\text{right}}(L_{29}) = \zeta(L_{29}) = \langle 0 \rangle$, $\text{char}(F) \neq 2$. Moreover, if $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, then $\lambda^2 + 2\lambda\mu\alpha + \mu^2\gamma + 2\mu\nu \neq 0$, L_{29} is non-nilpotent.

Suppose, now, that $[b, b] \neq 0$ for every element b such that $b \notin A$ and a subalgebra A is not nilpotent. As we have seen above, we can choose an element a_1 such that $[a_1, a_3] = a_3$. Then,

$$[b, a_3] = [b, [a_1, a_1]] = [[b, a_1], a_1] + [a_1, [b, a_1]] =$$
$$[a_1 + \alpha a_3, a_1] + [a_1, a_1 + \alpha a_3] =$$
$$[a_1, a_1] + [a_1, a_1] + \alpha[a_1, a_3] = (2 + \alpha)a_3.$$

If we suppose that $\alpha = -2$, then $[b, a_3] = 0$. Since $[a_3, b] = 0$, $b \in \text{Ann}_L(Fa_3) = \text{Ann}_L(\text{Leib}(L))$. But, $\text{Ann}_L(\text{Leib}(L))$ is an ideal of L. Then, $[b, a_1] \in Fb \oplus Fa_3$, and we obtain a contradiction. This contradiction shows that $\alpha \neq -2$. Further,

$$[b, [a_1, b]] = [[b, a_1], b] + [a_1, [b, b]] = [a_1 + \alpha a_3, b] + [a_1, \gamma a_3] =$$
$$[a_1, b] + \gamma[a_1, a_3] = -a_1 + \beta a_3 + \gamma a_3 = -a_1 + (\beta + \gamma)a_3.$$

On the other hand,

$$[b, [a_1, b]] = [b, -a_1 + \beta a_3] = -[b, a_1] + \beta[b, a_3] =$$
$$-(a_1 + \alpha a_3) + \beta(2 + \alpha)a_3 = -a_1 + (2\beta + \beta\alpha - \alpha)a_3.$$

It follows that $2\beta + \beta\alpha - \alpha = \beta + \gamma$ or $\beta(1 + \alpha) = \alpha + \gamma$. We consider separately the case when $\alpha = 0$ and $\alpha = -1$.

Let $\alpha = 0$, then $\beta = \gamma$ and $[b, a_3] = 2a_3$. Using the arguments given above, we can obtain that $\text{char}(F) \neq 2$. In this case, $[b, a_1] = a_1$ and $[a_1, b] = -a_1 + \gamma a_3$. Let $\lambda a_1 + \mu b + \nu a_3$ be the arbitrary element of L. We have:

$$[\lambda a_1 + \mu b + \nu a_3, \lambda a_1 + \mu b + \nu a_3] =$$
$$\lambda^2[a_1, a_1] + \lambda\mu[a_1, b] + \lambda\nu[a_1, a_3] + \lambda\mu[b, a_1] + \mu^2[b, b] + \mu\nu[b, a_3] =$$
$$\lambda^2 a_3 + \lambda\mu(-a_1 + \gamma a_3) + \lambda\mu a_3 + \lambda\mu a_1 + \mu^2\gamma a_3 + 2\mu\nu a_3 =$$
$$\lambda^2 a_3 - \lambda\mu a_1 + \lambda\mu\gamma a_3 + \lambda\nu a_3 + \lambda\mu a_1 + \mu^2\gamma a_3 + 2\mu\nu a_3 =$$
$$(\lambda^2 + \lambda\mu\gamma + \lambda\nu + \mu^2\gamma + 2\mu\nu)a_3.$$

Then, the condition $[b, b] \neq 0$ for every element $b \notin A$ yields that $\lambda^2 + \lambda\mu\gamma + \lambda\nu + \mu^2\gamma + 2\mu\nu \neq 0$. Put $a_2 = b$. Thus, we come to the following type of Leibniz algebras:

$$L_{30} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = [a_1, a_3] = a_3,$$
$$[a_1, a_2] = -a_1 + \gamma a_3 \ (\gamma \neq 0), [a_2, a_1] = a_1, [a_2, a_2] = \gamma a_3, [a_2, a_3] = 2a_3,$$
$$[a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{30}) = \zeta^{\mathrm{left}}(L_{30}) = Fa_3$, $[L_{30}, L_{30}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{30}) = \zeta(L_{30}) = \langle 0 \rangle$, $\mathrm{char}(F) \neq 2$. Moreover, if $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, then $\lambda^2 + \lambda\mu\gamma + \lambda\nu + \mu^2\gamma + 2\mu\nu \neq 0$, L_{30} is non-nilpotent.

Let now $\alpha = -1$. Then $\gamma = 1$, $[b, b] = [b, a_3] = a_3$, $[b, a_1] = a_1 - a_3$, $[a_1, b] = -a_1 + \beta a_3$. If $\beta = 0$, then $[a_1, b] = -a_1$. Let $\lambda a_1 + \mu b + \nu a_3$ be the arbitrary element of L. We have:

$$[\lambda a_1 + \mu b + \nu a_3, \lambda a_1 + \mu b + \nu a_3] =$$
$$\lambda^2[a_1, a_1] + \lambda\mu[a_1, b] + \lambda\nu[a_1, a_3] + \lambda\mu[b, a_1] + \mu^2[b, b] + \mu\nu[b, a_3] =$$
$$\lambda^2 a_3 - \lambda\mu a_1 + \lambda\nu a_3 + \lambda\mu(a_1 - a_3) + \mu^2 a_3 + \mu\nu a_3 =$$
$$\lambda^2 a_3 - \lambda\mu a_1 + \lambda\nu a_3 + \lambda\mu a_1 - \lambda\mu a_3 + \mu^2 a_3 + \mu\nu a_3 =$$
$$(\lambda^2 + \lambda\nu - \lambda\mu + \mu^2 + \mu\nu)a_3.$$

As above, $\lambda^2 + \lambda\nu - \lambda\mu + \mu^2 + \mu\nu \neq 0$. Put $a_2 = b$. Thus, we come to the following type of Leibniz algebras:

$$L_{31} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where}$$
$$[a_1, a_1] = [a_1, a_3] = [a_2, a_2] = [a_2, a_3] = a_3,$$
$$[a_1, a_2] = -a_1, [a_2, a_1] = a_1 - a_3,$$
$$[a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{31}) = \zeta^{\mathrm{left}}(L_{31}) = Fa_3$, $[L_{31}, L_{31}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{31}) = \zeta(L_{31}) = \langle 0 \rangle$. Moreover, if $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, then $\lambda^2 + \lambda\nu - \lambda\mu + \mu^2 + \mu\nu \neq 0$, L_{31} is non-nilpotent.

Suppose that $\beta \neq 0$. Let $\lambda a_1 + \mu b + \nu a_3$ be the arbitrary element of L. We have:

$$[\lambda a_1 + \mu b + \nu a_3, \lambda a_1 + \mu b + \nu a_3] =$$
$$\lambda^2[a_1, a_1] + \lambda\mu[a_1, b] + \lambda\nu[a_1, a_3] + \lambda\mu[b, a_1] + \mu^2[b, b] + \mu\nu[b, a_3] =$$
$$\lambda^2 a_3 + \lambda\mu(-a_1 + \beta a_3) + \lambda\nu a_3 + \lambda\mu(a_1 - a_3) + \mu^2 a_3 + \mu\nu a_3 =$$
$$\lambda^2 a_3 - \lambda\mu a_1 + \lambda\mu\beta a_3 + \lambda\nu a_3 + \lambda\mu a_1 - \lambda\mu a_3 + \mu^2 a_3 + \mu\nu a_3 =$$
$$(\lambda^2 + \lambda\mu\beta + \lambda\nu - \lambda\mu + \mu^2 + \mu\nu)a_3.$$

As above, $\lambda^2 + \lambda\mu\beta + \lambda\nu - \lambda\mu + \mu^2 + \mu\nu \neq 0$. Put $a_2 = b$. Thus, we come to the following type of Leibniz algebras:

$$L_{32} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where}$$
$$[a_1, a_1] = [a_1, a_3] = [a_2, a_2] = [a_2, a_3] = a_3,$$
$$[a_1, a_2] = -a_1 + \beta a_3\ (\beta \neq 0), [a_2, a_1] = a_1 - a_3,$$
$$[a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{32}) = \zeta^{\mathrm{left}}(L_{32}) = Fa_3$, $[L_{32}, L_{32}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{32}) = \zeta(L_{32}) = \langle 0 \rangle$. Moreover, if $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, then $\lambda^2 + \lambda\mu\beta + \lambda\nu - \lambda\mu + \mu^2 + \mu\nu \neq 0$, L_{32} is non-nilpotent.

Suppose now that $\alpha \neq 0$ and $\alpha \neq -1$. As we have noted above, $\alpha \neq -2$. We obtain $\beta = (\alpha + \gamma)(1 + \alpha)^{-1}$. If $\beta = 0$, then $\alpha = -\gamma$, $[a_1, b] = -a_1$, $[b, a_1] = a_1 - \gamma a_3$, $[b, a_3] = (2 - \gamma)a_3$. Let $\lambda a_1 + \mu b + \nu a_3$ be the arbitrary element of L. We have:

$$[\lambda a_1 + \mu b + \nu a_3, \lambda a_1 + \mu b + \nu a_3] =$$
$$\lambda^2[a_1, a_1] + \lambda\mu[a_1, b] + \lambda\nu[a_1, a_3] + \lambda\mu[b, a_1] + \mu^2[b, b] + \mu\nu[b, a_3] =$$
$$\lambda^2 a_3 - \lambda\mu a_1 + \lambda\nu a_3 + \lambda\mu(a_1 - \gamma a_3) + \mu^2\gamma a_3 + \mu\nu(2 - \gamma)a_3 =$$
$$\lambda^2 a_3 - \lambda\mu a_1 + \lambda\nu a_3 + \lambda\mu a_1 - \lambda\mu\gamma a_3 + \mu^2\gamma a_3 + \mu\nu(2 - \gamma)a_3 =$$
$$(\lambda^2 + \lambda\nu - \lambda\mu\gamma + \mu^2\gamma + \mu\nu(2 - \gamma))a_3.$$

As above, $\lambda^2 + \lambda\nu - \lambda\mu\gamma + \mu^2\gamma + \mu\nu(2 - \gamma) \neq 0$. Put $a_2 = b$. Thus, we come to the following type of Leibniz algebras:

$$L_{33} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = [a_1, a_3] = a_3,$$
$$[a_1, a_2] = -a_1, [a_2, a_1] = a_1 - \gamma a_3\ (\gamma \neq 0, \gamma \neq 1, \gamma \neq 2),$$
$$[a_2, a_2] = \gamma a_3, [a_2, a_3] = (2 - \gamma)a_3, [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{33}) = \zeta^{\mathrm{left}}(L_{33}) = Fa_3$, $[L_{33}, L_{33}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{33}) = \zeta(L_{33}) = \langle 0 \rangle$. Moreover, if $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, then $\lambda^2 + \lambda\nu - \lambda\mu\gamma + \mu^2\gamma + \mu\nu(2 - \gamma) \neq 0$, L_{33} is non-nilpotent.

Suppose that $\beta \neq 0$. Then $[a_1, b] = -a_1 + \beta a_3$ $(\beta = (\alpha + \gamma)(1 + \alpha)^{-1})$, $[b, a_1] = a_1 + \alpha a_3$, $[b, a_3] = (2 + \alpha)a_3$. Let $\lambda a_1 + \mu b + \nu a_3$ be the arbitrary element of L. We have:

$$[\lambda a_1 + \mu b + \nu a_3, \lambda a_1 + \mu b + \nu a_3] =$$
$$\lambda^2[a_1, a_1] + \lambda\mu[a_1, b] + \lambda\nu[a_1, a_3] + \lambda\mu[b, a_1] + \mu^2[b, b] + \mu\nu[b, a_3] =$$
$$\lambda^2 a_3 + \lambda\mu(-a_1 + \beta a_3) + \lambda\nu a_3 + \lambda\mu(a_1 + \alpha a_3) + \mu^2\gamma a_3 + \mu\nu(2 + \alpha)a_3 =$$
$$\lambda^2 a_3 - \lambda\mu a_1 + \lambda\mu\beta a_3 + \lambda\nu a_3 + \lambda\mu a_1 + \lambda\mu\alpha a_3 + \mu^2\gamma a_3 + \mu\nu(2 + \alpha)a_3 =$$
$$(\lambda^2 + \lambda\mu\beta + \lambda\nu + \lambda\mu\alpha + \mu^2\gamma + \mu\nu(2 + \alpha))a_3.$$

As above, $\lambda^2 + \lambda\mu\beta + \lambda\nu + \lambda\mu\alpha + \mu^2\gamma + \mu\nu(2+\alpha) \neq 0$. Put $a_2 = b$. Thus, we come to the following type of Leibniz algebras:

$$L_{34} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = [a_1, a_3] = a_3,$$
$$[a_1, a_2] = -a_1 + \beta a_3 \ (\beta = (\alpha + \gamma)(1 + \alpha)^{-1}),$$
$$[a_2, a_1] = a_1 + \alpha a_3 \ (\alpha \neq 0, \alpha \neq -1, \alpha \neq -2),$$
$$[a_2, a_2] = \gamma a_3 \ (\gamma \neq 0), [a_2, a_3] = (2 + \alpha)a_3,$$
$$[a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{34}) = \zeta^{\mathrm{left}}(L_{34}) = Fa_3$, $[L_{34}, L_{34}] = Fa_1 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{34}) = \zeta(L_{34}) = \langle 0\rangle$. Moreover, if $\lambda a_1 + \mu a_2 + \nu a_3 \notin A$, then $\lambda^2 + \lambda\mu\beta + \lambda\nu + \lambda\mu\alpha + \mu^2\gamma + \mu\nu(2 + \alpha) \neq 0$, L_{34} is non-nilpotent.

Suppose now that $[c, c] = 0$. Put again $\mathrm{Leib}(L) = Fa_3$. Let b be an element such that $b \notin C$.

Suppose that a subalgebra $C = Fc \oplus \mathrm{Leib}(L)$ is not abelian. As we have seen above, we can choose an element a_3 such that $[c, a_3] = a_3$. Since $\mathrm{Leib}(L) = Fa_3$ is an ideal, $[b, a_3] = \eta a_3$ for some element $\eta \in F$. Using the above arguments without loss of generality we can assume that $[b, c] \in c + Fa_3$, $[c, b] \in -c + Fa_3$, so that $[b, c] = c + \alpha a_3$, $[c, b] = -c + \beta a_3$ for some elements $\alpha, \beta \in F$. Then

$$a_3 = [c, a_3] = [c + \alpha a_3, a_3] = [[b, c], a_3] = [b, [c, a_3]] - [c, [b, a_3]] =$$
$$[b, a_3] - [c, \eta a_3] = \eta a_3 - \eta[c, a_3] = \eta a_3 - \eta a_3 = 0.$$

This contradiction shows that a subalgebra C is abelian.

Note that $\eta \neq 0$. In fact, otherwise $a_3 \in \zeta(L)$, and we obtain a contradiction. This contradiction shows that $[b, a_3] \neq 0$ for every element $b \notin C$. As we have seen above, we can choose an element b such that $[b, a_3] = a_3$.

Suppose first that there exists an element $b \notin C$ such that $[b, b] \neq 0$. It follows that $[b, b] = \gamma a_3$ where γ is a non-zero element of F. Then, subalgebra $B = Fb \oplus Fa_3$ is non-abelian. We have now

$$[c, a_3] = [c, \gamma^{-1}[b, b]] = \gamma^{-1}[c, [b, b]] = \gamma^{-1}([[c, b], b] + [b, [c, b]]) =$$
$$\gamma^{-1}([-c + \beta a_3, b] + [b, -c + \beta a_3]) = \gamma^{-1}(-[c, b] - [b, c] + \beta[b, a_3]) =$$
$$\gamma^{-1}(c - \beta a_3 - c - \alpha a_3 + \beta a_3) = -\alpha\gamma^{-1}a_3.$$

On the other hand, we proved above that $[c, a_3] = 0$. Since $\gamma \neq 0$, it follows that $\alpha = 0$. Hence, $[b, c] = c$.

If $\beta = 0$, then put $a_1 = b$, $a_2 = c$. A subalgebra Fc is an ideal of L and we come to the following type of Leibniz algebras:

$$L_{35} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = \gamma a_3 \ (\gamma \neq 0),$$
$$[a_1, a_2] = a_2, [a_1, a_3] = a_3, [a_2, a_1] = -a_2,$$
$$[a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{35}) = \zeta^{\mathrm{left}}(L_{35}) = Fa_3$, $[L_{35}, L_{35}] = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{35}) = \zeta(L_{35}) = \langle 0\rangle$, L_{35} is non-nilpotent.

If $\beta \neq 0$, then put $a_1 = b$, $a_2 = c$. A subalgebra Fc is not an ideal of L and we come to the following type of Leibniz algebras:

$$L_{36} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = \gamma a_3 \ (\gamma \neq 0),$$
$$[a_1, a_2] = a_2, [a_1, a_3] = a_3, [a_2, a_1] = -a_2 + \beta a_3 \ (\beta \neq 0),$$
$$[a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{36}) = \zeta^{\mathrm{left}}(L_{36}) = Fa_3$, $[L_{36}, L_{36}] = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{36}) = \zeta(L_{36}) = \langle 0\rangle$, L_{36} is non-nilpotent.

Suppose now that $[b, b] = 0$ for each element $b \notin C$. Let $u = \lambda c + \mu b + \nu a_3$ be an arbitrary element of L. Then,

$$[\lambda c + \mu b + \nu a_3, \lambda c + \mu b + \nu a_3] = \lambda\mu[c, b] + \lambda\mu[b, c] + \mu\nu[b, a_3] =$$
$$\lambda\mu(-c + \beta a_3) + \lambda\mu(c + \alpha a_3) + \mu\nu a_3 =$$
$$(\lambda\mu\beta + \lambda\mu\alpha + \mu\nu)a_3.$$

If $u \notin C$, then $\mu \neq 0$. If $\lambda = 0$, $\mu = \nu = 1$, then $[u, u] = a_3 \neq 0$, and we obtain a contradiction. This contradiction shows that this situation is not possible. $\square$

The following natural situation appears when $\dim_F(\mathrm{Leib}(L)) = 2$. Immediately, we obtain the following two subcases:

(IIA) the intersection $\zeta(L) \cap \mathrm{Leib}(L)$ is not trivial;

(IIB) $\zeta(L) \cap \mathrm{Leib}(L) = \langle 0\rangle$.

Theorem 5. *Let L be a nilpotent Leibniz algebra over a field F having dimension 3, which is not a Lie algebra. Suppose that $\dim_F(\mathrm{Leib}(L)) = 2$. Then, L is an algebra of the following type:*

$\mathrm{Lei}_{37}(3, F) = L_{37}$ *is a cyclic nilpotent Leibniz algebra, so that* $L_{37} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_2$, $[a_1, a_2] = a_3$, $[a_1, a_3] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{37}) = [L_{37}, L_{37}] = \zeta^{\mathrm{left}}(L_{37}) = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{37}) = \zeta(L_{37}) = Fa_3$, $\mathrm{ncl}(L_{37}) = 3$.

Proof. Since $\dim_F(L) = 3$, $\mathrm{ncl}(L) \leqslant 3$. Suppose first that $\mathrm{ncl}(L) = 3$. Then, L has an upper central series of a length 3:

$$\langle 0 \rangle = C_0 \leqslant C_1 \leqslant C_2 \leqslant C_3 = L.$$

Every factor of this series must be non-trivial. Therefore every factor of this series has dimension 1. Let a_1 be an element of L such that $a_1 \notin C_2$. The fact that L/C_2 is abelian implies that $a_2 = [a_1, a_1] \in C_2$. Suppose that $a_2 \in C_1$. Since $\dim_F(C_1) = 1$, $C_1 = Fa_2$. Choose an element $b \in C_2$ such that $b \notin C_1$. Then, $[a_1, b], [b, a_1], [b, b] \in C_1$, so that $[a_1, b] = \alpha a_2$, $[b, a_1] = \beta a_2$, $[b, b] = \gamma a_2$ for some elements $\alpha, \beta, \gamma \in F$. It is not hard to see that the elements a_1, b, a_2 generate L. Let $\lambda_1 a_1 + \lambda_2 b + \lambda_3 a_2$, $\mu_1 a_1 + \mu_2 b + \mu_3 a_2$ be two arbitrary elements of L. We have:

$$\begin{gathered}[\lambda_1 a_1 + \lambda_2 b + \lambda_3 a_2, \mu_1 a_1 + \mu_2 b + \mu_3 a_2] = \\ \lambda_1\mu_1[a_1, a_1] + \lambda_1\mu_2[a_1, b] + \lambda_1\mu_3[a_1, a_2] + \\ \lambda_2\mu_1[b, a_1] + \lambda_2\mu_2[b, b] + \lambda_2\mu_3[b, a_2] = \\ \lambda_1\mu_1 a_2 + \lambda_1\mu_2\alpha a_2 + \lambda_2\mu_1\beta a_2 + \lambda_2\mu_2\gamma a_2 \in C_1.\end{gathered}$$

It follows that $[L, L] \leqslant C_1 = \zeta(L)$ and hence, $\mathrm{ncl}(L) = 2$. This contradiction shows that a_2
$notinC_1$. Then, $a_3 = [a_1, a_2] \neq 0$. It follows that $C_2 = Fa_2 \oplus Fa_3$. We have $[a_2, a_3] = [a_3, a_2] = 0$, so that C_2 is an abelian subalgebra, $[a_1, a_3] = [a_3, a_1] = [a_2, a_1] = 0$. It follows that

$$\begin{gathered}[a_1 + a_2, a_1 + a_2] = [a_1, a_1] + [a_1, a_2] + [a_2, a_1] + [a_2, a_2] = \\ a_2 + a_3 \in \mathrm{Leib}(L),\end{gathered}$$

and hence, $\mathrm{Leib}(L) = C_2$. Thus, we come to the following type of Leibniz algebras:

$$\begin{gathered}L_{37} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_2, [a_1, a_2] = a_3, \\ [a_1, a_3] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.\end{gathered}$$

Note also that $\mathrm{Leib}(L_{37}) = [L_{37}, L_{37}] = \zeta^{\mathrm{left}}(L_{37}) = Fa_2 \oplus Fa_3$, $\zeta(L_{37}) = \zeta^{\mathrm{right}}(L_{37}) = Fa_3$, $\mathrm{ncl}(L_{37}) = 3$.

Suppose now that $\mathrm{ncl}(L) = 2$. Then, L has an upper central series of a length 2:

$$\langle 0\rangle = C_0 \leqslant C_1 \leqslant C_2 = L.$$

Here, we have two possibilities: $\dim_F(C_1) = 1$ or $\dim_F(C_1) = 2$. Since L/C_1 is abelian, $\mathrm{Leib}(L) \leqslant C_1$. Then, the fact that $\dim_F(\mathrm{Leib}(L)) = 2$ implies that $\dim_F(C_1) = 2$.

Since L is a not Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. Then, $a_1 \notin \zeta(L)$, $a_3 \in \mathrm{Leib}(L) \leqslant \zeta(L)$. It follows that $[a_1, a_3] = [a_3, a_1] = [a_3, a_3] = 0$. We can see that $A = \langle a_1\rangle = Fa_1 \oplus Fa_3$. In particular, $A \cap C_1 = Fa_3$. We have $C_1 = Fa_2 \oplus Fa_3$ for some element a_2. Then, $L = A \oplus Fa_2$. Since $a_2 \in \zeta(L)$, Fa_2 is an ideal of L. The choice of a_2 implies that $[a_1, a_2] = [a_2, a_1] = [a_2, a_3] = [a_3, a_2] = [a_2, a_2] = 0$. It follows that factor-algebra L/Fa_3 is abelian. Then, $\mathrm{Leib}(L) \leqslant Fa_3$, in particular, $\dim_F(\mathrm{Leib}(L)) = 1$, and we obtain a contradiction. This contradiction shows that the case $\mathrm{ncl}(L) = 2$ is not possible. □

Theorem 6. *Let L be a non-nilpotent Leibniz algebra over a field F having dimension 3, which is not a Lie algebra. Suppose that $\zeta(L) \neq \langle 0\rangle$ and that $\dim_F(\mathrm{Leib}(L)) = 2$. Then, L is an algebra of the following type:*

$\mathrm{Lei}_{38}(3, F) = L_{38}$ *is a cyclic Leibniz algebra,* $L_{24} = Fa_1 \oplus Fa_2 \oplus Fa_3$ *where* $[a_1, a_1] = a_2$, $[a_1, a_2] = a_2 + a_3$, $[a_1, a_3] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{38}) = \zeta^{\mathrm{left}}(L_{38}) = [L_{38}, L_{38}] = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{38}) = \zeta(L_{38}) = Fa_3$.

Proof. We note that a Leibniz algebra of dimension 1 is abelian. Then, by our conditions, we obtain that $\dim_F(\zeta(L)) = 1$. If we suppose that $\zeta(L) \cap \mathrm{Leib}(L) = \langle 0\rangle$, then $L = \zeta(L) \oplus \mathrm{Leib}(L)$, so that L is abelian, and we obtain a contradiction. This contradiction shows that $\zeta(L) \leqslant \mathrm{Leib}(L)$. Put $C = \zeta(L)$ and let c be an element such that $C = Fc$. Since $C \neq \mathrm{Leib}(L)$, the factor-algebra L/C is not a Lie algebra. Using the information above about the structure of the Leibniz algebras of dimension 2, we obtain that $L/C = F(b + C) \oplus F(d + C)$ where

$$[d + C, d + C] = b + C, [d + C, b + C] = b + C,$$
$$[b + C, d + C] = [b + C, b + C] = C.$$

Without loss of generality, we may assume that $[d, d] = b$. Then, we have $[d, b] = b + \alpha c$, $[b, d] = 0$ for some element $\alpha \in F$. The fact that $\mathrm{Leib}(L)$

is abelian implies that $[b, b] = [b, c] = [c, b] = 0$. Let $\lambda_1 d + \lambda_2 b + \lambda_3 c$ be an arbitrary element of L. We have:

$$[\lambda_1 d + \lambda_2 b + \lambda_3 c, \lambda_1 d + \lambda_2 b + \lambda_3 c] = \lambda_1^2 b + \lambda_1\lambda_2(b + \alpha c) = (\lambda_1^2 + \lambda_1\lambda_2)b + \lambda_1\lambda_2\alpha c.$$

Thus, we can see that if $\alpha = 0$, then $\mathrm{Leib}(L) = Fb$, in particular, $\dim_F(\mathrm{Leib}(L)) = 1$, and we obtain a contradiction. This contradiction shows that $\alpha \neq 0$. Put $a_1 = d$, $a_2 = b$, $a_3 = \alpha c$, then

$$[a_1 - a_2, a_1 - a_2] = [a_1, a_1] - [a_2, a_1] - [a_1, a_2] + [a_2, a_2] = a_2 - a_2 - a_3 = -a_3.$$

Thus, $\mathrm{Leib}(L) = Fa_2 \oplus Fa_3$, and we come to the following type of Leibniz algebras:

$$L_{38} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_2, [a_1, a_2] = a_2 + a_3,$$
$$[a_1, a_3] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{38}) = \zeta^{\mathrm{left}}(L_{38}) = [L_{38}, L_{38}] = Fa_2 \oplus Fa_3$, $\zeta(L_{38}) = \zeta^{\mathrm{right}}(L_{38}) = Fa_3$. □

Theorem 7. *Let L be a Leibniz algebra over a field F having dimension 3 and L not be a Lie algebra. Suppose that $\zeta(L) = \langle 0 \rangle$ and that $\dim_F(\mathrm{Leib}(L)) = 2$. Then, L is an algebra of one of the following types.*

(i) $\mathrm{Lei}_{39}(3, F) = L_{39}$ *is a direct sum of ideal $B = Fa_3$ and cyclic non-nilpotent subalgebra $A = Fa_1 \oplus Fa_2$, $[A, B] = Fa_3$, $[B, A] = \langle 0 \rangle$, so that $L_{39} = Fa_1 \oplus Fa_2 \oplus Fa_3$ where $[a_1, a_1] = [a_1, a_2] = a_2$, $[a_1, a_3] = \beta a_3$ $(\beta \neq 0)$, $[a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{39}) = \zeta^{\mathrm{left}}(L_{39}) = [L_{39}, L_{39}] = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{39}) = \zeta(L_{39}) = \langle 0 \rangle$.*

(ii) $\mathrm{Lei}_{40}(3, F) = L_{40}$ *is a cyclic Leibniz algebra, so that $L_{40} = Fa_1 \oplus Fa_2 \oplus Fa_3$ where $[a_1, a_1] = a_2$, $[a_1, a_2] = a_2 + \gamma a_3$ $(\gamma \neq 0)$, $[a_1, a_3] = \beta a_3$ $(\beta \neq 0)$, $[a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{40}) = \zeta^{\mathrm{left}}(L_{40}) = [L_{40}, L_{40}] = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{40}) = \zeta(L_{40}) = \langle 0 \rangle$.*

(iii) $\mathrm{Lei}_{41}(3, F) = L_{41}$ *is a cyclic Leibniz algebra, so that $L_{41} = Fa_1 \oplus Fa_2 \oplus Fa_3$ where $[a_1, a_1] = a_2$, $[a_1, a_2] = \gamma a_3$ $(\gamma \neq 0)$, $[a_1, a_3] = a_3$, $[a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{41}) = \zeta^{\mathrm{left}}(L_{41}) = [L_{41}, L_{41}] = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{41}) = \zeta(L_{41}) = \langle 0 \rangle$.*

(iv) $\mathrm{Lei}_{42}(3, F) = L_{42}$ *is a cyclic Leibniz algebra, so that $L_{42} = Fa_1 \oplus Fa_2 \oplus Fa_3$ where $[a_1, a_1] = a_2$, $[a_1, a_2] = a_3$, $[a_1, a_3] = \beta a_2 +$*

γa_3, $[a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0$, $\mathrm{Leib}(L_{42}) = \zeta^{\mathrm{left}}(L_{42}) = [L_{42}, L_{42}] = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_{42}) = \zeta(L_{42}) = \langle 0 \rangle$. Moreover, polynomial $X^2 - \gamma X - \beta$ is irreducible over field F.

Proof. Suppose first that $\mathrm{Leib}(L)$ includes an ideal K of dimension 1. Let c be an element such that $K = Fc$. Since $K \neq \mathrm{Leib}(L)$, the factor-algebra L/K is not a Lie algebra. Using the above information about the structure of the Leibniz algebras of dimension 2, we obtain that $L/K = F(b + K) \oplus F(d + K)$ where

$$[d + K, d + K] = b + K, [d + K, b + K] = b + K,$$
$$[b + K, d + K] = [b + K, b + K] = K$$

or

$$[d + K, d + K] = b + K,$$
$$[d + K, b + K] = [b + K, d + K] = [b + K, b + K] = K.$$

Consider the first situation. Without loss of generality, we may assume that $[d, d] = b$. Then, we have $[b, d] = 0$, $[d, b] = b + \alpha c$ for some element $\alpha \in F$. The fact that $\mathrm{Leib}(L)$ is abelian implies that $[b, b] = [b, c] = [c, b] = 0$. Since $\zeta(L) = \langle 0 \rangle$, $[d, c] = \beta c$ for some non-zero element $\beta \in F$. Put $a_3 = \beta c$, then $K = Fa_3$ and $[b, a_3] = [a_3, b] = [a_3, d] = [a_3, a_3] = 0$, $[d, b] = b + \gamma a_3$ where $\gamma = \alpha\beta^{-1}$. Let $u = \lambda_1 d + \lambda_2 b + \lambda_3 a_3$ be an arbitrary element of L. We have:

$$[u, u] = [\lambda_1 d + \lambda_2 b + \lambda_3 a_3, \lambda_1 d + \lambda_2 b + \lambda_3 a_3] =$$
$$\lambda_1^2[d, d] + \lambda_1\lambda_2[d, b] + \lambda_1\lambda_3[d, a_3] =$$
$$\lambda_1^2 b + \lambda_1\lambda_2(b + \gamma a_3) + \lambda_1\lambda_3\beta a_3 =$$
$$(\lambda_1^2 + \lambda_1\lambda_2)b + (\gamma\lambda_1\lambda_2 + \lambda_1\lambda_3\beta)a_3.$$

If we put $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = \beta^{-1}(1 + \gamma)$, then we obtain $[u, u] = a_3$. Put $a_1 = d$, $a_2 = b$.

If $\alpha = 0$ and hence, $\gamma = 0$, then a subalgebra Fa_2 is an ideal of L and $A = \langle a_1 \rangle = Fa_1 \oplus Fa_2$ is a cyclic subalgebra. Thus, we come to the following type of Leibniz algebras:

$$L_{39} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = [a_1, a_2] = a_2,$$
$$[a_1, a_3] = \beta a_3 \ (\beta \neq 0),$$
$$[a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{39}) = \zeta^{\mathrm{left}}(L_{39}) = [L_{39}, L_{39}] = Fa_2 \oplus Fa_3$, $\zeta(L_{39}) = \zeta^{\mathrm{right}}(L_{39}) = \langle 0 \rangle$.

If $\alpha \neq 0$, then $a_3 = \gamma^{-1}([a_1, a_2] - a_2)$. It follows that L is a cyclic algebra. Thus, we come to the following type of Leibniz algebras:

$$L_{40} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_2,$$
$$[a_1, a_2] = a_2 + \gamma a_3 \ (\gamma \neq 0), [a_1, a_3] = \beta a_3 \ (\beta \neq 0),$$
$$[a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{40}) = \zeta^{\mathrm{left}}(L_{40}) = [L_{40}, L_{40}] = Fa_2 \oplus Fa_3$, $\zeta(L_{40}) = \zeta^{\mathrm{right}}(L_{40}) = \langle 0 \rangle$.

Consider now a situation when $L/K = F(b + K) \oplus F(d + K)$ where

$$[d + K, d + K] = b + K,$$
$$[d + K, b + K] = [b + K, d + K] = [b + K, b + K] = K.$$

Without loss of generality we may assume that $[d, d] = b$. Then we have $[b, d] = 0$, $[d, b] = \alpha c$ for some element $\alpha \in F$. The fact that $\mathrm{Leib}(L)$ is abelian implies that $[b, b] = [b, c] = [c, b] = 0$. Since $\zeta(L) = \langle 0 \rangle$, $[d, c] = \beta c$ for some non-zero element $\beta \in F$. Put $a_3 = c$, then $K = Fa_3$ and $[b, a_3] = [a_3, b] = [a_3, d] = [a_3, a_3] = 0$, $[d, b] = \alpha a_3$.

If $\alpha = 0$, then Fb lies in the center of L, and we obtain a contradiction.

Suppose that $\alpha \neq 0$. Put $a_1 = \beta^{-1} d$, then $[a_1, c] = c$. Further $[a_1, a_1] = [\beta^{-1} d, \beta^{-1} d] = \beta^{-2} b = a_2$. We have $[a_2, a_2] = [a_2, a_1] = [a_2, c] = [c, a_2] = 0$, $[a_1, a_2] = [\beta^{-1} d, \beta^{-2} b] = \beta^{-3} \alpha c = \gamma c$. Thus we come to the following type of Leibniz algebra:

$$L_{41} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_2,$$
$$[a_1, a_2] = \gamma a_3 \ (\gamma \neq 0), [a_1, a_3] = a_3,$$
$$[a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{41}) = \zeta^{\mathrm{left}}(L_{41}) = [L_{41}, L_{41}] = Fa_2 \oplus Fa_3$, $\zeta(L_{41}) = \zeta^{\mathrm{right}}(L_{41}) = \langle 0 \rangle$.

Suppose now that $\mathrm{Leib}(L)$ does not include proper non-zero ideals. Since L is a non-Lie algebra, there is an element d such that $[d, d] = b \neq 0$. Then, $d \notin \mathrm{Leib}(L)$. Put $K = \mathrm{Leib}(L)$. By our assumption, $[d, b] = c \notin Fb$. The fact that $\dim_F(K) = 2$ implies that $K = Fb \oplus Fc$. Then, $[d, c] = \beta b + \gamma c$ for some elements $\beta, \gamma \in F$. The mapping $x \to [d, x]$, $x \in K$ is linear. Our conditions imply that a polynomial $X^2 - \gamma X - \beta$ is irreducible over a field F. Put $a_1 = d$, $a_2 = b$, $a_3 = c$. Thus, we come to the following type of Leibniz algebras:

$$L_{42} = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_2, [a_1, a_2] = a_3,$$
$$[a_1, a_3] = \beta a_2 + \gamma a_3,$$
$$[a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

Note also that $\mathrm{Leib}(L_{42}) = \zeta^{\mathrm{left}}(L_{42}) = [L_{42}, L_{42}] = Fa_2 \oplus Fa_3$, $\zeta(L_{42}) = \zeta^{\mathrm{right}}(L_{42}) = \langle 0 \rangle$, polynomial $X^2 - \gamma X - \beta$ is irreducible over field F. □

References

[1] Sh. Ayupov, B. Omirov, I. Rakhimov, *Leibniz Algebras: Structure and Classification*, CRC Press, Taylor & Francis Group, (2020).

[2] A. Blokh, *On a generalization of the concept of Lie algebra*, Dokl. Akad. Nauk SSSR, **165**(3), 471-473 (1965) (in Russian).

[3] J.M. Casas, M.A. Insua, M. Ladra, S. Ladra, *An algorithm for the classification of 3-dimensional complex Leibniz algebras*, Linear Algebra Appl., **436**(9), 3747-3756 (2012); DOI:10.1016/j.laa.2011.11.039.

[4] I. Demir, K.C. Misra, E. Stitzinger, *On some structures of Leibniz algebras*, Recent Advances in Representation Theory, Quantum Groups, Algebraic Geometry, and Related Topics, Contemporary Mathematics, **623**, 41-54 (2014). DOI:10.1090/conm/623/12456.

[5] A.Kh. Khudoyberdiyev, T.K. Kurbanbaev, B.A. Omirov, *Classification of three-dimensional solvable p-adic Leibniz algebras*, p-Adic Numbers Ultrametric Anal. Appl., **2**(3), 207-221 (2010); DOI:10.1134/S2070046610030039.

[6] V.V. Kirichenko, L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin, *Some aspects of Leibniz algebra theory*, Algebra Discrete Math., **24**(1), 1-33 (2017).

[7] L.A. Kurdachenko, J. Otal, A.A. Pypka, *Relationships between the factors of the canonical central series of Leibniz algebras*, Eur. J. Math., **2**(2), 565-577 (2016); DOI:10.1007/s40879-016-0093-5.

[8] J.-L. Loday, *Cyclic homology*, Grundlehren der Mathematischen Wissenschaften, **301**, Springer Verlag, (1992); DOI:10.1007/978-3-662-11389-9.

[9] J.-L. Loday, *Une version non commutative des algèbres de Lie; les algèbras de Leibniz*, Enseign. Math., **39**, 269-293 (1993).

[10] I.S. Rakhimov, I.M. Rikhsiboev, M.A. Mohammed, *An algorithm for classifications of three-dimensional Leibniz algebras over arbitrary fields*, JP Journal of Algebra, Number Theory and Applications, **40**(2), 181-198 (2018); DOI:10.17654/NT040020181.

[11] I.M. Rikhsiboev, I.S. Rakhimov, *Classification of three dimensional complex Leibniz algebras*, AIP Conference Proceedings, **1450**(1), 358-362 (2012); DOI:10.1063/1.4724168.

[12] V.S. Yashchuk, *On some Leibniz algebras, having small dimension*, Algebra Discrete Math., **27**(2), 292-308 (2019).

Contact information

L. A. Kurdachenko, O. O. Pypka — Oles Honchar Dnipro National University, Gagarin ave., 72, Dnipro, 49010, Ukraine
E-Mail(s): lkurdachenko@gmail.com, sasha.pypka@gmail.com

I. Ya. Subbotin — National University, 5245 Pacific Concourse Drive, Los Angeles, CA 90045-6904, USA
E-Mail(s): isubboti@nu.edu

Received by the editors: 02.11.2022.