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Constructions of BiHom-X algebras and bimodules of some BiHom-dialgebras

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Abstract. The aim of this paper is to introduce and to develop several methods for constructions of BiHom-X algebras by extending composition methods, and by using Rota-Baxter operators and some elements of centroids. The bimodules of BiHom-left symmetric dialgebras, BiHom-associative dialgebras and BiHom-tridendriform algebra are defined, and it is shown that a sequence of this kind of bimodules can be constructed. Their matched pairs of BiHom-left symmetric, BiHom-associative dialgebras BiHom-tridendriform algebra are introduced and methods for their constructions and properties are investigated.

Introduction

The theory of Hom-algebras has been initiated in [19, 27, 28] motivated by quasi-deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras. Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced first by Hartwig, Larsson and Silvestrov in [19] where a general approach to discretization of Lie algebras of vector fields using general twisted derivations (σ-derivations) and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed. The general

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quasi-Lie algebras, containing the quasi-Hom-Lie algebras and Hom-Lie algebras as subclasses, as well their graded color generalization, the color quasi-Lie algebras including color quasi-hom-Lie algebras, color hom-Lie algebras and their special subclasses the quasi-Hom-Lie superalgebras and hom-Lie superalgebras, have been first introduced in [19, 27–30, 52]. Subsequently, various classes of Hom-Lie admissible algebras have been considered in [42]. In particular, in [42], the Hom-associative algebras have been introduced and shown to be Hom-Lie admissible, that is leading to Hom-Lie algebras using commutator map as new product, and in this sense constituting a natural generalization of associative algebras as Lie admissible algebras leading to Lie algebras using commutator map. Furthermore, in [42], more general G-Hom-associative algebras including Hom-associative algebras, Hom-Vinberg algebras (Hom-left symmetric algebras), Hom-pre-Lie algebras (Hom-right symmetric algebras), and some other Hom-algebra structures, generalizing G-associative algebras, Vinberg and pre-Lie algebras respectively, have been introduced and shown to be Hom-Lie admissible, meaning that for these classes of Hom-algebras, the operation of taking commutator leads to Hom-Lie algebras as well. Also, flexible Hom-algebras have been introduced, connections to Hom-algebra generalizations of derivations and of adjoint maps have been noticed, and some low-dimensional Hom-Lie algebras have been described. In Hom-algebra structures, defining algebra identities are twisted by linear maps. Since the pioneering works [19, 27–30, 42], Hom-algebra structures have developed in a popular broad area with increasing number of publications in various directions. Hom-algebra structures include their classical counterparts and open new broad possibilities for deformations, extensions to Hom-algebra structures of representations, homology, cohomology and formal deformations, Hom-modules and hom-bimodules, Hom-Lie admissible Hom-coalgebras, Hom-coalgebras, Hom-bialgebras, Hom-Hopf algebras, L-modules, L-comodules and Hom-Lie quasi-bialgebras, n-ary generalizations of BiHom-Lie algebras and BiHom-associative algebras and generalized derivations, Rota-Baxter operators, Hom-dendriform color algebras, Rota-Baxter bisystems and covariant bialgebras, Rota-Baxter cosystems, coquasitriangular mixed bialgebras, coassociative Yang-Baxter pairs, coassociative Yang-Baxter equation and generalizations of Rota-Baxter systems and algebras, curved $\mathcal{O}$-operator systems and their connections with tridendriform systems and pre-Lie algebras, BiHom-algebras, BiHom-Frobenius algebras and double constructions, infinitesimal BiHom-bialgebras and Hom-dendriform D-bialgebras, Hom-algebras have been considered [1–10, 12–17, 20–27, 30–32, 34–41, 43–51, 53–62].

In this paper we introduce and develop methods of constructions of BiHom-X algebras by extending composition methods, and by using Rota-Baxter and some elements of centroids. The bimodules of BiHom-left symmetric dialgebras, BiHom-associative dialgebras and BiHom-tridendriform algebra are defined, and it is shown that a sequence of this kind of bimodules can be constructed. Their matched pairs are also introduced and related relevant properties are given. In section 1, we provide some results on constructions of BiHom-X algebras. Section 2 contains definitions and some key results about bimodules of BiHom-associative algebras and BiHom-left-symmetric algebras, and matched pairs of BiHom-left symmetric and BiHom-associative dialgebras. In section 3, devoted to bimodules of BiHom-tridendriform algebras, definitions and some constructions of BiHom-dendriform and BiHom-tridendriform algebras and the concepts of bimodules and matched pairs of BiHom-tridendriform algebra are investigated.

1. Constructions of BiHom-X algebras

Throughout this paper, all vector spaces are assumed to be over a field $\mathbb{K}$ of characteristic different from 2.

In this section, we provide some results on constructions of BiHom-X algebras.

Definition 1.1. A BiHom-algebra is a $(n+3)$-tuple $(A, \mu_1, \dots, \mu_n, \alpha, \beta)$ in which A is a linear space, $\mu_i : A \otimes A \to A$ $(i = 1, \dots, n)$ are bilinear maps, and $\alpha, \beta : A \to A$ are linear maps, called the twisting maps. If in addition,

$$\alpha \circ \mu_i = \mu_i \circ (\alpha \otimes \alpha), \ \beta \circ \mu_i = \mu_i \circ (\beta \otimes \beta), \quad (i = 1, \dots, n),$$

the BiHom-algebra $(A, \mu_1, \dots, \mu_n, \alpha, \beta)$ is said to be multiplicative.

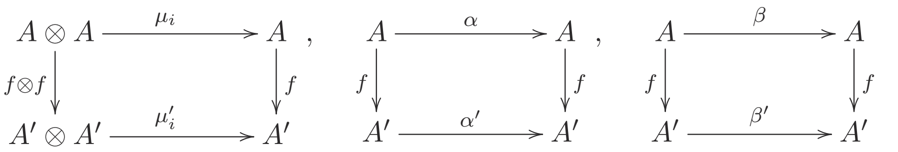
Definition 1.2. Let $(A, \mu_1, \dots, \mu_n, \alpha, \beta)$ be a BiHom-algebra.

1. BiHom-subalgebra of $(A, \mu_1, \dots, \mu_n, \alpha, \beta)$ is a linear subspace H of A, which is closed for the multiplication μ_i $(i = 1, \dots, n)$, and invariant by α and β, that is, $\mu_i(x, y) \in H$, $\alpha(x) \in H$ and $\beta(x) \in H$ for all $x, y \in H$. If furthermore $\mu_i(x, y) \in H$ and $\mu_i(y, x) \in H$ for all $(x, y) \in A \times H$, then H is called a two-sided BiHom-ideal of A.
2. $(A, \mu_1, \dots, \mu_n, \alpha, \beta)$ is said to be regular if α and β are algebra automorphisms.
3. $(A, \mu_1, \dots, \mu_n, \alpha, \beta)$ is said to be involutive if α and β are two involutions, that is $\alpha^2 = \beta^2 = id$.

Definition 1.3. Let $(A, \mu_1, \ldots, \mu_n, \alpha, \beta)$ and $(A', \mu'_1, \ldots, \mu'_n, \alpha', \beta')$ be two BiHom-algebras. Then a linear map $f : A \longrightarrow A'$ is said to be a BiHom-algebras morphism if the following conditions hold for all $i = 1, \ldots, n$:

$$f \circ \mu_i = \mu_i' \circ (f \otimes f), \quad f \circ \alpha = \alpha' \circ f, \quad f \circ \beta = \beta' \circ f,$$

as illustrated, by the following commutative diagrams:



Denote by $\Gamma_f = \{x + f(x); \quad x \in A\} \subset A \oplus A'$ the graph of a linear map $f : A \to A'$.

Definition 1.4. A BiHom-algebra $(A, \mu_1, \ldots, \mu_n, \alpha, \beta)$ is called a BiHom-X algebra if the axioms defining the structure of X are linear combination of the terms of the form $\mu_j \circ (\mu_i \otimes \beta)$ or $\mu_j \circ (\alpha \otimes \mu_i)$.

Proposition 1.5. Let $(A, \mu_1^A, \ldots, \mu_n^A, \alpha_1, \beta_1)$ and $(B, \mu_1^B, \ldots, \mu_n^B, \alpha_2, \beta_2)$ be BiHom-X algebras. Then, there is a BiHom-X algebra

$$(A \oplus B, \mu_1^{A\oplus B}, \ldots, \mu_n^{A\oplus B}, \alpha, \beta),$$

where for all $i = 1, \ldots, n$, the bilinear maps $\mu_i^{A\oplus B} : (A \oplus B)^{\times 2} \longrightarrow (A \oplus B)$ are given by

$$\mu_i^{A\oplus B}(a_1 + b_1, a_2 + b_2) = \mu_i^A(a_1, a_2) + \mu_i^B(b_1, b_2), \forall\, a_1, a_2 \in A,\ \forall\, b_1, b_2 \in B,$$

and the linear maps α and β are given, for all $(a, b) \in A \times B$, by

$$\alpha(a + b) = \alpha_1(a) + \alpha_2(b), \qquad \beta(a + b) = \beta_1(a) + \beta_2(b).$$

Proof. For any $a_1, b_1, c_1 \in A$, $a_2, b_2, c_2 \in B$ and $1 \leqslant i, j \leqslant n$,

$$\begin{aligned}
&\mu_i^{A\oplus B}(\mu_j^{A\oplus B}(a_1 + a_2, b_1 + b_2), \beta(c_1 + c_2)) \\
&\quad = \mu_i^{A\oplus B}(\mu_j^A(a_1, b_1) + \mu_j^B(a_2, b_2), \beta_1(c_1) + \beta_2(c_2)) \\
&\quad = \mu_i^A(\mu_j^A(a_1, b_1), \beta_1(c_1)) + \mu_i^B(\mu_j^B(a_2, b_2), \beta_2(c_2)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mu_i^{A\oplus B}(\alpha(a_1 + a_2), \mu_j^{A\oplus B}(b_1 + b_2, c_1 + c_2)) \\
&\quad = \mu_i^A(\alpha_1(a_1), \mu_j^A(b_1, c_1)) + \mu_i^B(\alpha_2(a_2), \mu_j^B(b_2, c_2)).
\end{aligned}$$

□

Proposition 1.6. Let $(A,\mu_1^A,\dots,\mu_n^A,\alpha_1,\beta_1)$ and $(B,\mu_1^B,\dots,\mu_n^B,\alpha_2,\beta_2)$ be BiHom-X algebras. Then a linear map $\varphi : A \to B$ is a morphism from the BiHom-X algebra $(A,\mu_1^A,\dots,\mu_n^A,\alpha_1,\beta_1)$ to the BiHom-X algebra $(B,\mu_1^B,\dots,\mu_n^B,\alpha_2,\beta_2)$ if and only if its graph Γ_φ is a BiHom-X subalgebra of $(A\oplus B,\mu_1^{A\oplus B},\dots,\mu_n^{A\oplus B},\alpha_1+\beta_1,\alpha_2+\beta_2)$.

Proof. Let $\varphi : (A,\mu_1^A,\dots,\mu_n^A,\alpha_1,\beta_1) \longrightarrow (B,\mu_1^B,\dots,\mu_n^B,\alpha_2,\beta_2)$ be a morphism of BiHom-X algebras. Then for all $u,v\in A$ and $1\leqslant i\leqslant n$,

$$\begin{aligned}\mu_i^{A\oplus B}((u+\varphi(u),v+\varphi(v)) &= (\mu_i^A(u,v)+\mu_i^B(\varphi(u),\varphi(v)))\\ &= (\mu_i^A(u,v)+\varphi(\mu_i^A(u,v))).\end{aligned}$$

Thus the graph Γ_φ is closed under the multiplication $\mu_i^{A\oplus B}$. Furthermore, $\varphi\circ\alpha_1=\alpha_2\circ\varphi$ yields $(\alpha_1\oplus\alpha_2)(u,\varphi(u))=(\alpha_1(u),\alpha_2\circ\varphi(u))=(\alpha_1(u),\varphi\circ\alpha_1(u))$. In the same way, $(\beta_1\oplus\beta_2)(u,\varphi(u))=(\beta_1(u),\beta_2\circ\varphi(u))=(\beta_1(u),\varphi\circ\beta_1(u))$, which implies that Γ_φ is closed under $\alpha_1\oplus\alpha_2$ and $\beta_1\oplus\beta_2$. Thus Γ_φ is a BiHom-X subalgebra of $(A\oplus B,\mu_1^{A\oplus B},\dots,\mu_n^{A\oplus B},\alpha_1+\beta_1,\alpha_2+\beta_2)$.

Conversely, if the graph $\Gamma_\varphi\subset A\oplus B$ is a BiHom-X subalgebra of

$$(A\oplus B,\mu_1^{A\oplus B},\dots,\mu_n^{A\oplus B},\alpha_1+\beta_1,\alpha_2+\beta_2),$$

then for all $1\leqslant i\leqslant n$,

$$\mu_i^{A\oplus B}((u+\varphi(u)),(v+\varphi(v)))=(\mu_i^A(u,v)+\mu_i^B(\varphi(u),\varphi(v)))\in\Gamma_\varphi,$$

which implies that $\mu_i^B(\varphi(u),\varphi(v))=\varphi(\mu_i^A(u,v))$.

Furthermore, $(\alpha_1\oplus\alpha_2)(\Gamma_\varphi)\subset\Gamma_\varphi,\ (\beta_1\oplus\beta_2)(\Gamma_\varphi)\subset\Gamma_\varphi$ implies

$$(\alpha_1\oplus\alpha_2)(u+\varphi(u))=(\alpha_1(u)+\alpha_2\circ\varphi(u))\in\Gamma_\varphi,$$

$$(\beta_1\oplus\beta_2)(u+\varphi(u))=(\beta_1(u)+\beta_2\circ\varphi(u))\in\Gamma_\varphi,$$

equivalent to the conditions $\alpha_2\circ\varphi(u)=\varphi\circ\alpha_1(u)$ and $\beta_2\circ\varphi(u)=\varphi\circ\beta_1(u)$, that is $\alpha_1\circ\varphi=\varphi\circ\alpha_2$ and $\beta_1\circ\varphi=\varphi\circ\beta_2$. Therefore, φ is a morphism of BiHom-X algebras. □

Theorem 1.7. Let $(A_1,\mu_1^{A_1},\dots,\mu_n^{A_1},\alpha_1,\beta_1)$ and $(A_2,\mu_1^{A_2},\dots,\mu_n^{A_2},\alpha_2,\beta_2)$ be some BiHom-X algebras. Then $A=A_1\otimes A_2$ is endowed with a BiHom-X algebra structure for twising maps $\alpha,\beta : A\to A$ and the product $*_i : A\otimes A\to A$ defined for any $a_1,b_1,c_1\in A_1$, $a_2,b_2,c_2\in A_2$ and $1\leqslant i\leqslant n$ by

$$\alpha(a_1\otimes a_2)=\alpha_1(a_1)\otimes\alpha_2(a_2),$$

$$\beta(a_1 \otimes a_2) = \beta_1(a_1) \otimes \beta_2(a_2),$$
$$(a_1 \otimes a_2) *_i (b_1 \otimes b_2) = \mu_i^{A_1}(a_1, b_1) \otimes \mu_i^{A_2}(a_2, b_2).$$

Proof. For any $a_1, b_1, c_1 \in A_1$, $a_2, b_2, c_2 \in A_2$ and $1 \leqslant i, j \leqslant n$,

$$\begin{aligned}
&((a_1 \otimes a_2) *_i (b_1 \otimes b_2)) *_j \beta(c_1 \otimes c_2)\\
&\quad = (\mu_i^{A_1}(a_1, b_1) \otimes \mu_i^{A_2}(a_2, b_2)) *_j \beta_1(c_1) \otimes \beta_2(c_2)\\
&\quad = \mu_j^{A_1}(\mu_i^{A_1}(a_1, b_1), \beta_1(c_1)) \otimes \mu_j^{A_2}(\mu_i^{A_2}(a_2, b_2)), \beta_2(c_2)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\alpha(a_1 \otimes a_2) *_j ((b_1 \otimes b_2) *_i (c_1 \otimes c_2))\\
&\quad = \mu_j^{A_1}(\alpha_1(a_1), \mu_i^{A_1}(b_1, c_1)) \otimes \mu_j^{A_2}(\alpha_2(a_2), \mu_i^{A_2}(b_2, c_2)).
\end{aligned}$$ □

Definition 1.8. Let $(A, \mu_1, \dots, \mu_n, \alpha, \beta)$ be a BiHom-algebra and $k \in \mathbb{N}^*$.

1) The kth derived BiHom-algebra of type 1 of A is defined by

$$A_1^k = (A, \mu_1^{(k)} = \mu_1 \circ (\alpha^k \otimes \beta^k), \dots, \mu_n^{(k)} = \mu_n \circ (\alpha^k \otimes \beta^k), \alpha^{k+1}, \beta^{k+1}).$$

2) The kth derived BiHom-algebra of type 2 of A is defined by

$$A_2^k = \Big(A, \mu_1^{(2^k-1)} = \mu_1 \circ (\alpha^{2^k-1} \otimes \beta^{2^k-1}), \dots, \mu_n^{(2^k-1)} = \mu_n \circ (\alpha^{2^k-1} \otimes \beta^{2^k-1}), \alpha^{2^k}, \beta^{2^k}\Big).$$

3) $A_1^0 = A_2^0 = (A, \mu_1, \dots, \mu_n, \alpha, \beta)$,
$A_1^1 = A_2^1 = (A, \mu_1 \circ (\alpha \otimes \beta), \dots, \mu_n \circ (\alpha \otimes \beta), \alpha^2, \beta^2)$.

Definition 1.9. A BiHom-algebra $(A, \mu_1, \dots, \mu_n, \alpha, \beta)$ endowed with a linear map $R : A \to A$ such that $\alpha \circ R = R \circ \alpha$, $\beta \circ R = R \circ \beta$, and for $x, y \in A$ and $i = 1, \dots, n$,

$$\mu_i(R(x), R(y)) = R\Big(\mu_i(R(x), y) + \mu_i(x, R(y)) + \lambda\mu_i(x, y)\Big), \tag{1}$$

is called a Rota-Baxter BiHom-algebra, and R is called a Rota-Baxter operator of weight $\lambda \in \mathbb{K}$ on A.

The below result allows to get BiHom-X algebras from either a BiHom-X algebra or an X-algebra.

Theorem 1.10. Let $(A, \mu_1, \dots, \mu_n, \alpha, \beta)$ be a Rota-Baxter BiHom-X algebra and let $\alpha', \beta' : A \to A$ be two endomorphisms of A such that any two of the maps $\alpha, \beta, \alpha', \beta'$ commute. Then, for any nonnegative integer p,

$$\begin{aligned} A_{\alpha',\beta'} &= (A, \mu^1_{\alpha',\beta'} = \mu_1 \circ (\alpha'^p \otimes \beta'^p), \dots, \mu^n_{\alpha',\beta'} \\ &= \mu_n \circ (\alpha'^p \otimes \beta'^p), \alpha'^p \circ \alpha, \beta'^p \circ \beta) \end{aligned}$$

is a Rota-Baxter BiHom-X algebra. Moreover, let $(A', \mu'_1, \dots, \mu'_n, \gamma, \delta)$ be another BiHom-X algebra and $\gamma', \delta' : A' \to A'$ be two endomorphisms such that any two of the maps $\gamma, \delta, \gamma', \delta'$ commute. If $f : A \to A'$ is a morphism of BiHom-X algebras that satisfies $f \circ \alpha' = \gamma' \circ f,\ f \circ \beta' = \delta' \circ f$, then $f : A_{\alpha',\beta'} \to A'_{\gamma',\delta'}$ is also a morphism of BiHom-X algebras.

Proof. The proof of the first part follows from the following facts.

For any $x, y, z \in A,\ 1 \leqslant i, j \leqslant n$,

$$\begin{aligned} \mu^i_{\alpha',\beta'}(\mu^j_{\alpha',\beta'}(x,y), (\beta'^p \circ \beta)(z)) &= \mu^i_{\alpha',\beta'}(\mu^j_{\alpha',\beta'}(x,y), \beta'^p(\beta(z))) \\ &= \mu_i\Big(\alpha'^p \mu_j(\alpha'^p(x), \beta'^p(y)), \beta'^{2p}(\beta(z))\Big) \\ &= \mu_i(\mu_j(\alpha'^{2p}(x), \alpha'^p\beta'^p(y)), \beta(\beta'^{2p}(z)) \\ &= \mu_i(\mu_j(X,Y), \beta(Z)), \\ \mu^i_{\alpha',\beta'}(\alpha'^p \circ \alpha(x), \mu^j_{\alpha',\beta'}(y,z)) &= \mu^i_{\alpha',\beta'}(\alpha(\alpha'^p(x)), \mu^j_{\alpha',\beta'}(y,z)) \\ &= \mu_i(\alpha'^{2p} \circ \alpha(x), \beta'^p(\mu_j(\alpha'^p(y), \beta'^p(z))) \\ &= \mu_i(\alpha(\alpha'^{2p}(x)), \beta'^p(\mu_j(\alpha'^p(y), \beta'^p(z))) \\ &= \mu_i(\alpha(\alpha'^{2p}(x)), \mu_j(\alpha'^p\beta'^p(y), \beta'^{2p}(z)) \\ &= \mu_i(\alpha(X), \mu_j(Y,Z)), \end{aligned}$$

where $X = \alpha'^{2p}(x),\ Y = \alpha'^p\beta'^p(y)$ and $Z = \beta'^{2p}(z)$.

The Rota-Baxter identity (1) for $\mu^i_{\alpha',\beta'}$ is proved by

$$\begin{aligned} &\mu^i_{\alpha',\beta'}(R(x), R(y)) = \mu_i(\alpha'^p(R(x)), \beta'^p(R(y))) \\ &\quad = \mu_i(R(\alpha'^p(x)), R(\beta'^p(y))) \\ &\quad = R\big(\mu_i(R(\alpha'^p(x)), \beta'^p(y)) + \mu_i(\alpha'^p(x), R(\beta'^p(y))) + \lambda\mu_i(\alpha'^p(x), \beta'^p(y))\big) \\ &\quad = R\big(\mu^i_{\alpha',\beta'}(R(x), y) + \mu^i_{\alpha',\beta'}(x, R(y)) + \lambda\mu^i_{\alpha',\beta'}(x,y)\big). \end{aligned}$$

The second assertion follows from

$$\begin{aligned} f(\mu^i_{\alpha',\beta'}(x,y)) &= f(\mu_i(\alpha'^p(x), \beta'^p(y))) = \mu'_i(f(\alpha'^p(x)), f(\beta'^p(y))) \\ &= \mu'_i(\gamma'^p(f(x)), \delta'^p(f(y))) = \mu^i_{\gamma',\delta'}(f(x), f(y)). \end{aligned}$$ □

We have the following series of consequence of Theorem 1.10.

Corollary 1.11. Let $(A, \mu_1, \ldots, \mu_n)$ be an X-algebra and $\alpha, \beta : A \to A$ be two endomorphisms of A. Then $A_{\alpha,\beta} = (A, \mu_1 \circ (\alpha \otimes \beta), \ldots, \mu_n(\alpha \otimes \beta), \alpha, \beta)$ is a multiplicative BiHom-X algebra.

Proof. Take $\alpha = \beta = \mathrm{Id}$ and $p = 1$ in Theorem 1.10. □

Corollary 1.12. Let $(A, \mu_1, \ldots, \mu_n, \alpha, \beta)$ be a BiHom-X algebra. Then the kth derived BiHom-algebra of type 1 and the kth derived BiHom-algebra of type 2 are BiHom-X algebras.

Proof. It is sufficient to take $\alpha' = \alpha$, $\beta' = \beta$, and $p = k$ and $p = 2^k - 1$ respectively in Theorem 1.10. □

Now we introduce the notion of centroids for BiHom-X algebras.

Definition 1.13. A Centroid of a BiHom-algebra $(A, \mu_1, \ldots, \mu_n, \alpha, \beta)$ is a linear map $\gamma : A \to A$ such that $\gamma \circ \alpha = \alpha \circ \gamma$, $\gamma \circ \beta = \beta \circ \gamma$ and for any $1 \leqslant i \leqslant n$ and $x, y \in A$,

$$\gamma(\mu_i(x, y)) = \mu_i(\gamma(x), y) = \mu_i(x, \gamma(y)).$$

Theorem 1.14. Let $(A, \mu_1, \ldots, \mu_n, R, \alpha, \beta)$ be a Rota-Baxter BiHom-X algebra, and $\gamma_1, \gamma_2 : A \to A$ be a pair of commuting elements of the centroid such that $\gamma_i \circ R = R \circ \gamma_i, i = 1, 2$. If $\mu_\gamma^i : A \times A \to A, i = 1, \ldots, n$ are bilinear maps defined for any $x, y \in A$ by $\mu_\gamma^i(x, y) = \mu_i(\gamma_2\gamma_1(x), y)$, then $A_{\gamma_1,\gamma_2} = (A, \mu_\gamma^1, \ldots, \mu_\gamma^n, R, \gamma_1\alpha, \gamma_2\beta)$ is a Rota-Baxter BiHom-X algebra.

Proof. For any $x, y \in A$ and $1 \leqslant i, j \leqslant n$,

$$\begin{aligned}
&\mu_\gamma^i(\mu_\gamma^j(x, y), \gamma_2\beta(z)) = \mu_\gamma^i(\mu_j(\gamma_1\gamma_2(x), y), \gamma_2\beta(z)) \\
&\quad = \mu_i(\gamma_1\gamma_2\mu_j(\gamma_1\gamma_2(x), y), \gamma_2\beta(z)) = \gamma_1^2\gamma_2^3\mu_i(\mu_j(x, y), \beta(z)), \\
&\mu_\gamma^i(\gamma_1\alpha(x), \mu_\gamma^j(y, z)) = \mu_\gamma^i(\gamma_1\alpha(x), \mu_j(\gamma_1\gamma_2(y), z)) \\
&\quad = \mu_i(\gamma_1^2\gamma_2\alpha(x), \mu_j(\gamma_1\gamma_2(y), z)) = \gamma_1^2\gamma_2^3\mu_i(\alpha(x), \mu_j(y, z)), \\
&\mu_\gamma^i(R(x), R(y)) = \mu_i(\gamma_1\gamma_2(R(x)), R(y)) = \gamma_1\gamma_2\mu_i(R(x), R(y)) \\
&\quad = \gamma_1\gamma_2 R\Big(\mu_i(R(x), y) + \mu_i(x, R(y)) + \lambda\mu_i(x, y)\Big) \\
&\quad = R\Big(\gamma_1\gamma_2\mu_i(R(x), y) + \gamma_1\gamma_2\mu_i(x, R(y)) + \lambda\gamma_1\gamma_2\mu_i(x, y)\Big) \\
&\quad = R\Big(\mu_i(\gamma_1\gamma_2R(x), y) + \mu_i(\gamma_1\gamma_2(x), R(y)) + \lambda\mu_i(\gamma_1\gamma_2(x), y)\Big) \\
&\quad = R\Big(\mu_\gamma^i(R(x), y) + \mu_\gamma^i(x, R(y)) + \lambda\mu_\gamma^i(x, y)\Big).
\end{aligned}$$

□

2. Bimodules and matched pairs of BiHom-left symmetric and BiHom-associative dialgebras

In this section, we recall definitions and some key results about bimodules of BiHom-associative algebras [18] and BiHom-left-symmetric algebras [11]. Next, we introduce the notions of BiHom-left-symmetric dialgebra and BiHom-associative dialgebra and we give some related relevant properties.

Definition 2.1. A BiHom-module is a triple (V, α_V, β_V) consisting of a $\mathbb{K}$-vector space V and two linear maps $\alpha_V, \beta_V : V \longrightarrow V$ such that $\alpha_V\beta_V = \beta_V\alpha_V$. A morphism $f : (V, \alpha_V, \beta_V) \to (W, \alpha_W, \beta_W)$ of BiHom-modules is a linear map $f : V \longrightarrow W$ such that $f\alpha_V = \alpha_W f$ and $f\beta_V = \beta_W f$.

Definition 2.2. A BiHom-associative algebra is a quadruple (A, μ, α, β) consisting of a vector space A on which the operation $\mu : A \otimes A \to A$ and $\alpha, \beta : A \to A$ are linear maps satisfying, for any $x, y, z \in A$,

$$\alpha \circ \beta = \beta \circ \alpha, \quad (2)$$
$$\alpha \circ \mu(x, y) = \mu(\alpha(x), \alpha(y)), \quad (3)$$
$$\beta \circ \mu(x, y) = \mu(\beta(x), \beta(y)), \quad (4)$$
$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \beta(z)). \quad (5)$$

Remarks 2.3. Clearly, there are the following connections between Hom-associative, BiHom-associative and BiHom-X algebra structures.

1. A Hom-associative algebra (A, μ, α) can be regarded as a BiHom-associative algebra (A, μ, α, α).
2. A BiHom-associative algebra is a BiHom-X algebra.

Example 2.4. Let $\{e_1, e_2\}$ be a basis of a 2-dimensional vector space A over $\mathbb{K}$. The following multiplication μ and maps α, β on A define a BiHom-associative algebra:

$$\begin{aligned} \alpha(e_1) &= 2e_1, & \alpha(e_2) &= -2a_1e_1 + (1-a)e_2, \\ \beta(e_1) &= 2e_1, & \beta(e_2) &= -ae_1 + (1-a)e_2, \\ \mu(e_1, e_1) &= 2e_1, & \mu(e_1, e_2) &= -ae_1 + (1-a)e_2, \\ \mu(e_2, e_1) &= -2ae_1 + (a-1)e_2, & \mu(e_2, e_2) &= 2a^2e_1 + ae_2. \end{aligned}$$

where $a \in \mathbb{K}\backslash\{0\}$.

Definition 2.5. Let $(A, \cdot, \alpha_1, \alpha_2)$ be a BiHom-associative algebra, and let (V, β_1, β_2) be a BiHom-module. Let $l, r : A \to gl(V)$ be two linear maps. Then $(l, r, \beta_1, \beta_2, V)$ is called a bimodule of A if, for all $x, y \in A, v \in V$,

$$\begin{aligned} l(x \cdot y)\beta_2(v) &= l(\alpha_1(x))l(y)v, & r(x \cdot y)\beta_1(v) &= r(\alpha_2(y))r(x)v,\\ l(\alpha_1(x))r(y)v &= r(\alpha_2(y))l(x)v, & \beta_1(l(x)v) &= l(\alpha_1(x))\beta_1(v),\\ \beta_1(r(x)v) &= r(\alpha_1(x))\beta_1(v), & \beta_2(l(x)v) &= l(\alpha_2(x))\beta_2(v),\\ \beta_2(r(x)v) &= r(\alpha_2(x))\beta_2(v). \end{aligned}$$

Proposition 2.6. Let $(l, r, \beta_1, \beta_2, V)$ be a bimodule of a BiHom-associative algebra $(A, \cdot, \alpha_1, \alpha_2)$. Then, the direct sum $A \oplus V$ of vector spaces is a BiHom-associative algebra with multiplication in $A \oplus V$, defined for all $x_1, x_2 \in A, v_1, v_2 \in V$, by

$$\begin{aligned} (x_1 + v_1) * (x_2 + v_2) &= x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1),\\ (\alpha_1 \oplus \beta_1)(x_1 + v_1) &= \alpha_1(x_1) + \beta_1(v_1),\\ (\alpha_2 \oplus \beta_2)(x_1 + v_1) &= \alpha_2(x_1) + \beta_2(v_1). \end{aligned}$$

We denote such BiHom-associative algebra by $(A \oplus V, *, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$, or $A \times_{l,r,\alpha_1,\alpha_2,\beta_1,\beta_2} V$.

Example 2.7. For any BiHom-associative algebra $(A, \cdot, \alpha, \beta)$, the quintuples $(L, 0, \alpha, \beta, A)$, $(0, R, \alpha, \beta, A)$ and (L, R, α, β, A) are bimodules of $(A, \cdot, \alpha, \beta)$, where $L(a)b = a \cdot b$ and $R(a)b = b \cdot a$ for all $a, b \in A$.

Theorem 2.8 ([22])**.** Let $(A, \cdot_A, \alpha_1, \alpha_2)$ and $(B, \cdot_B, \beta_1, \beta_2)$ be two BiHom-associative algebras. Suppose that there are linear maps $l_A, r_A : A \to gl(B)$ and $l_B, r_B : B \to gl(A)$ such that $(l_A, r_A, \beta_1, \beta_2, B)$ is a bimodule of A, $(l_B, r_B, \alpha_1, \alpha_2, A)$ is a bimodule of B, and for any $x, y \in A,\ a, b \in B$,

$$l_A(\alpha_1(x))(a \cdot_B b) = l_A(r_B(a)x)\beta_2(b) + (l_A(x)a) \cdot_B \beta_2(b), \tag{6}$$

$$r_A(\alpha_2(x))(a \cdot_B b) = r_A(l_B(b)x)\beta_1(a) + \beta_1(a) \cdot_B (r_A(x)b), \tag{7}$$

$$\begin{aligned} &l_A(l_B(a)x)\beta_2(b) + (r_A(x)a) \cdot_B \beta_2(b)\\ &\qquad - r_A(r_B(b)x)\beta_1(a) - \beta_1(a) \cdot_B (l_A(x)b) = 0, \end{aligned} \tag{8}$$

$$l_B(\beta_1(a))(x \cdot_A y) = l_B(r_A(x)a)\alpha_2(y) + (l_B(a)x) \cdot_A \alpha_2(y), \tag{9}$$

$$r_B(\beta_2(a))(x \cdot_A y) = r_B(l_A(y)a)\alpha_1(x) + \alpha_1(x) \cdot_A (r_B(a)y), \tag{10}$$

$$\begin{aligned} &l_B(l_A(x)a)\alpha_2(y) + (r_B(a)x) \cdot_A \alpha_2(y)\\ &\qquad - r_B(r_A(y)a)\alpha_1(x) - \alpha_1(x) \cdot_A (l_B(a)y) = 0. \end{aligned} \tag{11}$$

Then $(A, B, l_A, r_A, \beta_1, \beta_2, l_B, r_B, \alpha_1, \alpha_2)$ is called a matched pair of BiHom-associative algebras. In this case, there is a BiHom-associative algebra structure on the direct sum $A \oplus B$ of the underlying vector spaces of A and B given by

$$\begin{aligned}
&(x+a)\cdot(y+b)\\
&\qquad = x \cdot_A y + (l_A(x)b + r_A(y)a) + a \cdot_B b + (l_B(a)y + r_B(b)x),\\
&(\alpha_1 \oplus \beta_1)(x+a) = \alpha_1(x) + \beta_1(a),\\
&(\alpha_2 \oplus \beta_2)(x+a) = \alpha_2(x) + \beta_2(a).
\end{aligned}$$

Proof. For any $x, y, z \in A$ and $a, b, c \in B$,

$$\begin{aligned}
&(\alpha_1+\beta_1)(x+a)\cdot((y+b)\cdot(z+c))\\
&\quad = (\alpha_1(x)+\beta_1(a))(y \cdot_A z + l_B(b)z + r_B(c)y + b\cdot c + l_A(y)c + r_A(z)b)\\
&\quad = \alpha_1(x)\cdot_A (y\cdot_A z) + \alpha_1(x)\cdot_A l_B(b)z + \alpha_1(x)\cdot_A r_B(c)y\\
&\qquad + l_B(\beta_1(a))(y\cdot_A z) + l_B(\beta_1(a))l_B(b)z + l_B(\beta_1(a))r_B(c)y\\
&\qquad + r_B(b\cdot_B c)\alpha_1(x) + r_B(l_A(y)c)\alpha_1(x) + r_B(r_A(z)b)\alpha_1(x)\\
&\qquad + \beta_1(a)\cdot_B (b\cdot_B c) + \beta_1(a)\cdot_B l_A(y)c + \beta_A(\alpha_1(x))l_A(y)c\\
&\qquad + l_A(\alpha_1(x))r_A(z)b + r_A(y\cdot_A z)\beta_1(a) + r_A(l_A(b)z)\beta_1(a)\\
&\qquad + r_A(r_B(c)y)\beta_1(a),\\
&((x+a)\cdot(y+b))\cdot(\alpha_2+\beta_2)(z+c)\\
&\quad = (x\cdot_A y + l_B(a)y + r_B(b)x + a\cdot_B b + l_A(x)b + r_A(y)a)\cdot(\alpha_2(z)+\beta_2(c))\\
&\quad = (x\cdot_A y)\cdot_A \alpha_2(z) + l_B(a)y\cdot_A \alpha_2(z) + r_B(b)x\cdot_A \alpha_2(z) + l_B(a\cdot_B b)\alpha_2(z)\\
&+ l_B(l_A(x)b)\alpha_2(z) + l_B(r_A(y)a)\alpha_2(z) + r_B(\beta_2(c))(x\cdot_A y)\\
&+ r_A(\beta_2(c))l_B(a)y + r_B(\beta_2(c))r_B(b)x + (a\cdot_B b)\cdot_B \beta_2(c)\\
&+ (l_A(x)b)\cdot_B \beta_2(c) + (r_A(y)a)\cdot_B \beta_2(c) + r_A(\alpha_2(z))(a\cdot_B b)\\
&+ r_A(\alpha_2(z))(l_A(x)b) + r_A(\alpha_2(z))(r_A(y)a) + l_A(x\cdot_A y)\beta_2(c)\\
&+ l_A(l_B(a)y)\beta_2(c) + (r_B(b)x)\beta_2(c).
\end{aligned}$$

Then (5) and (6)–(11) yield

$$(\alpha_1+\beta_1)(x+a)\cdot((y+b)\cdot(z+c)) = ((x+a)\cdot(y+b))\cdot(\alpha_2+\beta_2)(z+c). \quad \square$$

We denote this BiHom-associative algebra by $A \bowtie^{l_A, r_A, \beta_1, \beta_2}_{l_B, r_B, \alpha_1, \alpha_2} B$.

Definition 2.9. A BiHom-left-symmetric algebra is a quadruple $(S, *, \alpha, \beta)$ consisting of a vector space S on which the operation $* : S \otimes S \to S$ and $\alpha, \beta : S \to S$ are linear maps satisfying, for all $x, y, z \in S$,

$$\alpha \circ \beta = \beta \circ \alpha,$$

$$\alpha(x * y) = \alpha(x) * \alpha(y),$$
$$\beta(x * y) = \beta(x) * \beta(y),$$
$$(\beta(x) * \alpha(y)) * \beta(z) - \alpha\beta(x) * (\alpha(y) * z)$$
$$= (\beta(y) * \alpha(x)) * \beta(z) - \alpha\beta(y) * (\alpha(x) * z).$$

Definition 2.10. Let $(S, *, \alpha_1, \alpha_2)$ be a BiHom-left-symmetric algebra, and (V, β_1, β_2) be a BiHom-module. Let $l, r : S \to gl(V)$ be two linear maps. The quintuple $(l, r, \beta_1, \beta_2, V)$ is called a bimodule of S if for all $x, y \in S, v \in V$,

$$l(\alpha_2(x) * \alpha_1(y))\beta_2(v) - l(\alpha_1\alpha_2(x))l(\alpha_1(y))v$$
$$= l(\alpha_2(y) * \alpha_1(x))\beta_2(v) - l(\alpha_1\alpha_2(y))l(\alpha_1(x))v,$$
$$r(\alpha_2(x))r(\alpha_1(y))\beta_2(v) - r(\alpha_1(y) * x)\alpha_1\alpha_2(v)$$
$$= r(\alpha_2(x))l(\alpha_2(y))\beta_1(v) - l(\alpha_1\alpha_2(y))r(x)\beta_1(v),$$
$$\beta_1(l(x)v) = l(\alpha_1(x))\beta_1(v), \quad \beta_1(r(x)v) = r(\alpha_1(x))\beta_1(v),$$
$$\beta_2(l(x)v) = l(\alpha_2(x))\beta_2(v), \quad \beta_2(r(x)v) = r(\alpha_2(x))\beta_2(v).$$

Proposition 2.11. Let $(l, r, \beta_1, \beta_2, V)$ be a bimodule of a BiHom-left-symmetric algebra $(S, *, \alpha_1, \alpha_2)$. Then, the direct sum $S \oplus V$ of vector spaces is turned into a BiHom-left-symmetric algebra by defining multiplication in $S \oplus V$ for all $x_1, x_2 \in S, v_1, v_2 \in V$ by

$$(x_1 + v_1) *' (x_2 + v_2) = x_1 * x_2 + (l(x_1)v_2 + r(x_2)v_1),$$
$$(\alpha_1 \oplus \beta_1)(x_1 + v_1) = \alpha_1(x_1) + \beta_1(v_1),$$
$$(\alpha_2 \oplus \beta_2)(x_1 + v_1) = \alpha_2(x_1) + \beta_2(v_1).$$

By $(S \oplus V, *, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$, or $S \times_{l,r,\alpha_1,\alpha_2,\beta_1,\beta_2} V$ we denote such a biHom-left-symmetric algebra.

Example 2.12. Let $(S, *, \alpha, \beta)$ be a BiHom-left-symmetric algebra. Then $(L, 0, \alpha, \beta, S)$, $(0, R, \alpha, \beta, S)$ and (L, R, α, β, S) are bimodules of $(S, *, \alpha, \beta)$, where $L(a)b = a * b$ and $R(a)b = b * a$ for all $a, b \in S$.

Theorem 2.13 ([25]). Let $(A, *_A, \alpha_1, \alpha_2)$ and $(B, *_B, \beta_1, \beta_2)$ be BiHom-left-symmetric algebras. Suppose that there are linear maps $l_{*_A}, r_{*_A} : A \to gl(B)$ and $l_{*_B}, r_{*_B} : B \to gl(A)$ such that

$$(l_{*_A}, r_{*_A}, \beta_1, \beta_2, B) \text{ is a bimodule of } A,$$
$$(l_{*_B}, r_{*_B}, \alpha_1, \alpha_2, A) \text{ is a bimodule of } B$$

and, for any $x, y \in A,\ a, b \in B$ with

$$\begin{aligned}\{\alpha_2(x), \alpha_1(y)\}_A &= \alpha_2(x) *_A \alpha_1(y) - \alpha_2(y) *_A \alpha_1(x),\\ (\rho_A \circ \alpha_2)\beta_1 &= (l_{*_A} \circ \alpha_2)\beta_1 - (r_{*_A} \circ \alpha_1)\beta_2,\\ \{\beta_2(a), \beta_1(b)\}_B &= \beta_2(a) *_B \beta_1(b) - \beta_2(b) *_B \beta_1(a),\\ (\rho_B \circ \beta_2)\alpha_1 &= (l_{*_B} \circ \beta_2)\alpha_1 - (r_{*_B} \circ \beta_1)\alpha_2,\end{aligned}$$

the following equalities hold:

$$\begin{aligned}&r_{*_A}(\alpha_2(x))\{\beta_2(a), \beta_1(b)\}_B = r_{*_A}(l_{*_B}(\beta_1(b))x)\beta_1\beta_2(a)\\ &\quad - r_{*_A}(l_{*_B}(\beta_1(a)x)\beta_1\beta_2(b) + \beta_1\beta_2(a) *_B r_{*_A}(x)\beta_1(b)\\ &\quad - \beta_1\beta_2(b) *_B r_{*_A}(x)\beta_1(a),\\ &l_{*_A}(\alpha_1\alpha_2(x))(\beta_1(a) *_B b) = (\rho_A(\alpha_2(x))\beta_1(a)) *_B \beta_2(b)\\ &\quad - l_{*_A}(\rho_B(\beta_2(a))\alpha_1(x))\beta_2(b) + \beta_1\beta_2(a) *_B (l_{*_A}(\alpha_1(x))b)\\ &\quad + r_{*_A}(r_{*_B}(b)\alpha_1(x))\beta_1\beta_2(a),\\ &r_{*_B}(\beta_2(a))\{\alpha_2(x), \alpha_1(y)\}_A = r_{*_B}(l_{*_A}(\alpha_1(y))a)\alpha_1\alpha_2(x)\\ &\quad - r_{*_B}(l_{*_A}(\alpha_1(x)a)\alpha_1\alpha_2(y) + \alpha_1\alpha_2(x) *_A r_{*_B}(a)\alpha_1(y)\\ &\quad - \alpha_1\alpha_2(y) *_A r_{*_B}(a)\alpha_1(x),\\ &l_{*_B}(\beta_1\beta_2(a))(\alpha_1(x) *_A y) = (\rho_B(\beta_2(a))\alpha_1(x)) *_A \alpha_2(y)\\ &\quad - l_{*_B}(\rho_A(\alpha_2(x))\beta_1(a))\alpha_2(y) + \alpha_1\alpha_2(x) *_A (l_{*_B}(\beta_1(a))y)\\ &\quad + r_{*_B}(r_{*_A}(y)\beta_1(a))\alpha_1\alpha_2(x).\end{aligned}$$

Then for $(A, B, l_{*_A}, r_{*_A}, \beta_1, \beta_2, l_{*_B}, r_{*_B}, \alpha_1, \alpha_2)$, called a matched pair of BiHom-left-symmetric algebras, there exists a BiHom-left-symmetric algebra structure on the vector space $A \oplus B$ of the underlying vector spaces of A and B given by

$$\begin{aligned}(x + a) * (y + b) &= x *_A y + (l_{*_A}(x)b + r_{*_A}(y)a)\\ &\qquad + a *_B b + (l_{*_B}(a)y + r_{*_B}(b)x),\\ (\alpha_1 \oplus \beta_1)(x + a) &= \alpha_1(x) + \beta_1(a),\\ (\alpha_2 \oplus \beta_2)(x + a) &= \alpha_2(x) + \beta_2(a).\end{aligned}$$

Proof. The proof is obtained in a similar way as for Theorem 2.8. □

We denote this BiHom-left-symmetric algebra by

$$A \bowtie^{l_{*_A}, r_{*_A}, \beta_1, \beta_2}_{l_{*_B}, r_{*_B}, \alpha_1, \alpha_2} B.$$

2.1. BiHom-left-symmetric dialgebra

Definition 2.14. A BiHom-left-symmetric dialgebra is a linear space S equipped with two bilinear products $\dashv, \vdash : S \times S \to S$ and two linear maps $\alpha, \beta : S \to S$ satisfying, for all $x, y, z \in S$,

$$\alpha \circ \beta = \beta \circ \alpha, \tag{12}$$

$$\alpha(x \dashv y) = \alpha(x) \dashv \alpha(y), \alpha(x \vdash y) = \alpha(x) \vdash \alpha(y), \tag{13}$$

$$\beta(x \dashv y) = \beta(x) \dashv \beta(y), \beta(x \vdash y) = \beta(x) \vdash \beta(y), \tag{14}$$

$$\alpha(x) \dashv (y \dashv z) = \alpha(x) \dashv (y \vdash z), \tag{15}$$

$$(x \vdash y) \vdash \beta(z) = (x \dashv y) \vdash \beta(z), \tag{16}$$

$$\begin{aligned} \alpha\beta(x) \dashv (\alpha(y) \dashv z) - (\beta(x) \dashv \alpha(y)) \dashv \beta(z) \\ = \alpha\beta(y) \vdash (\alpha(x) \dashv z) - (\beta(y) \vdash \alpha(x)) \dashv \beta(z), \end{aligned} \tag{17}$$

$$\begin{aligned} \alpha\beta(x) \vdash (\alpha(y) \vdash z) - (\beta(x) \vdash \alpha(y)) \vdash \beta(z) \\ = \alpha\beta(y) \vdash (\alpha(x) \vdash z) - (\beta(y) \vdash \alpha(x)) \vdash \beta(z). \end{aligned} \tag{18}$$

Example 2.15. Any BiHom-associative algebra (A, μ, α, β) is a BiHom-left-symmetric dialgebra with $\dashv = \vdash = \mu$.

Remark 2.16. Relation (18) means that $(S, \vdash, \alpha, \beta)$ is a BiHom-left-symmetric algebra. So, any BiHom-left-symmetric dialgebra is a BiHom-left-symmetric algebra.

Theorem 2.17. Given two BiHom-left-symmetric dialgebras $(S_1, \dashv_1, \vdash_1, \alpha_1, \beta_1)$ and $(S_2, \dashv_2, \vdash_2, \alpha_2, \beta_2)$, there is a BiHom-left-symmetric dialgebra $(S_1 \oplus S_2, \dashv = \dashv_1 + \dashv_2, \vdash = \vdash_1 + \vdash_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)$, where the bilinear maps $\dashv, \vdash : (S_1 \oplus S_2)^{\times 2} \longrightarrow (S_1 \oplus S_2)$ are given for all $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$ by

$$\begin{aligned} (a_1 + b_1) \dashv (a_2 + b_2) = a_1 \dashv_1 a_2 + b_1 \dashv_2 b_2, \\ (a_1 + b_1) \vdash (a_2 + b_2) = a_1 \vdash_1 a_2 + b_1 \vdash_2 b_2, \end{aligned}$$

and the linear maps $\alpha_1 + \alpha_2,\ \beta_1 + \beta_2 : (S_1 \oplus S_2) \longrightarrow (S_1 \oplus S_2)$ are given for all $(a, b) \in S_1 \times S_2$ by

$$\begin{aligned} (\alpha_1 + \alpha_2)(a + b) = \alpha_1(a) + \alpha_2(b), \\ (\beta_1 + \beta_2)(a + b) = \beta_1(a) + \beta_2(b). \end{aligned}$$

Proof. We prove only the axiom (15), as others are proved similarly. For any $a_1, b_1, c_1 \in S_1$ and $a_2, b_2, c_2 \in S_2$,

$$(\alpha_1 + \alpha_2)(a_1 + a_2) \dashv ((b_1 + b_2) \vdash (c_1 + c_2))$$

$$= (\alpha_1(a_1) + \alpha_2(a_2)) \dashv ((b_1 + b_2) \vdash (c_1 + c_2))$$
$$= \alpha_1(a_1) \dashv_1 (b_1 \vdash_1 c_1) + \alpha_2(a_2) \dashv_2 (b_2 \vdash_2 c_2)$$
$$= \alpha_1(a_1) \dashv_1 (b_1 \dashv_1 c_1) + \alpha_2(a_2) \dashv_2 (b_2 \dashv_2 c_2)$$
$$= (\alpha_1(a_1) + \alpha_2(a_2)) \dashv ((b_1 + b_2) \dashv (c_1 + c_2))$$
$$= (\alpha_1 + \alpha_2)(a_1 + a_2) \dashv ((b_1 + b_2) \dashv (c_1 + c_2)). \quad \square$$

Proposition 2.18. If $(S, \dashv, \vdash, \alpha, \beta)$ is a BiHom-left-symmetric dialgebra, and $\alpha^2 = \beta^2 = \alpha \circ \beta = \beta \circ \alpha = id$, then $(S, \dashv, \vdash, \alpha, \beta) \cong (S, \dashv, \vdash, \beta, \alpha)$.

Proof. We prove only one axiom, as others are proved similarly. For any $x, y, z \in S$,

$$\begin{aligned} \alpha(x) \dashv (y \dashv z) &= \alpha(x) \dashv (y \vdash z) \Leftrightarrow \\ \alpha(\alpha\beta(x)) \dashv (y \dashv z) &= \alpha(\alpha\beta(x)) \dashv (y \vdash z) \Leftrightarrow \\ \alpha^2\beta(x) \dashv (y \dashv z) &= \alpha^2\beta(x) \dashv (y \vdash z) \Leftrightarrow \\ \beta(x) \dashv (y \dashv z) &= \beta(x) \dashv (y \vdash z). \end{aligned}$$

Then $(S, \dashv, \vdash, \alpha, \beta) \cong (S, \dashv, \vdash, \beta, \alpha)$. $\square$

Theorem 2.19. Let $(S, \dashv, \vdash, \alpha, \beta)$ be a BiHom-left-symmetric dialgebra and $\alpha'\beta' : S \to S$ be two endomorphisms of S such that any two of the maps $\alpha, \beta, \alpha', \beta'$ commute. Then, $S_{\alpha',\beta'} = (S, \dashv_{\alpha',\beta'} = \dashv \circ(\alpha' \otimes \beta'), \vdash_{\alpha',\beta'} = \vdash \circ(\alpha' \otimes \beta'), \alpha' \circ \alpha, \beta' \circ \beta)$ is a BiHom-left-symmetric dialgebra.

Moreover, suppose that $(S', \dashv', \vdash', \gamma, \delta)$ is another BiHom-left-symmetric dialgebra and $\gamma', \delta' : S' \to S'$ be two endomorphisms of S' such that any two of the maps $\gamma, \delta, \gamma', \delta'$ commute. If $f : S \to S'$ is a morphism of BiHom-left-symmetric dialgebras such that $f \circ \alpha' = \gamma' \circ f,\ f \circ \beta' = \delta' \circ f$, then $f : S_{\alpha',\beta'} \to S'_{\gamma',\delta'}$ is a morphism of BiHom-left-symmetric dialgebras.

Proof. We only prove (15) in $S_{\alpha',\beta'}$, since the other axioms are proved analogously. For any $x, y, z \in S$,

$$\alpha\alpha'(x) \dashv_{\alpha',\beta'} (y \dashv_{\alpha',\beta'} z) = \alpha\alpha(x) \dashv_{\alpha',\beta'} (\alpha'^2(y) \dashv \beta'(z))$$
$$= \alpha\alpha'^2(x) \dashv (\alpha'\beta'(y) \dashv \beta'^2(z))$$
$$= \alpha\alpha'^2(x) \dashv (\alpha'\beta'(y) \vdash \beta'^2(z)) \quad (\textit{by } (15) \textit{ in } S)$$
$$= \alpha\alpha'(x) \dashv_{\alpha',\beta'} (\alpha'(y) \vdash \beta'(z))$$
$$= \alpha\alpha'(x) \dashv_{\alpha',\beta'} (y \vdash_{\alpha',\beta'} z).$$

For the second assertion, for any $x, y \in S$,

$$\begin{aligned} f(x \dashv_{\alpha',\beta'} y) &= f(\alpha'(x) \dashv \beta'(y)) = f(\alpha'(x)) \dashv' f(\beta'(y)) \\ &= \gamma'(f(x)) \dashv' \delta'(f(y)) = f(x) \dashv'_{\gamma',\delta'} f(y). \end{aligned}$$

Similarly, $f(x \vdash_{\alpha',\beta'} y) = f(x) \vdash'_{\gamma',\delta'} f(y)$. □

Definition 2.20. Let $(S, \dashv, \vdash, \alpha_1, \alpha_2)$ be a BiHom-left-symmetric dialgebra, and V be a vector space. Let $l_\dashv, r_\dashv, l_\vdash, r_\vdash : S \to gl(V)$, and $\beta_1, \beta_2 : V \to V$. Then, $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, \beta_1, \beta_2, V)$ is called a bimodule of S if the following equations hold for any $x, y \in S$ and $v \in V$:

$$\begin{aligned}
&l_\dashv(\alpha_1(x))l_\dashv(y)v = l_\dashv(\alpha_1(x))l_\vdash(y)v,\\
&r_\dashv(\alpha_1(x) \dashv y)\beta_1(v) = r \dashv (x \vdash y)\beta_1(v),\\
&l_\dashv(\alpha_1(x))r_\dashv(y)v = l_\dashv(\alpha_1(x))r_\vdash(y)v,\\
&l_\vdash(x \vdash y)\beta_2(v) = l_\vdash(x \dashv y)\beta_2(v),\\
&r_\vdash(\alpha_2(x))r_\vdash(y)v = r_\vdash(\alpha_2(x))r_\dashv(y)v,\\
&r_\dashv(\alpha_2(x))l_\vdash(y)v = r_\vdash(\alpha_2(x))l_\dashv(y)v,\\
&l_\dashv(\alpha_1\alpha_2(x))l_\dashv(\alpha_1(y))v - l_\dashv(\alpha_2(x) \dashv \alpha_1(y))\beta_2(v)\\
&\qquad = l_\vdash(\alpha_1\alpha_2(y))l_\dashv(\alpha_1(x))v - l_\dashv(\alpha_2(y) \vdash \alpha_1(x))\beta_2(v),\\
&r_\dashv(\alpha_1(x) \dashv y)\beta_1\beta_2(v) - r_\dashv(\alpha_2(y))r_\dashv(\alpha_1(x))\beta_2(v)\\
&\qquad = l_\vdash(\alpha_1\alpha_2(x))r_\dashv(y)\beta_1(v) - r_\dashv(\alpha_2(x))l_\vdash(\alpha_2(y))\beta_1(v),\\
&l_\vdash(\alpha_1\alpha_2(x))l_\vdash(\alpha_1(y))v - l_\vdash(\alpha_2(x) \vdash \alpha_1(y))\beta_2(v)\\
&\qquad = l_\vdash(\alpha_1\alpha_2(y))l_\vdash(\alpha_1(x))v - l_\vdash(\alpha_2(y) \vdash \alpha_1(x))\beta_2(v),\\
&r_\vdash(\alpha_1(x) \vdash y)\beta_1\beta_2(v) - r_\vdash(\alpha_2(y))r_\vdash(\alpha_1(x))\beta_2(v)\\
&\qquad = l_\vdash(\alpha_1\alpha_2(x))r_\vdash(y)\beta_1(v) - r_\vdash(\alpha_2(x))l_\vdash(\alpha_2(y))\beta_1(v),\\
&\beta_1(l_\vdash(x)v) = l_\vdash(\alpha_1(x))\beta_1(v), \beta_1(r_\vdash(x)v) = r_\vdash(\alpha_1(x))\beta_1(v),\\
&\beta_2(l_\vdash(x)v) = l_\vdash(\alpha_2(x))\beta_2(v), \beta_2(r_\vdash(x)v) = r_\vdash(\alpha_2(x))\beta_2(v),\\
&\beta_1(l_\dashv(x)v) = l_\dashv(\alpha_1(x))\beta_1(v), \beta_1(r_\dashv(x)v) = r_\dashv(\alpha_1(x))\beta_1(v),\\
&\beta_2(l_\dashv(x)v) = l_\dashv(\alpha_2(x))\beta_2(v), \beta_2(r_\dashv(x)v) = r_\dashv(\alpha_2(x))\beta_2(v).
\end{aligned}$$

Proposition 2.21. If $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, \beta_1, \beta_2, V)$ is a bimodule of a BiHom-left-symmetric dialgebra $(S, \dashv, \vdash, \alpha_1, \alpha_2)$, then there exists a BiHom-left-symmetric dialgebra structure on the direct sum $S \oplus V$ of the underlying vector spaces of S and V given for all $x, y \in S, u, v \in V$ by

$$\begin{aligned}
(x+u) \dashv' (y+v) &= x \dashv y + l_\dashv(x)v + r_\dashv(y)u,\\
(x+u) \vdash' (y+v) &= x \vdash y + l_\vdash(x)v + r_\vdash(y)u,\\
(\alpha_1+\beta_1)(x+u) &= \alpha_1(x) + \beta_1(u),\\
(\alpha_2+\beta_2)(x+u) &= \alpha_2(x) + \beta_2(u).
\end{aligned}$$

We denote this structure by $S \times_{l_\dashv, r_\dashv, l_\vdash, r_\vdash, \alpha_1, \alpha_2, \beta_1, \beta_2} V$.

Proof. We prove only the axiom (15), as the others are proved similarly. For any $x_1, x_2, x_3 \in S$ and $v_1, v_2, v_3 \in V$,

$$\begin{aligned}
&(\alpha_1 + \beta_1)(x_1 + v_1) \dashv' ((x_2 + v_2) \dashv' (x_3 + v_3)) \\
&\quad = (\alpha_1(x_1) + \beta_1(v_1)) \dashv' (x_1 \dashv x_3 + l_\dashv(x_2)v_3 + r_\dashv(x_3)v_2) \\
&\quad = \alpha_1(x_1) \dashv (x_2 \dashv x_3) + l_\dashv(\alpha_1(x_1))l_\dashv(x_2)v_3 \\
&\qquad + l_\dashv(\alpha_1(x_1))r_\dashv(x_3)v_2 + r_\dashv(x_2 \dashv x_3)\beta_1(v_1) \\
&\quad = \alpha_1(x_1) \dashv (x_2 \vdash x_3) + l_\dashv(\alpha_1(x_1))l_\vdash(x_2)v_3 \\
&\qquad + l_\dashv(\alpha_1(x_1))r_\vdash(x_3)v_2 + r_\dashv(x_2 \vdash x_3)\beta_1(v) \\
&\quad = (\alpha_1 + \beta_1)(x_1 + v_1) \dashv' ((x_2 + v_2) \vdash' (x_3 + v_3)).
\end{aligned}$$
□

Examples 2.22. 1. Let $(S, \dashv, \vdash, \alpha, \beta)$ be a BiHom-left-symmetric dialgebra. Then $(L_\dashv, R_\dashv, L_\vdash, R_\vdash, \alpha, \beta, S)$ and $(L_\dashv, 0, 0, R_\vdash, \alpha, \beta, S)$ are a bimodules of $(S, \dashv, \vdash, \alpha, \beta)$ where $L_\dashv(a)b = a \dashv b$, $R_\dashv(a)b = b \dashv a$, $L_\vdash(a)b = a \vdash b$ and $R_\vdash(a)b = b \vdash a$ for all $(a, b) \in S^2$. More generally, if B is a two-sided BiHom-ideal of $(S, \dashv, \vdash, \alpha, \beta)$, then $(L_\dashv, R_\dashv, L_\vdash, R_\vdash, \alpha, \beta, B)$ is a bimodule of S, where for all $x \in B$ and $(a, b) \in S^2$,

$$\begin{aligned}
L_\dashv(a)x = a \dashv x = x \dashv a = R_\dashv(a)x, \\
L_\vdash(a)x = a \vdash x = x \vdash a = R_\vdash(a)x.
\end{aligned}$$

2. If $(S, \dashv, \vdash)$ is a left-symmetric dialgebra and $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, V)$ are a bimodules of S in the usual sense, then $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, Id_V, Id_V, V)$ is a bimodule of $\mathbb{S}$, where $\mathbb{S} = (S, \dashv, \vdash, Id_S, Id_S)$ is a BiHom-left-symmetric dialgebra.

Proposition 2.23. If $f : (S, \dashv_1, \vdash_1, \alpha_1, \alpha_2) \longrightarrow (S', \dashv_2, \vdash_2, \beta_1, \beta_2)$ is a morphism of BiHom-left-symmetric dialgebras, then $(l_{\dashv_1}, r_{\dashv_1}, l_{\vdash_1}, r_{\vdash_1}, \beta_1, \beta_2, S')$ becomes a bimodule of S via f, that is, $l_{\dashv_1}(a)b = f(a) \dashv_2 b$, $r_{\dashv_1}(a)b = b \dashv_2 f(a)$, $l_{\vdash_1}(a)b = f(a) \vdash_2 b$ and $r_{\vdash_1}(a)b = b \vdash_2 f(a)$ for all $(a, b) \in S \times S'$.

Proof. We prove only first axiom, as the other axioms are proved similarly. For any $x, y \in S, z \in S'$,

$$\begin{aligned}
l_{\dashv_1}(\alpha_1(x))l_{\vdash_1}(y)z &= f(\alpha_1(x)) \dashv_2 (f(y) \vdash_2 z) \\
&= \beta_1 f(x) \dashv_2 (f(y) \vdash_2 z) \\
&= \beta_1 f(x) \dashv_2 (f(y) \dashv_2 z) \quad \text{(by (15))} \\
&= f(\alpha_1(x)) \dashv_2 (f(y) \dashv_2 z) = l_{\dashv_1}(\alpha_1(x))l_{\dashv_1}(y)z.
\end{aligned}$$
□

Definition 2.24. An abelian extension of BiHom-left-symmetric dialgebra is a short exact sequence of BiHom-left-symmetric dialgebras

$$0 \longrightarrow (V, \alpha_V, \beta_V) \xrightarrow{\text{i}} (A, \dashv_A, \vdash_A, \alpha_A, \beta_A) \xrightarrow{\pi} (B, \dashv_B, \vdash_B, \alpha_B, \beta_B) \longrightarrow 0,$$

where (V, α_V, β_V) is a trivial BiHom-left-symmetric dialgebra, and i and π are morphisms of BiHom-left-symmetric dialgebras. Furthermore, if there exists a morphism $s : (B, \dashv_B, \vdash_B, \alpha_B, \beta_B) \longrightarrow (A, \dashv_A, \vdash_A, \alpha_A, \beta_A)$ such that $\pi \circ s = id_B$ then the abelian extension is said to be split and s is called a section of π.

Remark 2.25. Consider the split null extension $S \oplus V$ determined by the bimodule $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, \alpha_V, \beta_V, V)$ of the BiHom-left-symmetric dialgebra $(S, \dashv, \vdash, \alpha, \beta)$ in the previous proposition. Write elements $a + v$ of $S \oplus V$ as (a, v). Then there is an injective homomorphism of BiHom-modules $i : V \to S \oplus V$ given by $i(v) = (0, v)$ and a surjective homomorphism of BiHom-modules $\pi : S \oplus V \to S$ given by $\pi(a, v) = a$. Moreover, $i(V)$ is a two-sided BiHom-ideal of $S \oplus V$ such that $S \oplus V/i(V) \cong S$. On the other hand, there is a morphism of BiHom-left-symmetric dialgebra $\sigma : S \to S \oplus V$ given by $\sigma(a) = (a, 0)$ which is clearly a section of π. Hence, we obtain the abelian split exact sequence of BiHom-left-symmetric dialgebra and $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, \alpha_V, \beta_V, V)$ is a bimodule for S via π.

Theorem 2.26. Let $(S, \dashv, \vdash, \alpha_1, \alpha_2)$ be a BiHom-left-symmetric dialgebra, and let $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, \beta_1, \beta_2, V)$ be a bimodule of S. Let α'_1, α'_2 be endomorphisms of S such that any two of the maps $\alpha_1, \alpha'_1, \alpha_2, \alpha'_2$ commute and $\beta'_1,\ \beta'_2$ be linear self-maps of V such that any two of the maps $\beta_1, \beta'_1, \beta_2, \beta'_2$ commute. Suppose furthermore that

$$\begin{cases} \beta'_1 \circ l_\dashv = (l_\dashv \circ \alpha'_1)\beta'_1, & \beta'_2 \circ l_\dashv = (l_\dashv \circ \alpha'_2)\beta'_2, \\ \beta'_1 \circ l_\vdash = (l_\vdash \circ \alpha'_1)\beta'_1, & \beta'_2 \circ l_\vdash = (l_\vdash \circ \alpha'_2)\beta'_2, \end{cases}$$

$$\begin{cases} \beta'_1 \circ r_\dashv = (r_\dashv \circ \alpha'_1)\beta'_1, & \beta'_2 \circ r_\dashv = (r_\dashv \circ \alpha'_2)\beta'_2, \\ \beta'_1 \circ r_\vdash = (r_\vdash \circ \alpha'_1)\beta'_1, & \beta'_2 \circ r_\vdash = (r_\vdash \circ \alpha'_2)\beta'_2. \end{cases}$$

For BiHom-left-symmetric dialgebra $S_{\alpha'_1,\alpha'_2} = (S, \dashv_{\alpha'_1,\alpha'_2}, \vdash_{\alpha'_1,\alpha'_2}, \alpha_1\alpha'_1, \alpha_2\alpha'_2)$ and $V_{\beta'_1,\beta'_2} = (\widetilde{l}_\dashv, \widetilde{r}_\dashv, \widetilde{l}_\vdash, \widetilde{r}_\vdash, \beta_1\beta'_1, \beta_2\beta'_2, V)$, where

$$\begin{aligned} \widetilde{l}_\dashv &= (l_\dashv \circ \alpha'_1)\beta'_2, & \widetilde{r}_\dashv &= (r_\dashv \circ \alpha'_2)\beta'_1, \\ \widetilde{l}_\vdash &= (l_\vdash \circ \alpha'_1)\beta'_2, & \widetilde{r}_\vdash &= (r_\vdash \circ \alpha'_2)\beta'_1, \end{aligned} \tag{19}$$

$V_{\beta'_1,\beta'_2}$ is a bimodule of $S_{\alpha'_1,\alpha'_2}$.

Proof. We prove only one axiom, as others are proved similarly. For any $x, y \in S, v \in V$,

$$\begin{aligned}\widetilde{l}_{\dashv}(\alpha_1\alpha_1'(x))\widetilde{l}_{\dashv}(y)v &= l_{\dashv}(\alpha_1\alpha_1'^2(x))\beta_2'(l_{\dashv}(\alpha_1'(y))\beta_2'(v)\\ &= l_{\dashv}(\alpha_1\alpha_1'^2(x))l_{\dashv}(\alpha_1'\alpha_2'(y))\beta_2'^2(v)\\ &= l_{\dashv}(\alpha_1\alpha_1'^2(x))l_{\vdash}(\alpha_1'\alpha_2'(y))\beta_2'^2(v) = \widetilde{l}_{\dashv}(\alpha_1\alpha_1'(x))\widetilde{l}_{\vdash}(y)v. \square\end{aligned}$$

Corollary 2.27. Let $(S, \dashv, \vdash, \alpha_1, \alpha_2)$ be a BiHom-left-symmetric dialgebra, and let $(l_{\dashv}, r_{\dashv}, l_{\vdash}, r_{\vdash}, \beta_1, \beta_2, V)$ be a bimodule of S. Then $V_{\beta_1^q,\beta_2^q}$ is a bimodule of $S_{\alpha_1^p,\alpha_2^p}$ for any nonnegative integers p and q.

Proof. Apply Theorem 2.26 with $\alpha_1' = \alpha_1^p$, $\alpha_2' = \alpha_2^p$, $\beta_1' = \beta_1^q$, $\beta_2' = \beta_2^q$. $\square$

Assume that $(l_{\dashv}, r_{\dashv}, l_{\vdash}, r_{\vdash}, \beta_1, \beta_2, V)$ be a bimodule of a BiHom-left-symmetric dialgebra $(S, \dashv, \vdash, \alpha_1, \alpha_2)$ and let $l_{\dashv}^*, r_{\dashv}^*, l_{\vdash}^*, r_{\vdash}^* : S \to gl(V^*)$. Let $\alpha_1^*, \alpha_2^* : S^* \to S^*$, and $\beta_1^*, \beta_2^* : V^* \to V^*$ be the dual maps of respectively $\alpha_1, \alpha_2, \beta_1$ and β_2 such that

$$\begin{aligned}&\langle l_{\dashv}^*(x)u^*, v\rangle = \langle u^*, l_{\dashv}(x)v\rangle, && \langle r_{\dashv}^*(x)u^*, v\rangle = \langle u^*, r_{\dashv}(x)v\rangle,\\ &\langle l_{\vdash}^*(x)u^*, v\rangle = \langle u^*, l_{\vdash}(x)v\rangle, && \langle r_{\vdash}^*(x)u^*, v\rangle = \langle u^*, r_{\vdash}(x)v\rangle,\\ &\alpha_1^*(x^*(y)) = x^*(\alpha_1(y)), && \alpha_2^*(x^*(y)) = x^*(\alpha_2(y)),\\ &\beta_1^*(u^*(v)) = u^*(\beta_1(v)), && \beta_2^*(u^*(v)) = u^*(\beta_2(v)).\end{aligned}$$

The following proposition holds.

Proposition 2.28. If $(l_{\dashv}, r_{\dashv}, l_{\vdash}, r_{\vdash}, \beta_1, \beta_2, V)$ is a bimodule of a BiHom-left-symmetric dialgebra $(S, \dashv, \vdash, \alpha_1, \alpha_2)$, then $(l_{\dashv}^*, r_{\dashv}^*, l_{\vdash}^*, r_{\vdash}^*, \beta_1^*, \beta_2^*, V^*)$ is a bimodule of $(S, \dashv, \vdash, \alpha_1, \alpha_2)$ provided that for all $x, y \in S$ and $u \in V$,

$$\begin{aligned}l_{\dashv}(y)l_{\dashv}(\alpha_1(x))u &= l_{\vdash}(y)l_{\dashv}(\alpha_1(x))u,\\ \beta_1(r_{\dashv}(\alpha_1(x) \dashv y)(v))u &= \beta_1(r \dashv (x \vdash y))u,\\ r_{\dashv}(y)l_{\dashv}(\alpha_1(x))u &= r_{\vdash}(y)l_{\dashv}(\alpha_1(x))u,\\ \beta_2(l_{\vdash}(x \vdash y))u &= \beta_2(l_{\vdash}(x \dashv y))u,\\ r_{\vdash}(y)r_{\vdash}(\alpha_2(x))u &= r_{\dashv}(y)r_{\vdash}(\alpha_2(x))u,\\ l_{\vdash}(y)r_{\dashv}(\alpha_2(x))u &= l_{\dashv}(y)r_{\vdash}(\alpha_2(x))u,\end{aligned}$$

$$\begin{aligned}&l_{\dashv}(\alpha_1(y))l_{\dashv}(\alpha_1\alpha_2(x))u - \beta_2(l_{\dashv}(\alpha_2(x) \dashv \alpha_1(y)))u\\ &\quad = l_{\dashv}(\alpha_1(x))l_{\vdash}(\alpha_1\alpha_2(y))u - \beta_2(l_{\dashv}(\alpha_2(y) \vdash \alpha_1(x)))u,\\ &\beta_2\beta_1(r_{\dashv}(\alpha_1(x) \dashv y))u - \beta_2 r_{\dashv}(\alpha_1(x))r_{\dashv}(\alpha_2(y))u\\ &\quad = r_{\dashv}(y)\beta_1 l_{\vdash}(\alpha_1\alpha_2(x))u - \beta_1 l_{\vdash}(\alpha_2(y))r_{\dashv}(\alpha_2(y))u,\end{aligned}$$

$$l_{\vdash}(\alpha_1(y))l_{\vdash}(\alpha_1\alpha_2(x))u - \beta_2(l_{\vdash}(\alpha_2(x) \vdash \alpha_1(y)))u$$
$$= l_{\vdash}(\alpha_1(x))l_{\vdash}(\alpha_1\alpha_2(y))u - \beta_2(l_{\vdash}(\alpha_2(y) \vdash \alpha_1(x)))u,$$
$$\beta_2(r_{\vdash}(\alpha_1(x) \vdash y)\beta_1)u - \beta_2 r_{\vdash}(\alpha_1(x))r_{\vdash}(\alpha_2(y))u$$
$$= \beta_1 r_{\vdash}(y)l_{\vdash}(\alpha_1\alpha_2(x))u - \beta_1 l_{\vdash}(\alpha_2(y))r_{\vdash}(\alpha_2(y))u.$$

The following theorem is proved in a similar way as for Theorem 2.8.

Theorem 2.29. Let $(A, \dashv_A, \vdash_A, \alpha_1, \alpha_2)$, $(B, \dashv_B, \vdash_B, \beta_1, \beta_2)$ be BiHom-left-symmetric dialgebras. If there are linear maps

$$l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A} : A \to gl(B), \quad l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B} : B \to gl(A)$$

such that

$$(l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, \beta_1, \beta_2, B) \text{ is a bimodule of } A,$$
$$(l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, \alpha_1, \alpha_2, A) \text{ is a bimodule of } B,$$

and for all $x, y \in A,\ a, b \in B$, the following equalities hold:

$$l_{\dashv_A}(\alpha_1(x))(a \dashv_B b) = l_{\dashv_A}(\alpha_1(x))(a \vdash_B b),$$
$$r_{\dashv_A}(r_{\dashv_B}(b)x)\beta_1(a) + \beta_1(a) \dashv_B (l_{\dashv_A}(x)b)$$
$$= r_{\dashv_A}(r_{\vdash_B}(b)x)\beta_1(a) + \beta_1(a) \dashv_B (l_{\vdash_A}(x)b),$$
$$r_{\dashv_A}(l_{\dashv_B}(b)x)\beta_1(a) + \beta_1(a) \dashv_B (r_{\dashv_A}(x)b)$$
$$= r_{\dashv_A}(l_{\vdash_B}(b)x)\beta_1(a) + \beta_1(a) \dashv_B (r_{\vdash_A}(x)b),$$
$$r_{\vdash_A}(\beta(x))(a \vdash_B b) = r_{\dashv_A}(\alpha_2(x))(a \dashv_B b),$$
$$l_{\vdash_A}(r_{\vdash_B}(a)x)\beta_2(b) + (l_{\vdash_A}(x)a) \vdash_B \beta_2(b)$$
$$= l_{\vdash_A}(r_{\dashv_B}(a)x)\beta_2(b) + (l_{\dashv_A}(x)a) \vdash_B \beta_2(b),$$
$$l_{\vdash_A}(l_{\dashv_B}(a)x)\beta_2(b) + (r_{\vdash_A}(x)a) \vdash_B \beta_2(b) =$$
$$l_{\vdash_A}(l_{\dashv_B}(a)x)\beta_2(b) + (r_{\dashv_A}(x)a) \vdash_B \beta_2(b),$$
$$l_{\dashv_A}(\alpha_1\alpha_2(x))(\alpha_1(a) \dashv_B b) - l_{\dashv_A}(r_{\dashv_B}(\beta_1(a))\alpha_2(x))\beta_2(b)$$
$$-(l_{\dashv_A}(\alpha_1(x))\beta_1(a)) \dashv_B \beta_2(b) =$$
$$\beta_1\beta_2(a) \vdash_B (l_{\dashv_A}(\alpha_1(x))b) + r_{\vdash_A}(r_{\dashv_B}(b)\alpha_1(x))\beta_1\beta_2(a)$$
$$-(r_{\vdash_A}(\alpha_1(x))\beta_2(a)) \dashv_B \beta_2(b) - l_{\dashv_A}(l_{\vdash_B}(\beta_2(a))\alpha_1(x))\beta_2(b),$$
$$\beta_1\beta_2(a) \dashv_B (l_{\dashv_A}(\alpha_1(x))b) + r_{\dashv_A}(r_{\dashv_B}(b)\alpha_1(x))\beta_1\beta_2(a)$$
$$-(r_{\dashv_A}(\alpha_1(x))\beta_2(a)) \dashv_B \beta_2(b) =$$
$$l_{\vdash_A}(\alpha_1\alpha_2(x))(\beta_2(a) \dashv_B b) - (l_{\vdash_A}(\alpha_2(x))\beta_2(a)) \dashv_B \beta_2(b)$$
$$-l_{\dashv_A}(l_{\vdash_B}(\alpha_2(x))\beta_2(a))\beta_2(b),$$

$$\beta_1\beta_2(a) \dashv_B (r_{\dashv_A}(x)\beta_2(b)) + r_{\dashv_A}(l_{\dashv_B}(\beta_2(b))x)\beta_1\beta_2(a)$$
$$-r_{\dashv_A}(\alpha_2(x))(\beta_2(a) \dashv_B \beta_1(b)) =$$
$$\beta_1\beta_2(b) \vdash_B (r_{\dashv_A}(x)\beta_1(a)) + r_{\vdash_A}(l_{\dashv_B}(\beta_1(a))x)\beta_1\beta_2(b)$$
$$-r_{\dashv_A}(\alpha_2(x))(\beta_2(b) \vdash_B \beta_1(a)),$$

$$l_{\vdash_A}(\alpha_1\alpha_2(x))(\beta_1(a) \vdash_B b) - l_{\vdash_A}(r_{\vdash_B}(\beta_1(a))\alpha_2(x))\beta_2(b)$$
$$-(l_{\vdash_A}(\alpha_2(x))\beta_1(a)) \vdash_B \beta_2(b) =$$
$$\beta_1\beta_2(a) \vdash_B (l_{\vdash_A}(\alpha_1(x))b) + r_{\vdash_A}(r_{\vdash_B}(b)\alpha_1(x))\beta_1\beta_2(a)$$
$$-(r_{\vdash_A}(\alpha_1(x))\beta_2(a)) \vdash_B \beta_2(b) - l_{\vdash_A}(l_{\vdash_B}(\beta_2(a))\alpha_1(x))\beta_2(b),$$

$$\beta_1\beta_2(a) \vdash_B (l_{\vdash_A}(\alpha_1(x))b) + r_{\vdash_A}(r_{\vdash_B}(b)\alpha_1(x))\beta_1\beta_2(a)$$
$$-(r_{\vdash_A}(\alpha_1(x))\beta_2(a)) \vdash_B \beta_2(b) =$$
$$l_{\vdash_A}(\alpha_1\alpha_2(x))(\beta_2(a) \vdash_B b) - (l_{\vdash_A}(\alpha_2(x))\beta_2(a)) \vdash_B \beta_2(b)$$
$$-l_{\vdash_A}(l_{\vdash_B}(\alpha_2(x))\beta_2(a))\beta_2(b),$$

$$\beta_1\beta_2(a) \vdash_B (r_{\vdash_A}(x)\beta_2(b)) + r_{\vdash_A}(l_{\vdash_B}(\beta_2(b))x)\beta_1\beta_2(a)$$
$$-r_{\vdash_A}(\alpha_2(x))(\beta_2(a) \vdash_B \beta_1(b)) =$$
$$\beta_1\beta_2(b) \vdash_B (r_{\vdash_A}(x)\beta_1(a)) + r_{\vdash_A}(l_{\vdash_B}(\beta_1(a))x)\beta_1\beta_2(b)$$
$$-r_{\vdash_A}(\alpha_2(x))(\beta_2(b) \vdash_B \beta_1(a)),$$

$$l_{\dashv_B}(\beta_1(a))(x \dashv_A y) = l_{\dashv_B}(\beta_1(a))(x \vdash_A y),$$

$$r_{\dashv_B}(r_{\dashv_A}(y)a)\alpha_1(x) + \alpha_1(x) \dashv_A (l_{\dashv_B}(a)y) =$$
$$r_{\dashv_B}(r_{\vdash_A}(y)a)\alpha_1(x) + \alpha_1(x) \dashv_A (l_{\vdash_B}(a)y),$$

$$r_{\dashv_B}(l_{\dashv_A}(y)a)\alpha_1(x) + \alpha_1(x) \dashv_A (r_{\dashv_B}(a)y) =$$
$$r_{\dashv_B}(l_{\vdash_A}(y)a)\alpha_1(x) + \alpha_1(x) \dashv_A (r_{\vdash_B}(a)y),$$

$$r_{\vdash_B}(\alpha(a))(x \vdash_A y) = r_{\dashv_B}(\beta_2(a))(x \dashv_A y),$$

$$l_{\vdash_B}(r_{\vdash_A}(x)a)\alpha_2(y) + (l_{\vdash_B}(a)x) \vdash_A \alpha_2(y) =$$
$$l_{\vdash_B}(r_{\dashv_A}(x)a)\alpha_2(y) + (l_{\dashv_B}(a)a) \vdash_A \alpha_2(y),$$

$$l_{\vdash_B}(l_{\dashv_A}(x)a)\alpha_2(y) + (r_{\vdash_B}(a)x) \vdash_A \alpha_2(y) =$$
$$l_{\vdash_B}(l_{\dashv_A}(x)a)\alpha_2(y) + (r_{\dashv_B}(a)x) \vdash_A \alpha_2(y),$$

$$l_{\dashv_B}(\beta_1\beta_2(a))(\alpha_1(x) \dashv_A y) - l_{\dashv_B}(r_{\dashv_A}(\alpha_1(x))\beta_2(a))\alpha_2(y)$$
$$-(l_{\dashv_B}(\beta_1(a))\alpha_1(x)) \dashv_A \alpha_2(y) =$$
$$\alpha_1\alpha_2(x) \vdash_A (l_{\dashv_B}(\beta_1(a))y) + r_{\vdash_B}(r_{\dashv_A}(y)\beta_1(a))\alpha_1\alpha_2(x)$$
$$-(r_{\vdash_B}(\beta_1(a))\alpha_2(x)) \dashv_A \alpha_2(y) - l_{\dashv_B}(l_{\vdash_A}(\alpha_2(x))\beta_1(a))\alpha_2(y),$$

$$\alpha_1\alpha_2(x) \dashv_A (l_{\dashv_B}(\beta_1(a))y) + r_{\dashv_B}(r_{\dashv_A}(y)\beta_1(a))\alpha_1\alpha_2(x)$$
$$-(r_{\dashv_B}(\beta_1(a))\alpha_2(x)) \dashv_A \alpha_2(y) =$$
$$l_{\vdash_B}(\beta_1\beta_2(xa))(\alpha_2(x) \dashv_A y) - (l_{\vdash_B}(\beta_2(a))\alpha_2(x)) \dashv_A \alpha_2(y)$$
$$-l_{\dashv_B}(l_{\vdash_A}(\beta_2(a))\alpha_2(x))\alpha_2(y),$$

$$\alpha_1\alpha_2(x) \dashv_A (r_{\dashv_B}(a)\alpha_2(y)) + r_{\dashv_B}(l_{\dashv_A}(\alpha_2(y))a)\alpha_1\alpha_2(x)$$
$$- r_{\dashv_B}(\beta_2(a))(\alpha_2(x) \dashv_A \alpha_1(y))$$

$$= \alpha_1\alpha_2(y) \vdash_A (r_{\dashv_B}(a)\alpha_1(x)) + r_{\vdash_B}(l_{\dashv_A}(\alpha_1(x))a)\alpha_1\alpha_2(y) - r_{\dashv_B}(\beta_2(a))(\alpha_2(y) \vdash_A \alpha_1(x)),$$

$$l_{\vdash_B}(\beta_1\beta_2(a))(\alpha_1(x) \vdash_A y) - l_{\vdash_B}(r_{\vdash_A}(\alpha_1(x))\beta_2(a))\alpha_2(y) -(l_{\vdash_B}(\beta_2(a))\alpha_1(x)) \vdash_A \alpha_2(y) = \alpha_1\alpha_2(x) \vdash_A (l_{\vdash_B}(\beta_1(a))y) + r_{\vdash_B}(r_{\vdash_A}(y)\beta_1(a))\alpha_1\alpha_2(x) -(r_{\vdash_B}(\beta_1(a))\alpha_2(x)) \vdash_A \alpha_2(y) - l_{\vdash_B}(l_{\vdash_A}(\alpha_2(x))\beta_1(a))\alpha_2(y),$$

$$\alpha_1\alpha_2(x) \vdash_A (l_{\vdash_B}(\beta_1(a))y) + r_{\vdash_B}(r_{\vdash_A}(y)\beta_1(a))\alpha_1\alpha_2(x) -(r_{\vdash_B}(\beta_1(a))\alpha_2(x)) \vdash_A \alpha_2(y) = l_{\vdash_B}(\beta_1\beta_2(a))(\alpha_2(x) \vdash_A y) - (l_{\vdash_B}(\beta_2(a))\alpha_2(x)) \vdash_A \alpha_2(y) -l_{\vdash_B}(l_{\vdash_A}(\beta_2(a))\alpha_2(x))\alpha_2(y),$$

$$\alpha_1\alpha_2(x) \vdash_A (r_{\vdash_B}(a)\alpha_2(y)) + r_{\vdash_B}(l_{\vdash_A}(\alpha_2(y))a)\alpha_1\alpha_2(x) -r_{\vdash_B}(\beta_2(a))(\alpha_2(x) \vdash_A \alpha_1(y)) = \alpha_1\alpha_2(y) \vdash_A (r_{\vdash_B}(a)\alpha_1(x)) + r_{\vdash_B}(l_{\vdash_A}(\alpha_1(x))a)\alpha_1\alpha_2(y) -r_{\vdash_B}(\beta_2(a))(\alpha_2(y) \vdash_A \alpha_1(x)),$$

then tuple $(A, B, l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, \beta_1, \beta_2, l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, \alpha_1, \alpha_2)$ is called a matched pair of BiHom-left-symmetric dialgebras. In this case, there exists a BiHom-left-symmetric dialgebra structure on the direct sum $A \oplus B$ of the underlying vector spaces of A and B given by

$$(x+a) \dashv (y+b) = x \dashv_A y + (l_{\dashv_A}(x)b + r_{\dashv_A}(y)a) + a \dashv_B b + (l_{\dashv_B}(a)y + r_{\dashv_B}(b)x),$$
$$(x+a) \vdash (y+b) = x \vdash_A y + (l_{\vdash_A}(x)b + r_{\vdash_A}(y)a) + a \vdash_B b + (l_{\vdash_B}(a)y + r_{\vdash_B}(b)x),$$
$$(\alpha_1 \oplus \beta_1)(x+a) = \alpha_1(x) + \beta_1(a), \quad (\alpha_2 \oplus \beta_2)(x+a) = \alpha_2(x) + \beta_2(a).$$

We denote this BiHom-left-symmetric dialgebra

$$A \bowtie^{l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, \beta_1, \beta_2}_{l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_A}, \alpha_1, \alpha_2} B.$$

2.2. BiHom-associative dialgebra

Definition 2.30. A BiHom-associative dialgebra $(D, \dashv, \vdash, \alpha, \beta)$ is a quintuple consisting of a vector space D on which the operations $\dashv, \vdash : D \otimes D \to D$ and $\alpha, \beta : D \to D$ are linear maps satisfying, for $x, y, z \in D$,

$$\alpha \circ \beta = \beta \circ \alpha, \tag{20}$$
$$\alpha(x \dashv y) = \alpha(x) \dashv \alpha(y), \qquad \alpha(x \vdash y) = \alpha(x) \vdash \alpha(y), \tag{21}$$
$$\beta(x \dashv y) = \beta(x) \dashv \beta(y), \qquad \beta(x \vdash y) = \beta(x) \vdash \beta(y), \tag{22}$$

$$(x \vdash y) \dashv \beta(z) = \alpha(x) \vdash (y \dashv z), \tag{23}$$
$$\alpha(x) \dashv (y \dashv z) = (x \dashv y) \dashv \beta(z), \tag{24}$$
$$(x \dashv y) \dashv \beta(z) = \alpha(x) \dashv (y \vdash z), \tag{25}$$
$$(x \vdash y) \vdash \beta(z) = \alpha(x) \vdash (y \vdash z), \tag{26}$$
$$\alpha(x) \vdash (y \vdash z) = (x \dashv y) \vdash \beta(z). \tag{27}$$

Remark 2.31. The considered structures are related in the following ways.

1. If $(D, \dashv, \vdash, \alpha, \beta)$ is a BiHom-associative dialgebra and $\dashv=\vdash=: \mu$, then (D, μ, α, β) is a BiHom-associative algebra. Any BiHom-associative algebra (A, μ, α, β) is a BiHom-associative dialgebra with $\dashv= \mu =:\vdash$.
2. A BiHom-associative dialgebra is a BiHom-X algebra.

Proposition 2.32. All BiHom-associative dialgebras are BiHom-left symmetric dialgebras.

Proof. Let $(D, \dashv, \vdash, \alpha, \beta)$ be a BiHom-associative dialgebra, then (15) and (16) are satisfied. Since the products $\dashv$ and $\vdash$ are associative with the condition (23), the equalities (17) and (18) are established. □

Remark 2.33. Any BiHom-left symmetric algebra is a BiHom-left symmetric dialgebra in which $\dashv=\vdash$. A nonassociative BiHom-left symmetric algebra is not a BiHom-left symmetric dialgebra.

Proposition 2.34. A BiHom-left-symmetric dialgebra S is a BiHom-associative dialgebra if and only if both products of S are BiHom-associative.

Proof. If a BiHom-left-symmetric dialgebra S is a BiHom-associative dialgebra, then both products $\dashv$ and $\vdash$ defined over S are BiHom-associative according to Definition 2.30. Conversely, if each product of a BiHom-left-symmetric dialgebra is BiHom-associative, then by Definition 2.14, S is a BiHom-associative dialgebra. □

Definition 2.35. An averaging operator over a BiHom-associative algebra (A, μ, α, β) is a linear map $\gamma : A \to A$ such that $\alpha\circ\gamma = \gamma\circ\alpha$ and $\beta\circ\gamma = \gamma\circ\beta$, and for all $x, y \in A$,

$$\gamma(\mu(\gamma(x), y) = \mu(\gamma(x), \gamma(y)) = \gamma(\mu(x, \gamma(y))). \tag{28}$$

Theorem 2.36. Let $(A, \cdot)$ be an associative algebra and $\alpha, \beta : A \to A$ two averaging operators such that $(A, \cdot, \alpha, \beta)$ be a BiHom-associative algebra. For any $x, y \in A$, define new operations on A by

$$x \vdash y = \alpha(x) \cdot \beta(y) \quad \text{and} \quad x \dashv y = \beta(x) \cdot \alpha(y).$$

Then $(A, \dashv, \vdash, \alpha, \beta)$ is a BiHom-associative dialgebra.

Proof. We prove only one axiom, as others are proved similarly. For any $x, y, z \in A$,

$$\begin{aligned}
&\alpha(x) \dashv (y \dashv z) - (x \dashv y) \dashv \beta(z)\\
&= \alpha\beta(x) \cdot \alpha(\beta(y) \cdot \alpha(z)) - \beta(\beta(x) \cdot \alpha(y)) \cdot \alpha\beta(z)\\
&= \alpha\beta(x) \cdot (\alpha\beta(y) \cdot \alpha(z)) - (\beta(x) \cdot \alpha\beta(y)) \cdot \alpha\beta(z) \qquad \text{(by (28))}\\
&= \alpha\beta(x) \cdot (\alpha\beta(y) \cdot \alpha(z)) - \alpha\beta(x) \cdot (\alpha\beta(y) \cdot \alpha(z)) = 0. \quad \text{(by (5))}
\end{aligned}$$

This proves the second axiom in Definition 2.30. □

Definition 2.37. Let $(D, \dashv, \vdash, \alpha_1, \alpha_2)$ be a BiHom-associative dialgebra, and V be a vector space. Let $l_\dashv, r_\dashv, l_\vdash, r_\vdash : D \to gl(V)$, and $\beta_1, \beta_2 : V \to V$ be six linear maps. Then, $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, \beta_1, \beta_2, V)$ is called a bimodule of D if for any $x, y \in D$ and $v \in V$:

$$\begin{aligned}
&l_\dashv(x \vdash y)\beta_2(v) = l_\vdash(\alpha_1(x))l_\dashv(y)v, && r_\dashv(\alpha_2(x))l_\vdash(y)v = l_\vdash(\alpha_1(y))r_\dashv(x)v,\\
&r_\dashv(\alpha_2(x))r_\vdash(y)(v) = r_\vdash(y \dashv x)\beta_1(v), && l_\dashv(x \dashv y)\beta_2(v) = l_\dashv(\alpha_1(x))l_\dashv(y)v,\\
&r_\dashv(\alpha_2(x))l_\dashv(y)v = l_\dashv(\alpha_1(y))r_\dashv(x)v, && r_\dashv(\alpha_2(x))r_\dashv(y)(v) = r_\dashv(y \dashv x)\beta_1(v),\\
&l_\dashv(x \dashv y)\beta_2(v) = l_\dashv(\alpha_1(x))l_\vdash(y)v, && r_\vdash(\alpha_2(x))l_\vdash(y)v = l_\vdash(\alpha_1(y))r_\vdash(x)v,\\
&r_\vdash(\alpha_2(x))r_\vdash(y)(v) = r_\vdash(y \vdash x)\beta_1(v), && l_\vdash(x \vdash y)\beta_2(v) = l_\vdash(\alpha_1(x))l_\vdash(y)v,\\
&r_\dashv(\alpha_2(x))l_\vdash(y)v = l_\vdash(\alpha_1(y))r_\dashv(x)v, && r_\dashv(\alpha_2(x))r_\vdash(y)(v) = r_\vdash(y \dashv x)\beta_1(v),\\
&l_\dashv(x \dashv y)\beta_2(v) = l_\vdash(\alpha_1(x))l_\dashv(y)v, && r_\vdash(\alpha_2(x))l_\dashv(y)v = l_\vdash(\alpha_1(y))r_\vdash(x)v,\\
&r_\vdash(\alpha_2(x))r_\dashv(y)(v) = r_\vdash(y \vdash x)\beta_1(v), && \beta_1(l_\vdash(x)v) = l_\vdash(\alpha_1(x))\beta_1(v),\\
&\beta_1(r_\vdash(x)v) = r_\vdash(\alpha_1(x))\beta_1(v), && \beta_2(l_\vdash(x)v) = l_\vdash(\alpha_2(x))\beta_2(v),\\
&\beta_2(r_\vdash(x)v) = r_\vdash(\alpha_2(x))\beta_2(v), && \beta_1(l_\dashv(x)v) = l_\dashv(\alpha_1(x))\beta_1(v),\\
&\beta_1(r_\dashv(x)v) = r_\dashv(\alpha_1(x))\beta_1(v), && \beta_2(l_\dashv(x)v) = l_\dashv(\alpha_2(x))\beta_2(v),\\
&\beta_2(r_\dashv(x)v) = r_\dashv(\alpha_2(x))\beta_2(v).
\end{aligned}$$

Proposition 2.38. Let $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, \beta_1, \beta_2, V)$ be a bimodule of a BiHom-associative dialgebra $(D, \dashv, \vdash, \alpha_1, \alpha_2)$. Then, there exists a BiHom-associative dialgebra structure on the direct sum $D \oplus V$ of the underlying vector spaces of D and V given for all $x, y \in D, u, v \in V$ by

$$\begin{aligned}
(x + u) \dashv' (y + v) &= x \dashv y + l_\dashv(x)v + r_\dashv(y)u,\\
(x + u) \vdash' (y + v) &= x \vdash y + l_\vdash(x)v + r_\vdash(y)u,\\
(\alpha_1 + \beta_1)(x + u) &= \alpha_1(x) + \beta_1(u),\\
(\alpha_2 + \beta_2)(x + u) &= \alpha_2(x) + \beta_2(u).
\end{aligned}$$

We denote such a BiHom-associative dialgebra by $D \times_{l_\dashv, r_\dashv, l_\vdash, r_\vdash, \alpha_1, \alpha_2, \beta_1, \beta_2} V$.

Proof. We prove only one axiom, as others are proved similarly. For any $x_1, x_2, x_3 \in D$ and $v_1, v_2, v_3 \in V$,

$$\begin{aligned}
&((x_1+v_1)\vdash'(x_2+v_2))\dashv'(\alpha_2+\beta_2)(x_3+v_3)\\
&\quad=(x_1\vdash x_2+l_\vdash(x_1)v_2+r_\vdash(x_2)v_1)\dashv'(\alpha_2(x_3)+\beta_2(v_3))\\
&\quad=(x_1\vdash x_2)\dashv\alpha_2(x_3)+l_\dashv(x_1\vdash x_2)\beta_2(v_3)\\
&\qquad+r_\dashv(\alpha_2(x_3))l_\vdash(x_1)v_1+r_\dashv(\alpha_2(x_3))r_\vdash(x_2)v_1.\\
&(\alpha_1+\beta_1)(x_1+v_1)\vdash'((x_2+v_2)\dashv'(x_3+v_3))\\
&\quad=(\alpha_1(x_1)+\beta_1(v_1))\vdash'(x_2\dashv x_3+l_\dashv(x_2)v_3+r_\dashv(x_3)v_2)\\
&\quad=\alpha_1(x_1)\vdash(x_2\dashv x_3)+l_\vdash(\alpha_1(x_1))l_\dashv(x_2)v_3\\
&\qquad+l_\vdash(\alpha_1(x_1))r_\dashv(x_3)v_2+r_\vdash(x_2+x_3)\beta_1(v_1),
\end{aligned}$$

which implies that

$$\begin{aligned}
&((x_1+v_1)\vdash'(x_2+v_2))\dashv'(\alpha_2+\beta_2)(x_3+v_3)=\\
&\qquad\qquad(\alpha_1+\beta_1)(x_1+v_1)\vdash'((x_2+v_2)\dashv'(x_3+v_3)).\ \square
\end{aligned}$$

Examples 2.39. Some examples can be obtained as follows.
1) Let $(D,\dashv,\vdash,\alpha,\beta)$ be a BiHom-associative dialgebra. Then the tuple $(L_\dashv, R_\dashv, L_\vdash, R_\vdash, \alpha, \beta, D)$ is a bimodule of D, where

$$L_\dashv(a)b = a\dashv b,\ \ R_\dashv(a)b = b\dashv a,\ \ L_\vdash(a)b = a\vdash b, R_\vdash(a)b = b\vdash a$$

for all $(a,b)\in D^2$. More generally, if B is a two-sided BiHom-ideal of $(D,\dashv,\vdash,\alpha,\beta)$, then $(L_\dashv, R_\dashv, L_\vdash, R_\vdash, \alpha, \beta, B)$ is a bimodule of D, where the structure maps are $L_\dashv(a)x = a\dashv x = x\dashv a = R_\dashv(a)x$ and $L_\vdash(a)x = a\vdash x = x\vdash a = R_\vdash(a)x$ for all $x\in B$ and $(a,b)\in D^2$.
2) If $(D,\dashv,\vdash)$ is an associative dialgebra and $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, \alpha, \beta, V)$ is a bimodule of D, then $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, Id_D, Id_D, V)$ is a bimodule of $\mathbb{D}$ where $\mathbb{D} = (D,\dashv,\vdash, Id_D, Id_D)$ is a BiHom-associative dialgebra.

Proposition 2.40. If $f : (D_1,\dashv_1,\vdash_1,\alpha_1,\alpha_2)\to(D_2,\dashv_2,\vdash_2,\beta_1,\beta_2)$ is a morphism of BiHom-associative dialgebra, then $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, \beta_1, \beta_2, D_2)$ is a bimodule of D_1 via f, that is, the structure maps are defined as

$$\begin{aligned}
l_{\dashv_1}(a)b &= f(a)\dashv_2 b, & r_{\dashv_1}(a)b &= b\dashv_2 f(a),\\
l_{\vdash_1}(a)b &= f(a)\vdash_2 b, & r_{\vdash_1}(a)b &= b\vdash_2 f(a)
\end{aligned}$$

for all $(a,b)\in D_1\times D_2$.

Proof. The proof is obtained in a similar way as for Proposition 2.23. $\square$

Theorem 2.41. Let $(A, \dashv_A, \vdash_A, \alpha_1, \alpha_2)$ and $(B, \dashv_B, \vdash_B, \beta_1, \beta_2)$ be BiHom-associative dialgebras. Suppose that there are linear maps

$$l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A} : A \to gl(B), \quad l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B} : B \to gl(A)$$

such that

$$(l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, \beta_1, \beta_2, B) \text{ is a bimodule of } A,$$
$$(l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, \alpha_1, \alpha_2, A) \text{ is a bimodule } of B,$$

and for any $x, y \in A,\ a, b \in B$,

$$r_{\dashv_A}(\alpha_2(x))(a \vdash_B b) = r_{\vdash_A}(l_{\dashv_B}(b)x)\beta_1(a) + \beta_1(a) \vdash_B (r_{\dashv_A}(x)b),$$
$$l_{\dashv_A}(l_{\vdash_B}(a)x)\beta_2(b) + (r_{\vdash_A}(x)a) \dashv_B \beta_2(b) =$$
$$\beta_1(a) \vdash_B (l_{\dashv_A}(x)b) + r_{\vdash_A}(r_{\dashv_B}(b)x)\beta_1(a),$$
$$l_{\dashv_A}(\alpha_1(x))(a \dashv_B b) = (l_{\vdash_A}(x)a) \dashv_B \beta_2(b) + l_{\dashv_A}(r_{\vdash_B}(a)x)\beta_2(b),$$
$$r_{\dashv_A}(\alpha_2(x))(a \dashv_B b) = r_{\dashv_A}(l_{\dashv_B}(b)x)\beta_1(a) + \beta_1(a) \dashv_B (r_{\dashv_A}(x)b),$$
$$l_{\dashv_A}(l_{\dashv_B}(a)x)\beta_2(b) + (r_{\dashv_A}(x)a) \dashv_B \beta_2(b) =$$
$$\beta_1(a) \dashv_B (l_{\dashv_A}(x)b) + r_{\dashv_A}(r_{\dashv_B}(b)x)\beta_1(a),$$
$$l_{\dashv_A}(\alpha_1(x))(a \dashv_B b) = (l_{\dashv_A}(x)a) \dashv_B \beta_2(b) + l_{\dashv_A}(r_{\dashv_B}(a)x)\beta_2(b),$$
$$r_{\dashv_A}(\alpha_2(x))(a \dashv_B b) = r_{\dashv_A}(l_{\vdash_B}(b)x)\beta_1(a) + \beta_1(a) \dashv_B (r_{\vdash_A}(x)b),$$
$$l_{\dashv_A}(l_{\dashv_B}(a)x)\beta_2(b) + (r_{\dashv_A}(x)a) \dashv_B \beta_2(b) =$$
$$\beta_1(a) \dashv_B (l_{\vdash_A}(x)b) + r_{\dashv_A}(r_{\vdash_B}(b)x)\beta_1(a),$$
$$l_{\dashv_A}(\alpha_1(x))(a \vdash_B b) = (l_{\dashv_A}(x)a) \dashv_B \beta_2(b) + l_{\dashv_A}(r_{\dashv_B}(a)x)\beta_2(b),$$
$$r_{\vdash_A}(\alpha_2(x))(a \vdash_B b) = r_{\vdash_A}(l_{\vdash_B}(b)x)\beta_1(a) + \beta_1(a) \vdash_B (r_{\vdash_A}(x)b),$$
$$l_{\vdash_A}(l_{\vdash_B}(a)x)\beta_2(b) + (r_{\vdash_A}(x)a) \vdash_B \beta_2(b) =$$
$$\beta_1(a) \vdash_B (l_{\vdash_A}(x)b) + r_{\vdash_A}(r_{\vdash_B}(b)x)\beta_1(a),$$
$$l_{\vdash_A}(\alpha_1(x))(a \vdash_B b) = (l_{\vdash_A}(x)a) \vdash_B \beta_2(b) + l_{\vdash_A}(r_{\vdash_B}(a)x)\beta_2(b),$$
$$r_{\vdash_A}(\alpha_2(x))(a \dashv_B b) = r_{\vdash_A}(l_{\vdash_B}(b)x)\beta_1(a) + \beta_1(a) \vdash_B (r_{\vdash_A}(x)b),$$
$$l_{\vdash_A}(l_{\dashv_B}(a)x)\beta_2(b) + (r_{\dashv_A}(x)a) \vdash_B \beta_2(b) =$$
$$\beta_1(a) \vdash_B (l_{\vdash_A}(x)b) + r_{\vdash_A}(r_{\vdash_B}(b)x)\beta_1(a),$$
$$l_{\vdash_A}(\alpha_1(x))(a \vdash_B b) = (l_{\dashv_A}(x)a) \vdash_B \beta_2(b) + l_{\vdash_A}(r_{\dashv_B}(a)x)\beta_2(b),$$
$$r_{\dashv_B}(\beta_2(a))(x \vdash_A y) = r_{\vdash_B}(l_{\dashv_A}(y)a)\alpha_1(x) + \alpha_1(x) \vdash_A (r_{\dashv_B}(a)y),$$
$$l_{\dashv_B}(l_{\vdash_A}(x)a)\alpha_2(y) + (r_{\vdash_B}(a)x) \dashv_A \alpha_2(y) =$$
$$\alpha_1(x) \vdash_B (l_{\dashv_B}(a)y) + r_{\vdash_B}(r_{\dashv_A}(y)a)\alpha_1(x),$$
$$l_{\dashv_B}(\beta_1(a))(x \dashv_A y) = (l_{\vdash_B}(a)x) \dashv_A \alpha_2(y) + l_{\dashv_B}(r_{\vdash_A}(x)a)\alpha_2(y),$$
$$r_{\dashv_B}(\beta_2(a))(x \dashv_A y) = r_{\dashv_B}(l_{\dashv_A}(y)a)\alpha_1(x) + \alpha_1(x) \dashv_A (r_{\dashv_B}(a)y),$$

$$
\begin{aligned}
& l_{\dashv_B}(l_{\dashv_A}(x)a)\alpha_2(y) + (r_{\dashv_B}(a)x) \dashv_A \alpha_2(y) = \\
& \qquad\qquad \alpha_1(x) \dashv_B (l_{\dashv_B}(a)y) + r_{\dashv_B}(r_{\dashv_A}(y)a)\alpha_1(x), \\
& l_{\dashv_B}(\beta_1(a))(x \dashv_A y) = (l_{\dashv_B}(a)x) \dashv_A \alpha_2(y) + l_{\dashv_B}(r_{\dashv_A}(x)a)\alpha_2(y), \\
& r_{\dashv_B}(\beta_2(a))(x \dashv_A y) = r_{\dashv_B}(l_{\vdash_A}(y)a)\alpha_1(x) + \alpha_1(x) \dashv_A (r_{\vdash_B}(a)y), \\
& l_{\dashv_B}(l_{\dashv_A}(x)a)\alpha_2(y) + (r_{\dashv_B}(a)x) \dashv_A \alpha_2(y) = \\
& \qquad\qquad \alpha_1(x) \dashv_B (l_{\vdash_B}(a)y) + r_{\dashv_B}(r_{\vdash_A}(y)a)\alpha_1(x), \\
& l_{\dashv_B}(\beta_1(a))(x \vdash_A y) = (l_{\dashv_B}(a)x) \dashv_A \alpha_2(y) + l_{\dashv_B}(r_{\dashv_A}(x)a)\alpha_2(y), \\
& r_{\vdash_B}(\beta_2(a))(x \vdash_A y) = r_{\vdash_B}(l_{\vdash_A}(y)a)\alpha_1(x) + \alpha_1(x) \vdash_A (r_{\vdash_B}(a)y), \\
& l_{\vdash_B}(l_{\vdash_A}(x)a)\alpha_2(y) + (r_{\vdash_B}(a)x) \vdash_A \alpha_2(y) = \\
& \qquad\qquad \alpha_1(x) \vdash_B (l_{\vdash_B}(a)y) + r_{\vdash_B}(r_{\vdash_A}(y)a)\alpha_1(x), \\
& l_{\vdash_B}(\beta_1(a))(x \vdash_A y) = (l_{\vdash_B}(a)x) \vdash_A \alpha_2(y) + l_{\vdash_B}(r_{\vdash_A}(x)a)\alpha_2(y), \\
& r_{\vdash_B}(\beta_2(a))(x \dashv_A y) = r_{\vdash_B}(l_{\vdash_A}(y)a)\alpha_1(x) + \alpha_1(x) \vdash_A (r_{\vdash_B}(a)y), \\
& l_{\vdash_B}(l_{\dashv_A}(x)a)\alpha_2(y) + (r_{\dashv_B}(a)x) \vdash_A \alpha_2(y) = \\
& \qquad\qquad \alpha_1(x) \vdash_B (l_{\vdash_B}(a)y) + r_{\vdash_B}(r_{\vdash_A}(y)a)\alpha_1(x), \\
& l_{\vdash_B}(\beta_1(a))(x \vdash_A y) = (l_{\dashv_B}(a)x) \vdash_A \alpha_2(y) + l_{\vdash_B}(r_{\dashv_A}(x)a)\alpha_2(y).
\end{aligned}
$$

Then, there is a BiHom-associative dialgebra structure on the direct sum $A \oplus B$ of the underlying vector spaces of A and B given by

$$
\begin{aligned}
(x + a) \dashv (y + b) &= (x \dashv_A y + r_{\dashv_B}(b)x + l_{\dashv_B}(a)y) \\
&\qquad + (l_{\dashv_A}(x)b + r_{\dashv_A}(y)a + a \dashv_B b), \\
(x + a) \vdash (y + b) &= (x \vdash_A y + r_{\vdash_B}(b)x + l_{\vdash_B}(a)y) \\
&\qquad + (l_{\vdash_A}(x)b + r_{\vdash_A}(y)a + a \vdash_B b), \\
(\alpha_1 + \beta_1)(x + a) &= \alpha_1(x) + \beta_1(a), \quad (\alpha_2 + \beta_2)(x + a) = \alpha_2(x) + \beta_2(a).
\end{aligned}
$$

By $A \bowtie^{l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, \beta_1, \beta_1}_{l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, \alpha_1, \alpha_2} B$ we denote this BiHom-associative dialgebra.

Proof. The proof is obtained in a similar way as for Theorem 2.8. □

3. Bimodules and matched pairs of BiHom-tridendriform algebras

In this section, we recall definitions of BiHom-dendriform and BiHom tridendriform algebras given in [33]. Next we study the concept of bimodules and matched pairs of BiHom-tridendriform alge- bra and we give some related properties.

Definition 3.1. A BiHom-dendriform algebra is a quintuple $(A, \dashv, \vdash, \alpha, \beta)$ consisting of a vector space A on which the operations $\dashv, \vdash \colon A \otimes A \to A$

and $\alpha, \beta : A \to A$ are linear maps satisfying $\alpha \circ \beta = \beta \circ \alpha$ and

$$\begin{aligned}
&\alpha(x \dashv y) = \alpha(x) \dashv \alpha(y), \alpha(x \vdash y) = \alpha(x) \vdash \alpha(y),\\
&\beta(x \dashv y) = \beta(x) \dashv \beta(y), \beta(x \vdash y) = \beta(x) \vdash \beta(y),\\
&(x \dashv y) \dashv \beta(z) = \alpha(x) \dashv (y \dashv z + y \vdash z),\\
&(x \vdash y) \dashv \beta(z) = \alpha(x) \vdash (y \dashv z),\\
&\alpha(x) \vdash (y \vdash z) = (x \dashv y + x \vdash y) \vdash \beta(z).
\end{aligned}$$

Remark 3.2. BiHom-dendriform algebras are BiHom-X algebras.

Proposition 3.3. If $(A, \dashv, \vdash, \alpha, \beta)$ is a BiHom-dendriform algebra, and $x * y = x \dashv y + x \vdash y$ for all $x, y \in A$, then $(A, *, \alpha, \beta)$ is a multiplicative BiHom-associative algebra.

Proof. For all $x, y, z \in A$,

$$\begin{aligned}
(x * y) * \beta(z) &= (x \dashv y) \dashv \beta(z) + (x \dashv y) \vdash \beta(z)\\
&\quad + (x \vdash y) \dashv \beta(z) + (x \vdash y) \vdash \beta(z)\\
&= (x \dashv y) \dashv \beta(z) + (x \vdash y) \dashv \beta(z)\\
&\quad + (x \dashv y) \vdash \beta(z) + (x \vdash y) \vdash \beta(z)\\
&= (x \dashv y) \dashv \beta(z) + (x \vdash y) \dashv \beta(z) + (x * y) \vdash \beta(z)\\
&= \alpha(x) \dashv (y * z) + \alpha(x) \vdash (y \dashv z) + \alpha(x) \vdash (y \vdash z)\\
&= \alpha(x) \dashv (y * z) + \alpha(x) \vdash (y * z) = \alpha(x) * (y * z),\\
\alpha(x * y) &= \alpha(x \vdash y) + \alpha(x \dashv y) = \alpha(x) \vdash \alpha(y) + \alpha(x) \dashv \alpha(y)\\
&= \alpha(x) * \alpha(y),\\
\beta(x * y) &= \beta(x \vdash y) + \beta(x \dashv y) = \beta(x) \vdash \beta(y) + \beta(x) \dashv \beta(y)\\
&= \beta(x) * \beta(y).
\end{aligned}$$

□

Definition 3.4 ([22]). If $(A, \dashv, \vdash, \alpha_1, \alpha_2)$ is a BiHom-dendriform algebra, V is a vector space, and $l_\dashv, r_\dashv, l_\vdash, r_\vdash : A \to gl(V)$, $\beta_1, \beta_2 : V \to V$ are six linear maps, then $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, \beta_1, \beta_2, V)$ is called a bimodule of A if for any $x, y \in A, v \in V$ and $x * y = x \dashv y + x \vdash y$, $l_* = l_\dashv + l_\vdash$, $r_* = r_\dashv + r_\vdash$, the following equalities hold

$$\begin{aligned}
&l_\dashv(x \dashv y)\beta_2(v) = l_\dashv(\alpha_1(x))l_*(y)v, && r_\dashv(\alpha_2(x))l_\dashv(y)v = l_\dashv(\alpha_1(y))r_*(x)v,\\
&r_\dashv(\alpha_2(y))r_\dashv(y)v = r_\dashv(x * y)\beta_1(v), && l_\dashv(x \vdash y)\beta_2(v) = l_\vdash(\alpha_1(x))l_\dashv(y)v,\\
&r_\dashv(\alpha_2(x))l_\vdash(y)v = l_\vdash(\alpha_1(y))r_\dashv(x)v, && r_\dashv(\alpha_2(x))r_\vdash(y)v = r_\vdash(y \dashv x)\beta_1(v),\\
&l_\vdash(x * y)\beta_2(v) = l_\vdash(\alpha_1(x))l_\vdash(y)v, && r_\vdash(\alpha_2(x))l_*(y)v = l_\vdash(\alpha_1(y))r_\vdash(x)v,
\end{aligned}$$

$$r_{\vdash}(\alpha_2(x))r_*(y)v = r_{\vdash}(y \vdash x)\beta_1(v), \quad \beta_1(l_{\vdash}(x)v) = l_{\vdash}(\alpha_1(x))\beta_1(v),$$
$$\beta_1(r_{\vdash}(x)v) = r_{\vdash}(\alpha_1(x))\beta_1(v), \quad \beta_2(l_{\vdash}(x)v) = l_{\vdash}(\alpha_2(x))\beta_2(v),$$
$$\beta_2(r_{\vdash}(x)v) = r_{\vdash}(\alpha_2(x))\beta_2(v), \quad \beta_1(l_{\dashv}(x)v) = l_{\dashv}(\alpha_1(x))\beta_1(v),$$
$$\beta_1(r_{\dashv}(x)v) = r_{\dashv}(\alpha_1(x))\beta_1(v), \quad \beta_2(l_{\dashv}(x)v) = l_{\dashv}(\alpha_2(x))\beta_2(v),$$
$$\beta_2(r_{\dashv}(x)v) = r_{\dashv}(\alpha_2(x))\beta_2(v).$$

Proposition 3.5 ([22])**.** Let $(l_{\dashv}, r_{\dashv}, l_{\vdash}, r_{\vdash}, \beta_1, \beta_2, V)$ be a bimodule of a BiHom-dendriform algebra $(A, \dashv, \vdash, \alpha_1, \alpha_2)$. Then, on the direct sum $A \oplus V$ of the underlying vector spaces of A and V, there exists a BiHom-dendriform algebra structure given for all $x, y \in A, u, v \in V$ by

$$(x+u) \dashv' (y+v) = x \dashv y + l_{\dashv}(x)v + r_{\dashv}(y)u$$
$$(x+u) \vdash' (y+v) = x \vdash y + l_{\vdash}(x)v + r_{\vdash}(y)u,$$
$$(\alpha_1+\beta_1)(x+a) = \alpha_1(x) + \beta_1(a), \quad (\alpha_2+\beta_2)(x+a) = \alpha_2(x) + \beta_2(a).$$

We denote it by $A \times_{l_{\dashv}, r_{\dashv}, l_{\vdash}, r_{\vdash}, \alpha_1, \alpha_2, \beta_1, \beta_2} V$.

Theorem 3.6 ([22])**.** Let $(A, \dashv_A, \vdash_A, \alpha_1, \alpha_2)$ and $(B, \dashv_B, \vdash_B, \beta_1, \beta_2)$ be two BiHom-dendriform algebras. Suppose that there are linear maps

$$l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A} : A \to gl(B), \quad l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B} : B \to gl(A)$$

such that for all $x, y \in A,\ a, b \in B$ and for

$$x *_A y = x \dashv_A y + x \vdash_A y, \quad l_A = l_{\dashv_A} + l_{\vdash_A}, \quad r_A = r_{\dashv_A} + r_{\vdash_A},$$
$$a *_B b = a \dashv_B b + a \vdash_B b, \quad l_B = l_{\dashv_B} + l_{\vdash_B}, \quad r_B = r_{\dashv_B} + r_{\vdash_B},$$

the following equalities hold:

$$r_{\dashv_A}(\alpha_2(x))(a \dashv_B b) = \beta_1(a) \dashv_B (r_A(x)b) + r_{\dashv_A}(l_B(x)\beta_1(a)),$$
$$l_{\dashv_A}(l_{\dashv_B}(x))\beta_2(b) \quad + \quad (r_{\dashv_A}(x)a) \dashv_B \beta_2(b) =$$
$$\beta_1(a) \dashv_B (l_{\dashv_A}(x)b) + r_{\dashv_A}(r_{\dashv_B}(b)x)\beta_1(a),$$
$$l_{\dashv_A}(\alpha_1(x))(a *_B b) = (l_{\dashv_A}(x)a) *_B \beta_2(b) + l_{\dashv_A}(r_{\dashv_A}(a)x)\beta_2(b),$$
$$r_{\dashv_A}(\alpha_2(x))(a \vdash_B b) = r_{\vdash_A}(l_{\dashv_B}(b)x)\beta_1(a) + \beta_1(a) \vdash_B (r_{\dashv_A}(x)b),$$
$$l_{\dashv_A}(l_{\vdash_B}(a)x)\beta_2(b) \quad + \quad (r_{\vdash_A}(x)a) \dashv_B \beta_2(b) =$$
$$\beta_1(a) \vdash_B (l_{\dashv_A}(x)b) + r_{\vdash_A}(r_{\dashv_B}(b)x)\beta_1(a),$$
$$l_{\vdash_A}(\alpha_1(x))(a \dashv_B b) = (l_{\vdash_A}(x)a) \dashv_B \beta_2(b) + l_{\dashv_A}(r_{\vdash_B}(a)x)\beta_2(b),$$
$$r_{\vdash_A}(\alpha_2(x))(a *_B b) = \beta_1(a) \vdash_B (r_{\vdash_A}(x)b) + r_{\vdash_A}(l_{\vdash_B}(b)x)\beta_1(a),$$

$$\begin{aligned}
\beta_1(a) \vdash_B (l_{\vdash_A}(x)b) \quad + \quad & r_{\vdash_A}(r_{\vdash_B}(b)x)\beta_1(a) = \\
& l_{\vdash_A}(l_B(a)x)\beta_2(b) + (r_A(x)a) \vdash_B \beta_2(b),
\end{aligned}$$

$$l_{\vdash_A}(\alpha_1(x))(a \vdash_B b) = (l_A(x)a) \vdash_B \beta_2(b) + l_{\vdash_A}(r_B(a)x)\beta_2(b),$$

$$r_{\dashv_B}(\beta_2(a))(x \dashv_A y) = \alpha_1(x) \dashv_A (r_B(a)y) + r_{\dashv_B}(l_A(y)a)\alpha_1(x),$$

$$\begin{aligned}
l_{\dashv_B}(l_{\dashv_A}(x)a)\alpha_2(y) \quad + \quad & (r_{\dashv_B}(a)x) \dashv_A \alpha_2(y) = \\
& \alpha_1(x) \dashv_A (l_B(a)y) + r_{\dashv_B}(r_A(y)a)\alpha_1(x),
\end{aligned}$$

$$l_{\dashv_B}(\beta_1(a))(x *_A y) = (l_{\dashv_B}(a)x) \dashv_A \alpha_2(y) + l_{\dashv_B}(r_{\dashv_A}(x)a)\alpha_2(y),$$

$$r_{\dashv_B}(\beta_2(a))(x \vdash_A y) = r_{\vdash_B}(l_{\dashv_B}(y)a)\alpha_1(x) + \alpha_1(x) \vdash_A (r_{\dashv_B}(a)y),$$

$$\begin{aligned}
l_{\dashv_B}(l_{\vdash_A}(x)a)\alpha_2(y) \quad + \quad & (r_{\vdash_B}(a)x) \dashv_A \alpha_2(y) = \\
& \alpha_1(x) \vdash_A (l_{\dashv_B}(a)y) + r_{\vdash_B}(r_{\dashv_A}(y)a)\alpha_1(x),
\end{aligned}$$

$$l_{\vdash_B}(\beta_1(a))(x \dashv_A y) = (l_{\vdash_B}(a)x) \dashv_A \alpha_2(y) + l_{\dashv_B}(r_{\vdash_A}(x)a)\alpha_2(y),$$

$$r_{\vdash_B}(\beta_2(a))(x *_A y) = \alpha_1(x) \vdash_A (r_{\vdash_B}(a)y) + r_{\vdash_B}(l_{\vdash_A}(y)a)\alpha_1(x),$$

$$\begin{aligned}
\alpha_1(x) \vdash_A (l_{\vdash_B}(a)y) \quad + \quad & r_{\vdash_B}(r_{\vdash_A}(y)a)\alpha_1(x) = \\
& l_{\vdash_B}(l_A(x)a)\alpha_2(y) + (r_B(a)x) \vdash_A \alpha_2(y),
\end{aligned}$$

$$l_{\vdash_B}(\beta_1(a))(x \vdash_A y) = (l_B(a)x) \vdash_A \alpha_2(y) + l_{\vdash_B}(r_A(x)a)\alpha_2(y).$$

Then tuple $(A, B, l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, \beta_1, \beta_2, l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, \alpha_1, \alpha_2)$ is called a matched pair of BiHom-dendriform algebras. In this case, on the direct sum $A \oplus B$ of the underlying vector spaces of A and B, there exists a BiHom-dendriform algebra structure given by

$$\begin{aligned}
(x + a) \dashv (y + b) = x \dashv_A y &+ (l_{\dashv_A}(x)b + r_{\dashv_A}(y)a) \\
&+ a \dashv_B b + (l_{\dashv_B}(a)y + r_{\dashv_B}(b)x), \\
(x + a) \vdash (y + b) = x \vdash_A y &+ (l_{\vdash_A}(x)b + r_{\vdash_A}(y)a) \\
&+ a \vdash_B b + (l_{\vdash_B}(a)y + r_{\vdash_B}(b)x), \\
(\alpha_1 \oplus \beta_1)(x + a) = \alpha_1(x) + \beta_1(a), &\quad (\alpha_2 \oplus \beta_2)(x + a) = \alpha_2(x) + \beta_2(a).
\end{aligned}$$

We denote this BiHom-dendriform algebra by $A \bowtie^{l_A, r_A, \beta_1, \beta_2}_{l_B, r_B, \alpha_1, \alpha_2} B$.

Definition 3.7. A BiHom-tridendriform algebra is a tuple $(T, \dashv, \vdash, \cdot, \alpha, \beta)$ consisting of a linear space T, three bilinear maps $\dashv, \vdash, \cdot : T \otimes T \to T$, and two linear maps $\alpha, \beta : T \to T$ satisfying for any $x, y, z \in T$,

$$\alpha \circ \beta = \beta \circ \alpha, \tag{29}$$

$$\alpha(x \dashv y) = \alpha(x) \dashv \alpha(y), \qquad \alpha(x \vdash y) = \alpha(x) \vdash \alpha(y), \tag{30}$$

$$\beta(x \dashv y) = \beta(x) \dashv \beta(y), \qquad \beta(x \vdash y) = \beta(x) \vdash \beta(y), \tag{31}$$

$$\beta(x \cdot y) = \beta(x) \cdot \beta(y), \qquad \beta(x \cdot y) = \beta(x) \cdot \beta(y), \tag{32}$$

$$(x \dashv y) \dashv \beta(z) = \alpha(x) \dashv (y \dashv z + y \vdash z + y \cdot z), \tag{33}$$
$$(x \vdash y) \dashv \beta(z) = \alpha(x) \vdash (y \dashv z), \tag{34}$$
$$\alpha(x) \vdash (y \vdash z) = (x \dashv y + x \vdash y + x \cdot y) \vdash \beta(z), \tag{35}$$
$$(x \dashv y) \cdot \beta(z) = \alpha(x) \cdot (y \vdash z), \tag{36}$$
$$(x \vdash y) \cdot \beta(z) = \alpha(x) \vdash (y \cdot z), \tag{37}$$
$$(x \cdot y) \dashv \beta(z) = \alpha(x) \cdot (y \dashv z), \tag{38}$$
$$(x \cdot y) \cdot \beta(z) = \alpha(x) \cdot (y \cdot z). \tag{39}$$

Remark 3.8. BiHom-tridendriform algebras are a BiHom-X algebras. Also, when the BiHom-associative product "$\cdot$" is identically null, we get a BiHom-dendriform algebra.

Proposition 3.9. Let $(T, \dashv, \vdash, \cdot, \alpha, \beta)$ be a BiHom-tridendriform algebra, and for any $x, y \in T$, $x \vdash' y = x \vdash y + x \cdot y$. Then $(T, \dashv, \vdash', \alpha, \beta)$ is a BiHom-dendriform algebra.

Proof. We prove only one axiom, as others are proved similarly. For any $x, y, z \in T$,

$$\begin{aligned}(x \vdash' y) \dashv \beta(z) &= (x \vdash y + x \cdot y) \dashv \beta(z)\\ &= \alpha(x) \vdash (y \dashv z) + \alpha(x) \cdot (y \dashv z) \quad \text{(by (34) and (38))}\\ &= \alpha(x) \vdash' (y \dashv z).\end{aligned}$$ □

Theorem 3.10. Let $(A, \cdot, \alpha, \beta, R)$ be some Rota-Baxter BiHom-associative algebra of weight λ, and three new operations $\dashv, \vdash$ and $*$ on A are defined by $x \dashv y = x \cdot R(y), \quad x \vdash y = R(x) \cdot y, \quad x * y = \lambda x \cdot y$. Then $(A, \dashv, \vdash, *, \alpha, \beta)$ is a BiHom-tridendriform algebra.

Proof. We prove only one axiom, as others are proved similarly. For any $x, y, z \in A$,

$$\begin{aligned}(x \vdash y) \dashv \beta(z) &= (R(x) \cdot y) \dashv \beta(z)\\ &= (R(x) \cdot y) \cdot R(\beta(z)) = \alpha(R(x)) \cdot (y \cdot R(z))\\ &= \alpha(R(x)) \cdot (y \dashv z) = \alpha(x) \vdash (y \dashv z) = \alpha(x) \vdash (y \dashv z).\end{aligned}$$ □

In Theorem 3.11, we show how to associate a BiHom-associative algebra to any BiHom-tridendriform algebra.

Theorem 3.11. Let $(T, \dashv, \vdash, \cdot, \alpha, \beta)$ be a BiHom-tridendriform algebra, and $x * y = x \vdash y + x \dashv y + x \cdot y$, for all $x, y \in A$. Then $(T, *, \alpha, \beta)$ is a BiHom-associative algebra.

Proof. For any $x, y, z \in T$,

$$\begin{aligned}(x * y) * \beta(z) - \alpha(x) * (y * z) &= (x \vdash y) \vdash \beta(z) + (x \dashv y) \vdash \beta(z) \\ &+ (x \cdot y) \vdash \beta(z) + (x \vdash y) \dashv \beta(z) + (x \dashv y) \dashv \beta(z) + (x \cdot y) \dashv \beta(z) \\ &+ (x \vdash y) \cdot \beta(z) + (x \dashv y) \cdot \beta(z) + (x \cdot y) \cdot \beta(z) \\ &- \alpha(x) \vdash (y \vdash z) - \alpha(x) \vdash (y \dashv z) - \alpha(x) \vdash (y \cdot z) \\ &- \alpha(x) \dashv (y \vdash z) - \alpha(x) \dashv (y \dashv z) - \alpha(x) \dashv (y \cdot z) \\ &- \alpha(x) \cdot (y \vdash z) - \alpha(x) \cdot (y \dashv z) - \alpha(x) \cdot (y \cdot z).\end{aligned}$$

The left hand side vanishes by axioms in Definition 3.7. This proves that $(T, *, \alpha, \beta)$ is a BiHom-associative algebra. □

Now, we introduce the notion of bimodule of BiHom-tridendriform algebra.

Definition 3.12. Let $(T, \dashv, \vdash, \cdot, \alpha_1, \alpha_2)$ be a BiHom-tridendriform algebra, and V be a vector space. Let $l_\dashv, r_\dashv, l_\vdash, r_\vdash, l_., r_. : T \to gl(V)$, and $\beta_1, \beta_2 : V \to V$ be eight linear maps. Then, $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, l_., r_., \beta_1, \beta_2, V)$ is called a bimodule of T if the following equations hold for any $x, y \in T$ and $v \in V$:

$$\begin{aligned}
&l_\dashv(x \dashv y)\beta_2(v) = l_\dashv(\alpha_1(x))l_*(y)v, && r_\dashv(\alpha_2(x))l_\dashv(y)v = l_\dashv(\alpha_1(y))r_*(x)v,\\
&r_\dashv(\alpha_2(y))r_\dashv(y)v = r_\dashv(x * y)\beta_1(v), && l_\dashv(x \vdash y)\beta_2(v) = l_\vdash(\alpha_1(x))l_\dashv(y)v,\\
&r_\dashv(\alpha_2(x))l_\vdash(y)v = l_\vdash(\alpha_1(y))r_\dashv(x)v, && r_\dashv(\alpha_2(x))r_\vdash(y)v = r_\vdash(y \dashv x)\beta_1(v),\\
&l_\vdash(x * y)\beta_2(v) = l_\vdash(\alpha_1(x))l_\vdash(y)v, && r_\vdash(\alpha_2(x))l_*(y)v = l_\vdash(\alpha_1(y))r_\vdash(x)v,\\
&r_\vdash(\alpha_2(x))r_*(y)v = r_\vdash(y \vdash x)\beta_1(v), && l_.(x \dashv y)\beta_2(v) = l_.(\alpha_1(x))l_\vdash(y)v,\\
&r_.(\alpha_2(x))l_\dashv(y)v = l_.(\alpha_1(y))r_\vdash(x)v, && r_.(\alpha_2(x))r_\dashv(y)v = r_.(y \vdash x)\beta_1(v),\\
&l_.(x \vdash y)\beta_2(v) = l_\vdash(\alpha_1(x))l_.(y)v, && r_.(\alpha_2(x))l_\vdash(y)v = l_\vdash(\alpha_1(y))r_.(x)v,\\
&r_.(\alpha_2(x))r_\vdash(y)v = r_\vdash(y \cdot x)\beta_1(v), && l_\dashv(x \cdot y)\beta_2(v) = l_.(\alpha_1(x))l_\dashv(y)v,\\
&r_\dashv(\alpha_2(x))l_.(y)v = l_.(\alpha_1(y))r_\dashv(x)v, && r_\dashv(\alpha_2(x))r_.(y)v = r_.(y \dashv x)\beta_1(v),\\
&l_.(x \cdot y)\beta_2(v) = l_.(\alpha_1(x))l_.(y)v, && r_.(\alpha_2(x))l_.(y)v = l_.(\alpha_1(y))r_.(x)v,\\
&r_.(\alpha_2(x))r_.(y)v = r_.(y \cdot x)\beta_1(v), && \beta_1(l_\vdash(x)v) = l_\vdash(\alpha_1(x))\beta_1(v),\\
&\beta_1(r_\vdash(x)v) = r_\vdash(\alpha_1(x))\beta_1(v), && \beta_2(l_\vdash(x)v) = l_\vdash(\alpha_2(x))\beta_2(v),\\
&\beta_2(r_\vdash(x)v) = r_\vdash(\alpha_2(x))\beta_2(v), && \beta_1(l_\dashv(x)v) = l_\dashv(\alpha_1(x))\beta_1(v),\\
&\beta_1(r_\dashv(x)v) = r_\dashv(\alpha_1(x))\beta_1(v), && \beta_2(l_\dashv(x)v) = l_\dashv(\alpha_2(x))\beta_2(v),\\
&\beta_2(r_\dashv(x)v) = r_\dashv(\alpha_2(x))\beta_2(v),
\end{aligned}$$

where $x * y = x \dashv y + x \vdash y + x \cdot y, l_* = l_\dashv + l_\vdash + l_., r_* = r_\dashv + r_\vdash + r_..$

Proposition 3.13. Let $(l_{\dashv}, r_{\dashv}, l_{\vdash}, r_{\vdash}, l_{\cdot}, r_{\cdot}, \beta_1, \beta_2, V)$ be a bimodule of a BiHom-tridendriform algebra $(T, \dashv, \vdash, \cdot, \alpha_1, \alpha_2)$. Then, on the direct sum $T \oplus V$ of the underlying vector spaces of T and V, there exists a BiHom-tridendriform algebra structure given, for all $x, y \in T, u, v \in V$, by

$$
\begin{aligned}
(x+u) \dashv' (y+v) &= x \dashv y + l_{\dashv}(x)v + r_{\dashv}(y)u, \\
(x+u) \vdash' (y+v) &= x \vdash y + l_{\vdash}(x)v + r_{\vdash}(y)u, \\
(x+u) \cdot (y+v) &= x \cdot y + l_{\cdot}(x)v + r_{\cdot}(y)u.
\end{aligned}
$$

We denote it by $T \times_{l_{\dashv}, r_{\dashv}, l_{\vdash}, r_{\vdash}, l_{\cdot}, r_{\cdot}, \alpha_1, \alpha_2, \beta_1, \beta_2} V$.

Proof. We prove only one axiom, as others are proved similarly. For any $x_1, x_2, x_3 \in T$ and $v_1, v_2, v_3 \in V$,

$$
\begin{aligned}
&((x_1+v_1) \vdash' (x_2+v_2)) \dashv' (\alpha_2+\beta_2)(x_3+v_3) \\
&\quad = (x_1 \vdash x_2 + l_{\vdash}(x_1)v_2 + r_{\vdash}(x_2)v_1) \dashv' (\alpha_2(x_3) + \beta_2(v_3)) \\
&\quad = (x_1 \vdash x_2) \dashv \alpha_2(x_3) + l_{\dashv}(x_1 \vdash x_2)\beta_2(v_3) \\
&\qquad + r_{\dashv}(\alpha_2(x_3))l_{\vdash}(x_1)v_1 + r_{\dashv}(\alpha_2(x_3))r_{\vdash}(x_2)v_1. \\
&(\alpha_1+\beta_1)(x_1+v_1) \vdash' ((x_2+v_2) \dashv' (x_3+v_3)) \\
&\quad = (\alpha_1(x_1) + \beta_1(v_1)) \vdash' (x_2 \dashv x_3 + l_{\dashv}(x_2)v_3 + r_{\dashv}(x_3)v_2) \\
&\quad = \alpha_1(x_1) \vdash (x_2 \dashv x_3) + l_{\vdash}(\alpha_1(x_1))l_{\dashv}(x_2)v_3 \\
&\qquad + l_{\vdash}(\alpha_1(x_1))r_{\dashv}(x_3)v_2 + r_{\vdash}(x_2+x_3)\beta_1(v_1).
\end{aligned}
$$

We deduce that $((x_1+v_1) \vdash' (x_2+v_2)) \dashv' (\alpha_2+\beta_2)(x_3+v_3) = ((x_1+v_1) \vdash' (x_2+v_2)) \dashv' (\alpha_2+\beta_2)(x_3+v_3)$. This ends the proof. □

Examples 3.14. Some examples of bimodules of BiHom-tridendriform algebra can be constructed as follows.
1) Let $(T, \dashv, \vdash, \cdot, \alpha, \beta)$ be a BiHom-tridendriform algebra. Then

$$(L_{\dashv}, R_{\dashv}, L_{\vdash}, R_{\vdash}, L_{\cdot}, R_{\cdot}, \alpha, \beta, T) \text{ is a bimodule of } T,$$

where for all $(a, b) \in T^{\times 2}$,

$$
\begin{aligned}
L_{\dashv}(a)b &= a \dashv b, & R_{\dashv}(a)b &= b \dashv a, \\
L_{\vdash}(a)b &= a \vdash b, & R_{\vdash}(a)b &= b \vdash a, \\
L_{\cdot}(a)b &= a \cdot b, & R_{\cdot}(a)b &= b \cdot a.
\end{aligned}
$$

More generally, if B is a two-sided BiHom-ideal of $(T, \dashv, \vdash, \cdot, \alpha, \beta)$, then

$$(L_{\dashv}, R_{\dashv}, L_{\vdash}, R_{\vdash}, L_{\cdot}, R_{\cdot}, \alpha, \beta, B) \text{ is a bimodule of } T,$$

where for all $x \in B$ and $(a,b) \in T^{\times 2}$,

$$L_{\dashv}(a)x = a \dashv x = x \dashv a = R_{\dashv}(a)x,$$
$$L_{\vdash}(a)x = a \vdash x = x \vdash a = R_{\vdash}(a)x,$$
$$L_{\cdot}(a)x = a \cdot x = x \cdot a = R_{\cdot}(a)x.$$

2) If $(l_{\dashv}, r_{\dashv}, l_{\vdash}, r_{\vdash}, l_{\cdot}, r_{\cdot}, V)$ is a bimodule of a BiHom-tridendriform algebra $(T, \dashv, \vdash, \cdot)$, then $(l_{\dashv}, r_{\dashv}, l_{\vdash}, r_{\vdash}, l_{\cdot}, r_{\cdot}, Id_V, Id_V, V)$ is a bimodule of $\mathbb{T}$, where $\mathbb{T} = (T, \dashv, \vdash, \cdot, Id_T, Id_T)$ is a BiHom-tridendriform algebra.

Proposition 3.15. Let $f : (T, \dashv_1, \vdash_1, \cdot_1, \alpha_1, \alpha_2) \to (T', \dashv_2, \vdash_2, \cdot_2, \beta_1, \beta_2)$ be a morphism of BiHom-tridendriform algebras.

Then $(l_{\dashv_1}, r_{\dashv_1}, l_{\vdash_1}, r_{\vdash_1}, l_{\cdot_1}, r_{\cdot_1}, \beta_1, \beta_2, T')$ is a bimodule of T via f, that is, for all $(a,b) \in T \times T'$,

$$l_{\dashv_1}(a)b = f(a) \dashv_2 b, \qquad r_{\dashv_1}(a)b = b \dashv_2 f(a),$$
$$l_{\vdash_1}(a)b = f(a) \vdash_2 b, \qquad r_{\vdash_1}(a)b = b \vdash_2 f(a),$$
$$l_{\cdot_1}(a)b = f(a) \cdot_2 b, \qquad r_{\cdot_1}(a)b = b \cdot_2 f(a).$$

Proof. We prove only one axiom, since other axioms are proved similarly. For any $x, y \in T$ and $z \in T'$,

$$\begin{aligned} l_{\vdash_1}(x \dashv_1 y)\beta_2(z) &= f(x \dashv_1 y) \dashv_2 \beta_2(z) \\ &= (f(x) \dashv_2 f(y)) \dashv_2 \beta_2(z) = \beta_1 f(x) \dashv_2 (f(y) *_2 z) \\ &= f(\alpha_1(x)) \dashv_2 l_{*_1}(y)z = l_{\dashv_1}(\alpha_1(x)) l_{*_1}(y)z. \end{aligned}$$ □

Definition 3.16. An abelian extension of BiHom-tridendriform algebra is a short exact sequence of BiHom-tridendriform algebra

$$0 \to (V, \alpha_V, \beta_V) \xrightarrow{\text{i}} (T, \dashv_T, \vdash_T, \cdot_T, \alpha_T, \beta_T) \xrightarrow{\pi} (T', \dashv_{T'}, \vdash_{T'}, \cdot_{T'}, \alpha_{T'}, \beta_{T'}) \to 0$$

where (V, α_V, β_V) is a trivial BiHom-tridendriform algebra, i and π are morphisms of BiHom-algebras. Furthermore, if there is a morphism $s : (T', \dashv_{T'}, \vdash_{T'}, \cdot_{T'}, \alpha_{T'}, \beta_{T'}) \longrightarrow (T, \dashv_T, \vdash_T, \cdot_T, \alpha_T, \beta_T)$ such that $\pi \circ s = id_{T'}$, then the abelian extension is said to be split and s is called a section of π.

Remark 3.17. Consider the split null extension $T \oplus V$ determined by the bimodule $(l_{\dashv}, r_{\dashv}, l_{\vdash}, r_{\vdash}, l_{\cdot}, r_{\cdot}, \alpha_V, \beta_V, V)$ for the BiHom-tridendriform algebra $(T, \dashv_T, \vdash_T, \cdot_T, \alpha, \beta)$ in the previous proposition. Write elements $a + v$ of $T \oplus V$ as (a, v). Then there is an injective homomorphism of BiHom-modules $i : V \to T \oplus V$ given by $i(v) = (0, v)$ and a surjective homomorphism of BiHom-modules $\pi : T \oplus V \to T$ given by $\pi(a, v) = a$. Moreover, $i(V)$ is a two-sided BiHom-ideal of $T \oplus V$ such that $T \oplus V/i(V) \cong T$. On the other hand, there is a morphism of BiHom-algebras $\sigma : T \to T \oplus V$ given by $\sigma(a) = (a, 0)$ which is clearly a section of π. Hence, we obtain the abelian split exact sequence of BiHom-tridendriform algebra and $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, l_., r_., \alpha_V, \beta_V, V)$ is a bimodule for T via π.

Proposition 3.18. Let $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, l_., r_., \beta_1, \beta_2, V)$ be a bimodule of a BiHom-tridendriform algebra $(T, \dashv, \vdash, \cdot, \alpha_1, \alpha_2)$. Let $(T, *, \alpha_1, \alpha_2)$ be the associated BiHom-associative algebra. Then, $(l_\dashv + l_\vdash + l_., r_\dashv + r_\vdash + r_., \beta_1, \beta_2, V)$ is a bimodule of $(T, *, \alpha_1, \alpha_2)$.

Proof. We prove only one axiom. The other axioms are proved similarly. For any $x, y \in A, v \in V$,

$$\begin{aligned}
&l_*(x * y)\beta_2(v) = (l_\dashv + l_\vdash + l_.)(x * y)\beta_2(v)\\
&= (l_\dashv + l_\vdash + l_.)(x \dashv y + x \vdash y + x \cdot y)\beta_2(v)\\
&= l_\dashv(x \dashv y)\beta_2(v) + l_\dashv(x \vdash y)\beta_2(v) + l_\dashv(x \cdot y)\beta_2(v) + l_\vdash(x * y)\beta_2(v)\\
&\qquad + l_.(x \dashv y)\beta_2(v) + l_.(x \vdash y)\beta_2(v) + l_.(x \cdot y)\beta_2(v)\\
&= l_\dashv(\alpha_1(x))l_*(y)v + l_\vdash(\alpha_1(x))l_\dashv(y)v\\
&\qquad + l_.(\alpha_1(x))l_\dashv(y)v + l_\vdash(\alpha_1(x))l_\vdash(y)v\\
&\qquad + l_.(\alpha_1(x))l_\vdash(y)v + l_\vdash(\alpha_1(x))l_.(y)v + l_.(\alpha_1(x))l_.(y)v\\
&= (l_\dashv + l_\vdash + l_.)(\alpha_1(x))(l_\dashv + l_\vdash + l_.)(y)v = l_*(\alpha_1(x))l_*(y)v. \qquad \square
\end{aligned}$$

Theorem 3.19. Let $(T, \dashv, \vdash, \cdot, \alpha_1, \alpha_2)$ be a BiHom-tridendriform algebra, and $V_{\beta_1,\beta_2} = (l_\dashv, r_\dashv, l_\vdash, r_\vdash, l_., r_., \beta_1, \beta_2, V)$ be a bimodule of T. Let α'_1, α'_2 be two endomorphisms of T such that any two of the maps $\alpha_1, \alpha'_1, \alpha_2, \alpha'_2$ commute and $\beta'_1,\ \beta'_2$ be linear maps of V such that any two of the maps $\beta_1, \beta'_1, \beta_2, \beta'_2$ commute. Suppose furthermore that

$$\begin{cases}
\beta'_1 \circ l_\dashv = (l_\dashv \circ \alpha'_1)\beta'_1, & \beta'_2 \circ l_\dashv = (l_\dashv \circ \alpha'_2)\beta'_2,\\
\beta'_1 \circ l_\vdash = (l_\vdash \circ \alpha'_1)\beta'_1, & \beta'_2 \circ l_\vdash = (l_\vdash \circ \alpha'_2)\beta'_2,\\
\beta'_1 \circ l_. = (l_. \circ \alpha'_1)\beta'_1, & \beta'_2 \circ l_. = (l_. \circ \alpha'_2)\beta'_2,
\end{cases}$$

$$\begin{cases}
\beta'_1 \circ r_\dashv = (r_\dashv \circ \alpha'_1)\beta'_1, & \beta'_2 \circ r_\dashv = (r_\dashv \circ \alpha'_2)\beta'_2,\\
\beta'_1 \circ r_\vdash = (r_\vdash \circ \alpha'_1)\beta'_1, & \beta'_2 \circ r_\vdash = (r_\vdash \circ \alpha'_2)\beta'_2,\\
\beta'_1 \circ r_. = (r_. \circ \alpha'_1)\beta'_1, & \beta'_2 \circ r_. = (r_. \circ \alpha'_2)\beta'_2,
\end{cases}$$

and write $T_{\alpha'_1,\alpha'_2} = (T, \dashv_{\alpha'_1,\alpha'_2}, \vdash_{\alpha'_1,\alpha'_2}, \cdot_{\alpha'_1,\alpha'_2}, \alpha_1\alpha'_1, \alpha_2\alpha'_2)$ for the BiHom-tridendriform algebra, and $V_{\beta'_1,\beta'_2} = (\widetilde{l}_\dashv, \widetilde{r}_\dashv, \widetilde{l}_\vdash, \widetilde{r}_\vdash, \widetilde{l}_., \widetilde{r}_., \beta_1\beta'_1, \beta_2\beta'_2, V)$, where

$$\widetilde{l}_\dashv = (l_\dashv \circ \alpha'_1)\beta'_2, \qquad \widetilde{r}_\dashv = (r_\dashv \circ \alpha'_2)\beta'_1,$$
$$\widetilde{l}_\vdash = (l_\vdash \circ \alpha'_1)\beta'_2, \qquad \widetilde{r}_\vdash = (r_\vdash \circ \alpha'_2)\beta'_1,$$
$$\widetilde{l}_\cdot = (l_\cdot \circ \alpha'_1)\beta'_2, \qquad \widetilde{r}_\cdot = (r_\cdot \circ \alpha'_2)\beta'_1.$$

This gives the BiHom-module $V_{\beta'_1,\beta'_2}$ the structure of $T_{\alpha'_1,\alpha'_2}$-bimodule.

Proof. We prove only one axiom, since other axioms are proved similarly. For any $x, y \in T$ and $v \in V$,

$$\widetilde{l}_\dashv(x \dashv_{\alpha'_1\alpha'_2} y)\beta_2\beta'_2(v) = \widetilde{l}_\dashv(\alpha'(x) \dashv_{\alpha'_1\alpha'_2} \alpha'(y))\beta_2\beta'_2(v)$$
$$= l_\dashv(\alpha'2^{(}_1x) \dashv \alpha'_1\alpha'_2(y))\beta_2\beta'2^{(}_2v) = l_\dashv(\alpha_1\alpha'^2_1(x))l_*(\alpha'_1\alpha'_2(y))\beta'^2_2(v)$$
$$= \widetilde{l}_\dashv(\alpha_1\alpha'_1(x))l_*(\alpha'_1(y))\beta'_2(v) = \widetilde{l}_\dashv(\alpha_1\alpha'_1(x))\widetilde{l}_*(y)v. \qquad \square$$

Let $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, l_\cdot, r_\cdot, \beta_1, \beta_2, V)$ be a bimodule of a BiHom-tridendriform algebra $(T, \dashv, \vdash, \cdot, \alpha_1, \alpha_2)$ and $l^*_\dashv, r^*_\dashv, l^*_\vdash, r^*_\vdash, l^*_\cdot, r^*_\cdot : T \to gl(V^*)$. Let $\alpha^*_1, \alpha^*_2 : T^* \to T^*$, $\beta^*_1, \beta^*_2 : V^* \to V^*$ be the dual maps of respectively $\alpha_1, \alpha_2, \beta_1$ and β_2 such that

$$\langle l^*_\dashv(x)u^*, v\rangle = \langle u^*, l_\dashv(x)v\rangle, \qquad \langle r^*_\vdash(x)u^*, v\rangle = \langle u^*, r_\vdash(x)v\rangle,$$
$$\langle l^*_\vdash(x)u^*, v\rangle = \langle u^*, l_\vdash(x)v\rangle, \qquad \langle r^*_\vdash(x)u^*, v\rangle = \langle u^*, r_\vdash(x)v\rangle,$$
$$\langle l^*_\cdot(x)u^*, v\rangle = \langle u^*, l_\cdot(x)v\rangle, \qquad \langle r^*_\cdot(x)u^*, v\rangle = \langle u^*, r_\cdot(x)v\rangle,$$
$$\alpha^*_1(x^*(y) = x^*(\alpha_1(y)), \qquad \alpha^*_2(x^*(y) = x^*(\alpha_2(y)),$$
$$\beta^*_1(u^*(v) = u^*(\beta_1(v)), \qquad \beta^*_2(u^*(v) = u^*(\beta_2(v)).$$

Proposition 3.20. Let $(l_\dashv, r_\dashv, l_\vdash, r_\vdash, l_\cdot, r_\cdot, \beta_1, \beta_2, V)$ be a bimodule of a BiHom-tridendriform algebra $(T, \dashv, \vdash, \cdot, \alpha_1, \alpha_2)$.

Then $(l^*_\dashv, r^*_\dashv, l^*_\vdash, r^*_\vdash, l^*_\cdot, r^*_\cdot, \beta^*_1, \beta^*_2, V^*)$ is a bimodule of $(T, \dashv, \vdash, \cdot, \alpha_1, \alpha_2)$, provided that for all $x, y \in T$ and $u \in V$,

$$\beta_2(l_\dashv(x \dashv y))u = l_*(y)l_\dashv(\alpha_1(x))u, \quad l_\dashv(y)r_\dashv(\alpha_2(x))u = r_*(x)l_\dashv(\alpha_1(y))u,$$
$$r_\dashv(y)r_\dashv(\alpha_2(y))u = \beta_1(r_\dashv(x * y))u, \quad \beta_2(l_\dashv(x \vdash y))u = l_\dashv(y)l_\vdash(\alpha_1(x))u,$$
$$l_\vdash(y)r_\dashv(\alpha_2(x))u = r_\dashv(x)l_\vdash(\alpha_1(y))u, \quad r_\vdash(y)r_\dashv(\alpha_2(x))u = \beta_1(r_\vdash(y \dashv x))u,$$
$$\beta_2(l_\vdash(x * y))u = l_\vdash(y)l_\vdash(\alpha_1(x))u, \quad l_*(y)r_\vdash(\alpha_2(x))u = r_\vdash(x)l_\vdash(\alpha_1(y))u,$$
$$r_*(y)r_\vdash(\alpha_2(x))u = \beta_1(r_\vdash(y \vdash x))u, \quad \beta_2(l_\cdot(x \dashv y))u = l_\vdash(y)l_\cdot(\alpha_1(x))u,$$
$$l_\dashv(y)r_\cdot(\alpha_2(x))u = r_\vdash(x)l_\cdot(\alpha_1(y))u, \quad r_\dashv(y)r_\cdot(\alpha_2(x))u = \beta_1(r_\cdot(y \vdash x))u,$$
$$\beta_2(l_\cdot(x \vdash y))u = l_\cdot(y)l_\vdash(\alpha_1(x))u, \quad l_\vdash(y)r_\cdot(\alpha_2(x))u = r_\cdot(x)l_\vdash(\alpha_1(y))u,$$
$$r_\vdash(y)r_\cdot(\alpha_2(x))u = \beta_1(r_\vdash(y \cdot x))u, \quad \beta_2(l_\dashv(x \cdot y))u = l_\dashv(y)l_\cdot(\alpha_1(x))u,$$

$$l_{\cdot}(y)r_{\dashv}(\alpha_2(x))u = r_{\dashv}(x)l_{\cdot}(\alpha_1(y))u, \quad r_{\cdot}(y)r_{\dashv}(\alpha_2(x))u = \beta_1(r_{\cdot}(y \dashv x))u,$$
$$\beta_2(l_{\cdot}(x \cdot y))u = l_{\cdot}(y)l_{\cdot}(\alpha_1(x))u, \quad l_{\cdot}(y)r_{\cdot}(\alpha_2(x))u = r_{\cdot}(x)l_{\cdot}(\alpha_1(y))u,$$
$$r_{\cdot}(y)r_{\cdot}(\alpha_2(x))u = \beta_1(r_{\cdot}(y \cdot x))u,$$

where $x * y = x \dashv y + x \vdash y + x \cdot y, l_* = l_{\dashv} + l_{\vdash} + l_{\cdot}, r_* = r_{\dashv} + r_{\vdash} + r_{\cdot}$.

Theorem 3.21. Let $(A, \dashv_A, \vdash_A, \cdot_A, \alpha_1, \alpha_2)$ and $(B, \dashv_B, \vdash_B, \cdot_B, \beta_1, \beta_2)$ be two BiHom-tridendriform algebras. Suppose that there are linear maps

$$l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, l_{\cdot_A}, r_{\cdot_A} : A \to gl(B),$$
$$l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, l_{\cdot_B}, r_{\cdot_B} : B \to gl(A),$$

such that

$$(l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, l_{\cdot_A}, r_{\cdot_A}, \beta_1, \beta_2, B) \text{ is a bimodule of } A,$$
$$(l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, l_{\cdot_B}, r_{\cdot_B}\alpha_1, \alpha_2, A) \text{ is a bimodule of } B,$$

and for

$$x *_A y = x \dashv_A y + x \vdash_A y + x \cdot_A y, \quad l_A = l_{\dashv_A} + l_{\vdash_A} + l_{\cdot_A},$$
$$r_A = r_{\dashv_A} + r_{\vdash_A} + r_{\cdot_A},$$
$$a *_B b = a \dashv_B b + a \vdash_B b + a \cdot_B b, \quad l_B = l_{\dashv_B} + l_{\vdash_B} + l_{\cdot_B},$$
$$r_B = r_{\dashv_B} + r_{\vdash_B} + r_{\cdot_B}.$$

and for any $x, y \in A,\ a, b \in B$,

$$r_{\dashv_A}(\alpha_2(x))(a \dashv_B b) = r_{A(l_B}(b)x)\beta_1(a) + \beta_1(a) \dashv_B (r_{\dashv_A}(x)b), \tag{40}$$
$$l_{\dashv_A}(l_{\dashv_B}(a)x)\beta_2(b) + (r_{\dashv_A}(x)a) \dashv_B \beta_2(b) = \beta_1(a) \dashv_B (l_{\dashv_A}(x)b) + r_{\dashv_A}(r_{\dashv_B}(b)x)\beta_1(a), \tag{41}$$
$$l_{\dashv_A}(\alpha_1(x))(a *_B b) = (l_{\dashv_A}(x)a) *_B \beta_2(b) + l_{\dashv_A}(r_{\dashv_B}(a)x)\beta_2(b), \tag{42}$$
$$r_{\dashv_A}(\alpha_2(x))(a \vdash_B b) = r_{\vdash_A}(l_{\dashv_B}(b)x)\beta_1(a) + \beta_1(a) \vdash_B (r_{\dashv_A}(x)b), \tag{43}$$
$$l_{\dashv_A}(l_{\vdash_B}(a)x)\beta_2(b) + (r_{\vdash_A}(x)a) \dashv_B \beta_2(b) = \beta_1(a) \vdash_B (l_{\dashv_A}(x)b) + r_{\vdash_A}(r_{\dashv_B}(b)x)\beta_1(a), \tag{44}$$
$$l_{\dashv_A}(\alpha_1(x))(a \dashv_B b) = (l_{\vdash_A}(x)a) \dashv_B \beta_2(b) + l_{\dashv_A}(r_{\vdash_B}(a)x)\beta_2(b), \tag{45}$$
$$r_{\vdash_A}(\alpha_2(x))(a *_B b) = r_{\vdash_A}(l_{\vdash_B}(b)x)\beta_1(a) + \beta_1(a) \vdash_B (r_{\vdash_A}(x)b), \tag{46}$$
$$l_{\vdash_A}(l_A(a)x)\beta_2(b) + (r_A(x)a) \vdash_B \beta_2(b) = \beta_1(a) \vdash_B (l_{\vdash_A}(x)b) + r_{\vdash_A}(r_{\vdash_B}(b)x)\beta_1(a), \tag{47}$$
$$l_{\vdash_A}(\alpha_1(x))(a \vdash_B b) = (l_{\vdash_A}(x)a) \vdash_B \beta_2(b) + l_A(r_B(a)x)\beta_2(b), \tag{48}$$
$$r_{\cdot_A}(\alpha_2(x))(a \dashv_B b) = r_{\cdot_A}(l_{\vdash_B}(b)x)\beta_1(a) + \beta_1(a) \cdot_B (r_{\vdash_A}(x)b), \tag{49}$$

$$\begin{aligned} l_{\cdot_A}(l_{\dashv_B}(a)x)\beta_2(b) \quad &+(r_{\dashv_A}(x)a)\cdot_B \beta_2(b) = \\ &\beta_1(a)\cdot_B (l_{\vdash_A}(x)b) + r_{\cdot_A}(r_{\vdash_B}(b)x)\beta_1(a), \end{aligned} \tag{50}$$

$$l_{\cdot_A}(\alpha_1(x))(a\vdash_B b) = (l_{\dashv_A}(x)a)\cdot_B \beta_2(b) + l_{\cdot_A}(r_{\dashv_B}(a)x)\beta_2(b), \tag{51}$$

$$r_{\cdot_A}(\alpha_2(x))(a\vdash_B b) = r_{\vdash_A}(l_{\cdot_B}(b)x)\beta_1(a) + \beta_1(a)\vdash_B (r_{\cdot_A}(x)b), \tag{52}$$

$$\begin{aligned} l_{\cdot_A}(l_{\vdash_B}(a)x)\beta_2(b) \quad &+(r_{\vdash_A}(x)a)\cdot_B \beta_2(b) = \\ &\beta_1(a)\vdash_B (l_{\cdot_A}(x)b) + r_{\vdash_A}(r_{\cdot_B}(b)x)\beta_1(a), \end{aligned} \tag{53}$$

$$l_{\vdash_A}(\alpha_1(x))(a\cdot_B b) = (l_{\vdash_A}(x)a)\cdot_B \beta_2(b) + l_{\cdot_A}(r_{\vdash_B}(a)x)\beta_2(b), \tag{54}$$

$$r_{\dashv_A}(\alpha_2(x))(a\cdot_B b) = r_{\cdot_A}(l_{\dashv_B}(b)x)\beta_1(a) + \beta_1(a)\cdot_B (r_{\dashv_A}(x)b), \tag{55}$$

$$\begin{aligned} l_{\dashv_A}(l_{\cdot_B}(a)x)\beta_2(b) \quad &+(r_{\cdot_A}(x)a)\dashv_B \beta_2(b) = \\ &\beta_1(a)\cdot_B (l_{\dashv_A}(x)b) + r_{\cdot_A}(r_{\dashv_B}(b)x)\beta_1(a), \end{aligned} \tag{56}$$

$$l_{\cdot_A}(\alpha_1(x))(a\dashv_B b) = (l_{\cdot_A}(x)a)\dashv_B \beta_2(b) + l_{\dashv_A}(r_{\cdot_B}(a)x)\beta_2(b), \tag{57}$$

$$r_{\cdot_A}(\alpha_2(x))(a\cdot_B b) = r_{\cdot_A}(l_{\cdot_B}(b)x)\beta_1(a) + \beta_1(a)\cdot_B (r_{\cdot_A}(x)b), \tag{58}$$

$$\begin{aligned} l_{\cdot_A}(l_{\cdot_B}(a)x)\beta_2(b) \quad &+(r_{\cdot_A}(x)a)\cdot_B \beta_2(b) = \\ &\beta_1(a)\cdot_B (l_{\cdot_A}(x)b) + r_{\cdot_A}(r_{\cdot_B}(b)x)\beta_1(a), \end{aligned} \tag{59}$$

$$l_{\cdot_A}(\alpha_1(x))(a\cdot_B b) = (l_{\cdot_A}(x)a)\cdot_B \beta_2(b) + l_{\cdot_A}(r_{\cdot_B}(a)x)\beta_2(b), \tag{60}$$

$$r_{\dashv_B}(\beta_2(a))(x\dashv_A y) = r_{B(l_A}(y)a)\alpha_1(x) + \alpha_1(x)\dashv_A (r_{\dashv_B}(a)y), \tag{61}$$

$$\begin{aligned} l_{\dashv_B}(l_{\dashv_A}(x)a)\alpha_2(y) \quad &+(r_{\dashv_B}(a)x)\dashv_A \alpha_2(y) = \\ &\alpha_1(x)\dashv_B (l_{\dashv_B}(a)y) + r_{\dashv_B}(r_{\dashv_A}(y)a)\alpha_1(x), \end{aligned} \tag{62}$$

$$l_{\dashv_B}(\beta_1(a))(x *_A y) = (l_{\dashv_B}(a)x) *_A \alpha_2(y) + l_{\dashv_B}(r_{\dashv_A}(x)a)\alpha_2(y), \tag{63}$$

$$r_{\dashv_B}(\beta_2(a))(x\vdash_A y) = r_{\vdash_B}(l_{\dashv_A}(y)a)\alpha_1(x) + \alpha_1(x)\vdash_A (r_{\dashv_B}(a)y), \tag{64}$$

$$\begin{aligned} l_{\dashv_B}(l_{\vdash_A}(x)a)\alpha_2(y) \quad &+(r_{\vdash_B}(a)x)\dashv_A \alpha_2(y) = \\ &\alpha_1(x)\vdash_B (l_{\dashv_B}(a)y) + r_{\vdash_B}(r_{\dashv_A}(y)a)\alpha_1(x), \end{aligned} \tag{65}$$

$$l_{\dashv_B}(\beta_1(a))(x\dashv_A y) = (l_{\vdash_B}(a)x)\dashv_A \alpha_2(y) + l_{\dashv_B}(r_{\vdash_A}(x)a)\alpha_2(y), \tag{66}$$

$$r_{\vdash_B}(\beta_2(a))(x *_A y) = r_{\vdash_B}(l_{\vdash_A}(y)a)\alpha_1(x) + \alpha_1(x)\vdash_A (r_{\vdash_B}(a)y), \tag{67}$$

$$\begin{aligned} l_{\vdash_B}(l_B(x)a)\alpha_2(y) \quad &+(r_B(a)x)\vdash_A \alpha_2(y) = \\ &\alpha_1(x)\vdash_B (l_{\vdash_B}(a)y) + r_{\vdash_B}(r_{\vdash_A}(y)a)\alpha_1(x), \end{aligned} \tag{68}$$

$$l_{\vdash_B}(\beta_1(a))(x\vdash_A y) = (l_{\vdash_B}(a)x)\vdash_A \alpha_2(y) + l_B(r_A(x)a)\alpha_2(y), \tag{69}$$

$$r_{\cdot_B}(\beta_2(a))(x\dashv_A y) = r_{\cdot_B}(l_{\vdash_A}(y)x)\alpha_1(x) + \alpha_1(x)\cdot_A (r_{\vdash_B}(a)y), \tag{70}$$

$$\begin{aligned} l_{\cdot_B}(l_{\dashv_A}(x)a)\alpha_2(y) \quad &+(r_{\dashv_B}(a)x)\cdot_A \alpha_2(y) = \\ &\alpha_1(x)\cdot_B (l_{\vdash_B}(a)y) + r_{\cdot_B}(r_{\vdash_A}(y)a)\alpha_1(x), \end{aligned} \tag{71}$$

$$l_{\cdot_B}(\beta_1(a))(x\vdash_A y) = (l_{\dashv_B}(a)x)\cdot_A \alpha_2(y) + l_{\cdot_B}(r_{\dashv_A}(x)a)\alpha_2(y), \tag{72}$$

$$r_{\cdot_B}(\beta_2(a))(x\vdash_A y) = r_{\vdash_B}(l_{\cdot_A}(y)a)\alpha_1(x) + \alpha_1(x)\vdash_A (r_{\cdot_B}(a)y), \tag{73}$$

$$\begin{aligned} l_{\cdot_B}(l_{\vdash_A}(x)a)\alpha_2(y) \quad &+(r_{\vdash_B}(a)x)\cdot_A \alpha_2(y) = \\ &\alpha_1(x)\vdash_B (l_{\cdot_B}(a)y) + r_{\vdash_B}(r_{\cdot_A}(y)a)\alpha_1(x), \end{aligned} \tag{74}$$

$$l_{\vdash_B}(\beta_1(a))(x \cdot_A y) = (l_{\vdash_B}(a)x) \cdot_A \alpha_2(y) + l_{\cdot_B}(r_{\vdash_A}(x)a)\alpha_2(y), \tag{75}$$

$$r_{\dashv_B}(\beta_2(a))(x \cdot_A y) = r_{\cdot_B}(l_{\dashv_A}(y)a)\alpha_1(x) + \alpha_1(x) \cdot_A (r_{\dashv_B}(a)y), \tag{76}$$

$$\begin{aligned} l_{\dashv_B}(l_{\cdot_A}(x)a)\alpha_2(y) \quad &+ (r_{\cdot_B}(a)x) \dashv_A \alpha_2(y) = \\ &\alpha_1(x) \cdot_B (l_{\dashv_B}(a)y) + r_{\cdot_B}(r_{\dashv_A}(y)a)\alpha_1(x), \end{aligned} \tag{77}$$

$$l_{\cdot_B}(\beta_1(a))(x \dashv_A y) = (l_{\cdot_B}(a)x) \dashv_A \alpha_2(y) + l_{\dashv_B}(r_{\cdot_A}(x)a)\alpha_2(y), \tag{78}$$

$$r_{\cdot_B}(\beta_2(a))(x \cdot_A y) = r_{\cdot_B}(l_{\cdot_A}(y)a)\alpha_1(x) + \alpha_1(x) \cdot_A (r_{\cdot_B}(a)y), \tag{79}$$

$$\begin{aligned} l_{\cdot_B}(l_{\cdot_A}(x)a)\alpha_2(y) \quad &+ (r_{\cdot_B}(a)x) \cdot_A \alpha_2(y) = \\ &\alpha_1(x) \cdot_B (l_{\cdot_B}(a)y) + r_{\cdot_B}(r_{\cdot_A}(y)a)\alpha_1(x), \end{aligned} \tag{80}$$

$$l_{\cdot_B}(\beta_1(a))(x \cdot_A y) = (l_{\cdot_B}(a)x) \cdot_A \alpha_2(y) + l_{\cdot_B}(r_{\cdot_A}(x)x)\alpha_2(y). \tag{81}$$

Then, there is a BiHom-tridendriform algebra structure on the direct sum $A \oplus B$ of the underlying vector spaces of A and B given for any $x, y \in A, a, b \in B$ by

$$\begin{aligned} (x + a) \dashv (y + b) &= (x \dashv_A y + r_{\dashv_B}(b)x + l_{\dashv_B}(a)y) \\ &\quad + (l_{\dashv_A}(x)b + r_{\dashv_A}(y)a + a \dashv_B b), \\ (x + a) \vdash (y + b) &= (x \vdash_A y + r_{\vdash_B}(b)x + l_{\vdash_B}(a)y) \\ &\quad + (l_{\vdash_A}(x)b + r_{\vdash_A}(y)a + a \vdash_B b), \\ (x + a) \cdot (y + b) &= (x \cdot_A y + r_{\cdot_B}(b)x + l_{\cdot_B}(a)y) \\ &\quad + (l_{\cdot_A}(x)b + r_{\cdot_A}(y)a + a \cdot_B b). \end{aligned}$$

Proof. The proof is obtained in a similar way as for Theorem 2.8. □

By $A \bowtie^{l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, l_{\cdot_A}, r_{\cdot_A}, \beta_1, \beta_1}_{l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, l_{\cdot_B}, r_{\cdot_B}, \alpha_1, \alpha_2} B$ we denote this BiHom-tridendriform algebra.

Definition 3.22. Let $(A, \dashv_A, \vdash_A, \cdot_A, \alpha_1, \alpha_2)$ and $(B, \dashv_B, \vdash_B, \cdot_B, \beta_1, \beta_2)$ be two BiHom-tridendriform algebras. Suppose there exist linear maps

$$\begin{aligned} l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, l_{\cdot_A}, r_{\cdot_A} &: A \to gl(B), \\ l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, l_{\cdot_B}, r_{\cdot_B} &: B \to gl(A) \end{aligned}$$

such that $(l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, l_{\cdot_A}, r_{\cdot_A}, \beta_1, \beta_2)$ is a bimodule of A, and

$$(l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, l_{\cdot_B}, r_{\cdot_B}, \alpha_1, \alpha_2) \text{ is a bimodule of } B.$$

If (40) - (81) are satisfied, then

$$(A, B, l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, l_{\cdot_A}, r_{\cdot_A}, \beta_1, \beta_2, l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, l_{\cdot_B}, r_{\cdot_B}, \alpha_1, \alpha_2)$$

is called a matched pair of BiHom-tridendriform algebras.

Corollary 3.23. For any matched pair of BiHom-tridendriform algebras

$$(A, B, l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, l_{\cdot_A}, r_{\cdot_A}, \beta_1, \beta_2, l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, l_{\cdot_B}, r_{\cdot_B}, \alpha_1, \alpha_2),$$

the tuple

$$(A, B, l_{\dashv_A} + l_{\vdash_A} + l_{\cdot_A}, r_{\dashv_A} + r_{\vdash_A} + r_{\cdot_A}, \beta_1, \beta_2,$$
$$l_{\dashv_B} + l_{\vdash_B} + l_{\cdot_B}, r_{\dashv_B} + r_{\vdash_B} + r_{\cdot_B}, \alpha_1, \alpha_2)$$

is a matched pair of associated BiHom-associative algebras $(A, *_A, \alpha_1, \alpha_2)$ and $(B, *_B, \beta_1, \beta_2)$.

Proof. For a matched pair

$$(A, B, l_{\dashv_A}, r_{\dashv_A}, l_{\vdash_A}, r_{\vdash_A}, l_{\cdot_A}, r_{\cdot_A}, \beta_1, \beta_2, l_{\dashv_B}, r_{\dashv_B}, l_{\vdash_B}, r_{\vdash_B}, l_{\cdot_B}, r_{\cdot_B}, \alpha_1, \alpha_2)$$

of a BiHom-tridendriform algebras

$$(A, \dashv_A, \vdash_A, \cdot_A, \alpha_1, \alpha_2) \text{ and } (B, \dashv_B, \vdash_B, \cdot_B, \beta_1, \beta_2),$$

in view of Proposition 3.18, the linear maps

$$\begin{aligned} &l_{\dashv_A} + l_{\vdash_A} + l_{\cdot_A}, \quad r_{\dashv_A} + r_{\vdash_A} + r_{\cdot_A} : A \to gl(B) \\ &l_{\dashv_B} + l_{\vdash_B} + l_{\cdot_B}, \quad r_{\dashv_B} + r_{\vdash_B} + r_{\cdot_B} : B \to gl(A) \end{aligned}$$

are bimodules of the underlying BiHom-associative algebras $(A, *_A, \alpha_1, \alpha_2)$ and $(B, *_B, \beta_1, \beta_2)$, respectively. Thus, (6)–(8) are equivalent to (40)–(60), and (9)–(11) are equivalent to (61)–(81). □

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