© Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **35** (2023). Number 1, pp. 86–110
DOI:10.12958/adm2020

Cohomology and deformation of an associative superalgebra

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Communicated by I. P. Shestakov

Abstract. In this paper we generalize to associative superalgebras Gerstenhaber's work on cohomology structure of an associative algebra. We introduce formal deformation theory of associative superalgebras.

1. Introduction

The cohomology theory of associative algebra was studied by G. Hochschild in [1], [2], [3]; and by Murray Gerstenhaber in [4]. Gerstenhaber proved that there exists a cup product multiplication $\cup$ in $H^*(A; A)$ with respect to which it is a commutative graded associative algebra. It was shown that if P is a two sided module over A, then $H^*(A; P)$ is a two sided module over $H^*(A; A)$. He introduced a bracket product $[-, -]$ with respect to which $H^*(A; A)$ is a graded Lie algebra.

The deformation is a tool to study a mathematical object by deforming it into a family of the same kind of objects depending on a certain parameter. Deformation theory of algebraic structures was introduced by Gerstenhaber for rings and algebras in a series of papers [4], [5], [6], [7]. Recently, deformation theory of superalgebraic structures has been studied by many authors [8], [9], [10], [11], [12].

Graded algebras are of interest in physics in the context of 'supersymmetries' relating particles of differing statistics. In mathematics, graded algebras are known for some time in the context of deformation theory [13].

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*I thank the referee for the valuable comments.

2020 MSC: superalgebra, module, cohomology, graded Lie superalgebra, differential graded superalgebra, formal deformation.

Key words and phrases: 17B70, 17B60, 16E40, 17B10, 17C70, 16S80.

A superalgebra is a $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$ (that is, if $a \in A_\alpha$, $b \in A_\beta$, then $ab \in A_{\alpha+\beta}$, $\alpha, \beta \in \mathbb{Z}_2 = \{0, 1\}$). An associative superalgebra is a superalgebra $A = A_0 \oplus A_1$ such that $(ab)c = a(bc)$, for all a, b, c in A.

The goal of this paper is to study different algebraic structures on the cochain complex $C^*(A; A)$, the cohomology $H^*(A; A)$ of an associative superalgebra and application of this study in the formal deformation theory of A. Organization of the paper is as follows. In section 2, we recall some basic definitions. In section 3, we introduce $\mathbb{Z}$-graded Lie and pre-Lie superalgebras. In section 4, we introduce supermodules over superalgebras. In section 5, we introduce derivations of $\mathbb{Z}$-graded superalgebras. In section 6, we discuss cohomology of associative superalgebras. In this section we establish a fundamental isomorphism between $H^n_i(A; P)$ and $H^{n-1}_i(A; C^1(A; P))$, for $n \geqslant 2$, $i = 0, 1$ as in [1] for associative algebras. In section 7, we compute cohomology of associative superalgebras in dimensions 0, 1 and 2. In section 8, we introduce a cup product $\cup$ for the cohomology of an associative superalgebra A. In this section we prove that $\{C^*(A; A), \cup\}$ is a $\mathbb{Z}$-graded associative superalgebra and coboundary map δ is a derivation on it. Also, we prove that $\{H^*(A; A), \cup\}$ is a $\mathbb{Z}$-graded associative superalgebra. In section 9, we introduce $\mathbb{Z}$-graded right pre-Lie supersystem and discuss the $\mathbb{Z}$-graded right pre-Lie superalgebra given by it. We show the existence of a bracket product $[-, -]$ on $C^*(A; A)$ with respect to which it is a $\mathbb{Z}$-graded Lie superalgebra and δ is a derivation on $\{C^*(A; A), [-, -]\}$. We prove that $\{H^*(A; A), [-, -]\}$ $\mathbb{Z}$-graded Lie superalgebra. If P is a two sided module over an associative superalgebra A, then $H^*(A; P)$ is a two sided module over $\{H^*(A; A), \cup\}$. In section 10, we introduce formal deformation theory of associative superalgebras. We prove that obstruction cochain to the deformations are 3-cocycles. We discuss equivalence of deformations and prove that cohomology class of the infinitesimal of a deformation depends only on its equivalence class.

2. Associative superalgebra

In this section, we recall definitions of graded algebra, associative superalgebra, Lie superalgebra. We give some examples of associative superalgebras. Throughout the paper we denote a fixed field of characteristic 0 by K.

Definition 2.1. Let Δ be any nonempty set and K be a field. A Δ-graded vector space is a vector space V over K together with a family of subspaces $\{V^\alpha\}_{\alpha\in\Delta}$, indexed by Δ such that $V = \bigoplus_{\alpha\in\Delta} V^\alpha$, the direct sum of V^α's. An element a in V^α is called homogeneous of degree α, we write $\deg(a) = \alpha$.

Let Δ be a commutative group. A Δ-graded algebra over K is a Δ-graded vector space $E = \bigoplus_{\alpha \in \Delta} E^{\alpha}$ together with a bilinear map $m : E \times E \to E$ such that $m(E^{\alpha} \times E^{\beta}) \subset E^{\alpha+\beta}$ for all $\alpha,\ \beta \in \Delta$. An associative superalgebra is a $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$ such that $m(m(a,b),c) = m(a,m(b,c))$, for all $a,b,c \in A$. A Lie superalgebra is a $\mathbb{Z}_2$-graded algebra such that following conditions are satisfied:

1) $m(a,b) = -(-1)^{\alpha\beta} m(b,a)$,
2) $(-1)^{\alpha\gamma} m(m(a,b),c) + (-1)^{\beta\alpha} m(m(b,c),a) + (-1)^{\gamma\beta} m(m(c,a),b) = 0$,

for all $a \in E_{\alpha}$, $b \in E_{\beta}$, $c \in E_{\gamma}$, $\alpha, \beta, \gamma \in \mathbb{Z}_2$.

In any $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ we use a notation in which we replace degree $deg(a)$ of a homogeneous element $a \in V$ by 'a' whenever $deg(a)$ appears in an exponent; thus, for example $(-1)^{ab} = (-1)^{deg(a)deg(b)}$. Let $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ be $\mathbb{Z}_2$-graded vector spaces over a field K. A linear map $f : V \to W$ is said to be homogeneous of degree α if $f(a) \in W$ is homogeneous and $\deg(f(a)) - \deg(a) = \alpha$, for all $a \in V_{\beta}$, $\beta \in \{0,1\}$. We denote degree of f by $\deg(f)$.

Example 2.1. Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded vector space. Consider the vector space A of all homogeneous endomorphisms of V. Then $A = End_0(V) \oplus End_1(V)$, where

$$End_{\alpha}(V) = \{f \in End(V) : f(V_{\beta}) \subset V_{\alpha+\beta}, \forall \beta \in \mathbb{Z}_2\}, \quad \alpha \in \mathbb{Z}_2.$$

A is an associative superalgebra with respect to composition operation.

3. Graded Lie and pre-Lie superalgebras

Definition 3.1. We call a $\mathbb{Z} \times \mathbb{Z}_2$-graded algebra $E = \bigoplus_{(\alpha,\beta) \in \mathbb{Z} \times \mathbb{Z}_2} E^{\alpha}_{\beta}$ a $\mathbb{Z}$-graded superalgebra. An element a in E^{α}_{β} is said to be homogeneous of degree (α, β), for all $(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}_2$. The anti-isomorph or opposite A' of a $\mathbb{Z}$-graded superalgebra is the superalgebra which as a K-vector space, is identical with A, but in which multiplication m' is given by $m'(a,b) = m(b,a)$, where m is the multiplication in A.

For all $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{Z}_2$, from here onwards whenever $\alpha + \beta$ appears in an exponent we understand it as $\alpha + \beta \mod 2$. Since the exponentes will be of (-1), it is well defined. We call a $\mathbb{Z}$-graded superalgebra E a $\mathbb{Z}$-graded associative superalgebra if

$$m(m(a,b),c) = m(a,m(b,c)),$$

for all homogeneous $a, b, c \in E$. Clearly every $\mathbb{Z}$-graded associative superalgebra is an associative algebra. We call a $\mathbb{Z}$-graded superalgebra E unital if there exists an element $e \in E_0^0$ such that $m(e, a) = m(a, e) = a$, for every $a \in E_\beta^\alpha$. We call the element e the unity of E. We call a $\mathbb{Z}$-graded superalgebra E a $\mathbb{Z}$-graded commutative superalgebra if

$$m(a, b) = (-1)^{\alpha_1\alpha_2+\beta_1\beta_2} m(b, a),$$

for all $a \in E_{\beta_1}^{\alpha_1}$, $b \in E_{\beta_2}^{\alpha_2}$. We call a $\mathbb{Z}$-graded superalgebra E a $\mathbb{Z}$-graded Lie superalgebra if following conditions are satisfied

1) If $a \in E_{\beta_1}^{\alpha_1}$ and $b \in E_{\beta_2}^{\alpha_2}$ then

$$m(a, b) = -(-1)^{\alpha_1\alpha_2+\beta_1\beta_2} m(b, a).$$

2) If $a \in E_{\beta_1}^{\alpha_1}$, $b \in E_{\beta_2}^{\alpha_2}$ and $c \in E_{\beta_3}^{\alpha_3}$, then

$$(-1)^{\alpha_1\alpha_3+\beta_1\beta_3} m(m(a, b), c) + (-1)^{\alpha_2\alpha_1+\beta_2\beta_1} m(m(b, c), a) \\ + (-1)^{\alpha_3\alpha_2+\beta_3\beta_2} m(m(c, a), b) = 0. \quad (1)$$

Definition 3.2. We call a $\mathbb{Z}$-graded superalgebra E a $\mathbb{Z}$-graded right pre-Lie superalgebra if

$$m(m(c, a), b) - (-1)^{\alpha_1\alpha_2+\beta_1\beta_2} m(m(c, b), a) \\ = m(c, m(a, b)) - (-1)^{\alpha_1\alpha_2+\beta_1\beta_2} m(c, m(b, a)), \quad (2)$$

for all $a \in E_{\beta_1}^{\alpha_1}$, $b \in E_{\beta_2}^{\alpha_2}$ and $c \in E_{\beta_3}^{\alpha_3}$. An antiisomorph A' of a $\mathbb{Z}$-graded right pre-Lie superalgebra is called $\mathbb{Z}$-graded left pre-Lie superalgebra.

Theorem 3.1. *Let A be a $\mathbb{Z}$-graded pre-Lie superalgebra. Define a multiplication $[-, -] : A \times A \to A$ by*

$$[a, b] = m(a, b) - (-1)^{\alpha_1\alpha_2+\beta_1\beta_2} m(b, a), \quad (3)$$

for all $a \in A_{\beta_1}^{\alpha_1}$, $b \in A_{\beta_2}^{\alpha_2}$. Then in the bracket product $[-, -]$ A is a $\mathbb{Z}$-graded Lie superalgebra.

Proof. Clearly the bracket product $[-, -]$ satisfies

$$[a, b] = -(-1)^{\alpha_1\alpha_2+\beta_1\beta_2} [b, a]$$

for all $a \in A^{\alpha_1}_{\beta_1}$, $b \in A^{\alpha_2}_{\beta_2}$. For all $a \in A^{\alpha_1}_{\beta_1}$, $b \in A^{\alpha_2}_{\beta_2}$ and $c \in A^{\alpha_3}_{\beta_3}$, by using relations 2, 3, we have

$$
\begin{aligned}
&(-1)^{\alpha_1\alpha_3+\beta_1\beta_3}[[a,b],c] \\
&\qquad = (-1)^{\alpha_1\alpha_3+\beta_1\beta_3}\{m(m(a,b),c) - (-1)^{\alpha_1\alpha_2+\beta_1\beta_2}m(m(b,a),c)\} \\
&(-1)^{\alpha_3\alpha_2+\beta_3\beta_2}\{-m(c,m(a,b)) + (-1)^{\alpha_1\alpha_2+\beta_1\beta_2}m(c,m(b,a))\} \\
&\qquad = (-1)^{\alpha_1\alpha_3+\beta_1\beta_3}\{m(m(a,b),c) - (-1)^{\alpha_1\alpha_2+\beta_1\beta_2}m(m(b,a),c)\} \\
&(-1)^{\alpha_3\alpha_2+\beta_3\beta_2}\{-m(m(c,a),b) + (-1)^{\alpha_1\alpha_2+\beta_1\beta_2}m(m(c,b),a)\}
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
&(-1)^{\alpha_2\alpha_1+\beta_2\beta_1}[[b,c],a] \\
&\qquad = (-1)^{\alpha_2\alpha_1+\beta_2\beta_1}\{m(m(b,c),a) - (-1)^{\alpha_2\alpha_3+\beta_2\beta_3}m(m(c,b),a)\} \\
&+ (-1)^{\alpha_1\alpha_3+\beta_1\beta_3}\{-m(a,m(b,c)) + (-1)^{\alpha_2\alpha_3+\beta_2\beta_3}m(a,m(c,b))\} \\
&\qquad = (-1)^{\alpha_2\alpha_1+\beta_2\beta_1}\{m(m(b,c),a) - (-1)^{\alpha_2\alpha_3+\beta_2\beta_3}m(m(c,b),a)\} \\
&\qquad\qquad\qquad 1^{\alpha_3+\beta} \qquad\qquad {}_{+\beta} \\
&+ (-1)^{\alpha_1\alpha_3+\beta_1\beta_3}\{-m(m(a,b),c) + (-
\end{aligned}
$$

Definition 4.2. If A is a $\mathbb{Z}$-graded Lie superalgebra, then we say that P is a right supermodule over A, denoting $\rho(x,a)$ by $[x,a]$ and $m(a,b)$ by $[a,b]$, provided $[[x,a],b] = [x,[a,b]] + (-1)^{\alpha_3\alpha_2+\beta_3\beta_2}[[x,b],a]$, for all $x \in P^{\alpha_1}_{\beta_1}$, $a \in A^{\alpha_2}_{\beta_2}$, $b \in A^{\alpha_3}_{\beta_3}$.

If A is a $\mathbb{Z}$-graded right pre-Lie superalgebra, then we say that P is a right supermodule over A, denoting $\rho(x,a)$ by $x \circ a$ and $m(a,b)$ by ab, provided

$$(x \circ a) \circ b - (-1)^{\alpha_3\alpha_2+\beta_3\beta_2}(x \circ b) \circ a \ = x \circ (ab) - (-1)^{\alpha_3\alpha_2+\beta_3\beta_2}(x \circ (ba)),$$

for all $x \in P^{\alpha_1}_{\beta_1}$, $a \in A^{\alpha_2}_{\beta_2}$ and $b \in A^{\alpha_3}_{\beta_3}$. Every $\mathbb{Z}$-graded right pre-Lie superalgebra is a right supermodule over itself.

We call a right supermodule P over the anti-isomorph A' of a $\mathbb{Z}$-graded associative or Lie superalgebra A as left supermodule over it. We say that P is a (two sided) supermodule over a $\mathbb{Z}$-graded (associative or Lie) superalgebra A if $A \oplus P$ is a $\mathbb{Z}$-graded (associative or Lie) superalgebra such that A is subsuperalgebra of $A \oplus P$ and $m(x,y) = 0$, for all $x, y \in P$.

Clearly, if P is a (two sided) supermodule over a $\mathbb{Z}$-graded (associative or Lie) superalgebra A then it is a right as well as left supermodule over A. If P is a right supermodule over a $\mathbb{Z}$-graded commutative superalgebra A, then if we define a left action $A \times P \to P$ of A on P by $ax = (-1)^{\alpha_1\alpha_2+\beta_1\beta_2}xa$, for all $x \in P^{\alpha_1}_{\beta_1}$, $a \in A^{\alpha_2}_{\beta_2}$, then P becomes a (two sided) supermodule over A. If P is a right supermodule over a $\mathbb{Z}$-graded Lie superalgebra A, then if we define a left action $A \times P \to P$ of A on P given by $[a,x] = -(-1)^{\alpha_1\alpha_2+\beta_1\beta_2}[x,a]$, for all $x \in P^{\alpha_1}_{\beta_1}$, $a \in A^{\alpha_2}_{\beta_2}$, P becomes a (two sided) supermodule over A.

5. Derivations of $\mathbb{Z}$-graded superalgebras

Definition 5.1. Let $A = \bigoplus_{(\alpha,\beta)\in\mathbb{Z}\times\mathbb{Z}_2} A^{\alpha}_{\beta}$ be a $\mathbb{Z}$-graded superalgebra. A K-linear map $D : A \to A$ is called left derivation of degree (α,β) of A if D is homogeneous of degree (α,β) and

$$D(ab) = (Da)b + (-1)^{\alpha\alpha_1+\beta\beta_1}a(Db), \tag{7}$$

for all $a \in A^{\alpha_1}_{\beta_1}$ and $b \in A^{\alpha_2}_{\beta_2}$. A K-linear map $D : A \to A$ is called a right derivation of degree (α,β) of A if D is homogeneous of degree (α,β) and

$$D(ab) = (-1)^{\alpha\alpha_2+\beta\beta_2}(Da)b + a(Db), \tag{8}$$

for all $a \in A^{\alpha_1}_{\beta_1}$ and $b \in A^{\alpha_2}_{\beta_2}$. Let $\mathcal{D} = \bigoplus_{(\alpha,\beta)\in\mathbb{Z}\times\mathbb{Z}_2} \mathcal{D}^{\alpha}_{\beta}$ be the vector space obtained by taking direct sum of the vector spaces $\mathcal{D}^{\alpha}_{\beta}$ of right derivations of A of degree (α, β), $(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}_2$. For $D_1 \in \mathcal{D}^{\alpha_1}_{\beta_1}$, $D_2 \in \mathcal{D}^{\alpha_2}_{\beta_2}$, if we define

$$[D_1, D_2] = D_1 D_2 - (-1)^{\alpha_1 \alpha_2 + \beta_1 \beta_2} D_2 D_1,$$

then it can be easily verified that $[D_1, D_2]$ is a right derivation of A of degree $(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$ and with this multiplication $\mathcal{D}$ is a $\mathbb{Z}$-graded Lie superalgebra. Similar statement can be given if $\mathcal{D} = \bigoplus_{(\alpha,\beta) \in \mathbb{Z} \times \mathbb{Z}_2} \mathcal{D}^\alpha_\beta$ is the vector space obtained by taking direct sum of the vector spaces $\mathcal{D}^\alpha_\beta$ of left derivations of A of degree (α, β), $(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}_2$. If $A = \bigoplus_{(\alpha,\beta) \in \mathbb{Z} \times \mathbb{Z}_2} A^\alpha_\beta$ is a $\mathbb{Z}$-graded associative superalgebra and $a \in A^\alpha_\beta$, then if we define two K-linear maps $D^a_1, D^a_2 : A \to A$ by

$$D^a_1 b = ab - (-1)^{\alpha\alpha' + \beta\beta'} ba \tag{9}$$

$$D^a_2 b = ba - (-1)^{\alpha\alpha' + \beta\beta'} ab, \tag{10}$$

for all $b \in A^{\alpha'}_{\beta'}$, then D^a_1 and D^a_2 are left and right derivations of A, respectively of degree (α, β). Similarly if $A = \bigoplus_{(\alpha,\beta) \in \mathbb{Z} \times \mathbb{Z}_2} A^\alpha_\beta$ is a $\mathbb{Z}$-graded Lie superalgebra and $a \in A^\alpha_\beta$, then if we define

$$D^a_1 b = [a, b] \tag{11}$$

$$D^a_2 b = [b, a], \tag{12}$$

for all $b \in A^{\alpha'}_{\beta'}$, then D^a_1 and D^a_2 are left and right derivations of A, respectively of degree (α, β). D^a_1 and D^a_2 are called inner derivations of A induced by a.

Let A be a $\mathbb{Z}$-graded superalgebra and P be a (two sided) module over A. A K-linear map $D : A \to P$ is called a left derivation, respectively a right derivation, of degree α of A into P if 7, respectively 8 holds, for all $a \in A^{\alpha_1}_{\beta_1}$ and $b \in A^{\alpha_2}_{\beta_2}$. If A is $\mathbb{Z}$-graded associative or Lie superalgebra, then we can define left and right inner derivations using relations 9, 10,11, 12. In this case, we choose $a \in P^\alpha_\beta$.

Definition 5.2. We call a $\mathbb{Z}$-graded superalgebra A as a differential graded superalgebra if it is equipped with a (right or left) derivation $D : A \to A$ of degree $(1, 0)$ such that $D^2 = 0$.

6. Cohomology of associative superalgebras

Let $V = V_0 \oplus V_1$, $W = W_0 \oplus W_1$ be $\mathbb{Z}_2$-graded K-vector spaces. An n-linear map $f : \underbrace{V \times \cdots \times V}_{n\ times} \to W$ is said to be homogeneous of degree α if for all homogeneous $x_i \in V$, $1 \leqslant i \leqslant n$, $f(x_1, \cdots, x_n)$ is a homogeneous element in W and $\deg(f(x_1, \cdots, x_n)) - \sum_{i=1}^{n} \deg(x_i)) = \alpha$. We denote the degree of a homogeneous f by $\deg(f)$. We use a notation in which we replace degree $\deg(f)$ by 'f' whenever $\deg(f)$ appears in an exponent; thus, for example $(-1)^{\deg(f)} = (-1)^f$. For each $n \geqslant 0$, we define a K-vector space $C^n(V; W)$ as follows: For $n \geqslant 1$, $C^n(V; W)$ consists of n-linear maps $f : \underbrace{V \times \cdots \times V}_{n\ times} \to W$, and $C^0(V; W) = W$. Clearly, $C^n(V; W) = C_0^n(V; W) \oplus C_1^n(V; W)$, where $C_i^n(V; W)$ is the K-vector subspace of $C^n(A; P)$ consisting of elements of degree i with $i = 0, 1$.

Let $A = A_0 \oplus A_1$ be an associative superalgebra and $P = P_0 \oplus P_1$ be a (two sided) supermodule over A. We define two K-bilinear maps

$$A \times C^1(A; P) \to C^1(A; P) \text{ and } C^1(A; P) \times A \to C^1(A; P)$$

(we use same symbol $*$ for both the maps and differentiate them from context) by

$$(a * f)(a_1) = af(a_1), \tag{13}$$

$$(f * a)(a_1) = f(aa_1) - f(a)a_1, \tag{14}$$

for all $a, a_1 \in A$, $f \in C^1(A; P)$. We have following proposition:

Proposition 6.1. *$C^1(A; P)$ is a (two sided) supermodule over A.*

Proof. Proof is a direct consequence of the two actions of A on $C^1(A; P)$ given by relations 13, 14 and the definition of supermodule. □

We define a K-linear map $\delta^n : C^n(A; P) \to C^{n+1}(A; P)$ by

$$\begin{aligned} \delta^n f(x_1, \cdots, x_{n+1}) = (-1)^{x_1 f} x_1.f(x_2, \cdots, x_{n+1}) \\ + \sum_{i=1}^{n} (-1)^i f(x_1, \cdots, x_i.x_{i+1}, \cdots, x_{n+1}) \\ + (-1)^{n+1} f(x_1, \cdots, \cdots, x_n).x_{n+1}, \end{aligned} \tag{15}$$

for all f in $C^n(A; P)$, $n \geqslant 1$, and $\delta^0 f(x_1) = (-1)^{x_1 f} x_1.f - f.x_1$, for all f in $C^0(A; P) = P$. Clearly, for each $f \in C^n(A; P)$, $n \geqslant 0$, $\deg(\delta f) = \deg(f)$.

Lemma 6.1. *For $n > 0$, $C^n(A; P) \cong C^{n-1}(A; C^1(A; P))$.*

Proof. Define $\phi : C^n(A; P) \to C^{n-1}(A; C^1(A; P))$ by $\phi(f) = f_{n-1}$, where

$$f_{n-1}(a_1, \cdots, a_{n-1})(a_n) = f(a_1, \cdots, a_n),$$

for each $f \in C^n(A; P)$, $n > 0$. Clearly, ϕ is linear and bijection. □

Theorem 6.1. *$\delta\delta = 0$, that is, $(C^*(A;P),\delta)$ is a cochain complex.*

Proof. For $f \in C^0(A;P)$, we have

$$\begin{aligned}\delta\delta f(x_1,x_2) &= (-1)^{x_1 f} x_1 \delta f(x_2) - \delta f(x_1.x_2) + \delta f(x_1).x_2\\ &= (-1)^{x_1 f} x_1.((-1)^{x_2 f} x_2.f - f.x_2) - (-1)^{x_2 f + x_1 f}(x_1.x_2).f\\ &\quad\quad + f.(x_1.x_2) + (-1)^{x_1 f}(x_1.f).x_2 - (f.x_1).x_2\\ &= 0.\end{aligned}$$

For $f \in C^n(A;P)$, $n \geqslant 1$, we have

$$\begin{aligned}(\delta f)_n(x_1,\cdots,x_n)(x_{n+1}) &= \delta f(x_1,\cdots,x_{n+1})\\ &= (-1)^{x_1 f} x_1.f(x_2,\cdots,x_{n+1})\\ &\quad + \sum_{i=1}^{n}(-1)^i f(x_1,\cdots,x_i.x_{i+1},\cdots,x_{n+1})\\ &\quad + (-1)^{n+1} f(x_1,\cdots,x_n).x_{n+1}\\ &= (-1)^{x_1 f}(x_1 * f_{n-1}(x_2,\cdots,x_n)(x_{n+1})\\ &\quad + \sum_{i=1}^{n-1}(-1)^i f_{n-1}(x_1,\cdots,x_i.x_{i+1},\cdots,x_n)(x_{n+1})\\ &\quad + (-1)^n (f_{n-1}(x_1,\cdots,x_{n-1}) * x_n)x_{n+1}\\ &= \delta f_{n-1}(x_1,\cdots,x_n)(x_{n+1}) \qquad (16)\end{aligned}$$

Thus $(\delta f)_n = \delta(f_{n-1})$. This implies that for all $f \in C^n(A;P)$, $n \geqslant 1$,

$$(\delta\delta f)_{n+1} = \delta((\delta f)_n) = \delta\delta(f_{n-1}) \qquad (17)$$

Assume that $\delta\delta f = 0$ holds, for all $f \in C^q(A;P)$, where P is an arbitrary supermodule over A, $0 \leqslant q \leqslant n$. Using Equation 17, for $f \in C^{n+1}(A;P)$ we have $\delta\delta(f_n) = (\delta\delta f)_{n+2}$. By induction hypothesis $\delta\delta(f_n) = 0$. This implies that $(\delta\delta f)_{n+2} = 0$. Since $f = 0$ if and only if $f_{n-1} = 0$, for all $f \in C^n(A;P)$, $n \geqslant 1$, we conclude that $\delta\delta f = 0$. So, by using mathematical induction we conclude that $\delta\delta = 0$. □

We denote $\ker(\delta^n)$ by $Z^n(A;P)$ and image of δ^{n-1} by $B^n(A;P)$. We call the n-th cohomology $Z^n(A;P)/B^n(A;P)$ of the cochain complex $(C^*(A;P),\delta)$ as the n-th cohomology of A with coefficients in P and denote it by $H^n(A;P)$. Since A is a supermodule over itself. So we can consider cohomology groups $H^n(A;A)$. We call $H^n(A;A)$ as the n-th cohomology group of A. We have

$$Z^n(A;P) = Z_0^n(A;P) \oplus Z_1^n(A;P),\ B^n(A;P) = B_0^n(A;P) \oplus B_1^n(A;P),$$

$Z_i^n(A;P) = \{f \in Z^n(A;P) : \deg(f) = i\}$, $B_i^n(A;P) = \{f \in B^n(A;P) : \deg(f) = i\}$ are vector subspaces of $Z^n(A;P)$ and $B^n(A;P)$,respectively, $i = 0,1$. Since boundary map $\delta^n : C^n(A;P) \to C^{n+1}(A;P)$ is homogeneous of degree 0, we conclude that $H^n(A;P)$ is $\mathbb{Z}_2$-graded and

$$H^n(A;P) \cong H_0^n(A;P) \oplus H_1^n(A;P),$$

where $H_i^n(A;P) = Z_i^n(A;P)/B_i^n(A;P)$, $i = 0,1$.

Theorem 6.2. *For $n \geqslant 2$, $H_i^n(A;P) \cong H_i^{n-1}(A;C^1(A;P))$, $i = 0,1$.*

Proof. Clearly, for $i = 0,1$, the mapping $f \mapsto f_{n-1}$ is an isomorphism from $C_i^n(A;P)$ onto $C_i^{n-1}(A;C^1(A;P))$. Since $(\delta f)_n = \delta(f_{n-1})$, $Z_i^n(A;P)$ and $B_i^n(A;P)$ are mapped by the mapping $f \mapsto f_{n-1}$ onto $Z_i^{n-1}(A;C^1(A;P))$ and $B_i^{n-1}(A;C^1(A;P))$, respectively. Hence, for $i = 0,1$, the mapping $f \mapsto f_{n-1}$ induces an isomorphism from $H_i^n(A;P)$ onto $H_i^{n-1}(A;C^1(A;P))$. □

7. Cohomology of associative superalgebras in low dimensions

Let $A = A_0 \oplus A_1$ be an associative superalgebra and $P = P_0 \oplus P_1$ be a supermodule over A. For any $m \in C_i^0(A;P) = P_i$, $f \in C_i^1(A;P)$ and $g \in C_i^2(A;P)$, $i = 0,1$

$$\delta^0 m(x) = (-1)^{mx} x.m - m.x, \tag{18}$$

$$\delta^1 f(x_1,x_2) = -f(x_1.x_2) + (-1)^{x_1 f} x_1.f(x_2) + f(x_1).x_2, \tag{19}$$

$$\begin{aligned}\delta^2 g(x_1,x_2,x_3) = &-g(x_1.x_2,x_3) + g(x_1,x_2.x_3)\\ &+ (-1)^{x_1 g} x_1.g(x_2,x_3) - g(x_1,x_2).x_3.\end{aligned} \tag{20}$$

We denote the set $\{m \in P_i | m.x = (-1)^{mx} x.m, \text{for all homogeneous } x \in A\}$ by $Com_{P_i}A$. We have

$$\begin{aligned}H_i^0(A;P) &= \{m \in P_i | (-1)^{mx} x.m - m.x = 0 \text{ for all homogeneous } x \in A\}\\ &= Com_{P_i}A.\end{aligned}$$

For every $m \in P_i$ the map $x \mapsto (-1)^{ix}x.m - m.x$ is a left inner derivation of A into P of degree i and induced by $-m$. We denote the vector spaces of left derivations and left inner derivations of degree i of A into P by $Der_i(A;P)$ and $Der_i^{Inn}(A;P)$ respectively. By using 18, 19 we have

$$H_i^1(A;P) = Der_i(A;P)/Der_i^{Inn}(A;P).$$

Let A be an associative superalgebra and P be a supermodule over A. We regard P as an associative superalgebra with trivial product $xy = 0$, for all $x, y \in P$. We define an exact sequence of superalgebras as a sequence of superalgebra morphisms

$$\cdots \longrightarrow A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \cdots$$

such that $\mathrm{Im} f_n = \ker f_{n+1}$, for every n. We define extension of A by P of degree 0 to be an exact sequence

$$0 \longrightarrow P \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} A \longrightarrow 0 \qquad (*)$$

of associative superalgebras such that $\deg(i) = 0 = \deg(\pi)$ and

$$x.i(m) = \pi(x).m, \; i(m).x = m.\pi(x), \qquad (21)$$

for all homogeneous $x \in \mathcal{E}$, $m \in P$. The exact sequence $(*)$ regarded as a sequence of K-vector spaces splits. Therefore without any loss of generality we may assume that $\mathcal{E}$ as a K-vector space coincides with the direct sum $A \oplus P$ and that $i(m) = (0, m)$, $\pi(x, m) = x$. Hence relation 21 is meaningful. Thus we have $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, where $\mathcal{E}_0 = A_0 \oplus P_0$, $\mathcal{E}_1 = A_1 \oplus P_1$. The multiplication in $\mathcal{E} = A \oplus P$ has then necessarily the form

$$(0, m_1).(0, m_2) = (0, 0), \; (x_1, 0).(0, m_1) = (0, x_1.m_1),$$

$$(0, m_2).(x_2, 0) = (0, m_2.x_2), \; (x_1, 0).(x_2, 0) = (x_1.x_2, h(x_1, x_2)),$$

for some $h \in C_0^2(A;P)$, for all homogeneous $x_1, x_2 \in A$, $m_1, m_2 \in P$. Thus, in general, we have

$$(x, m).(y, n) = (x.y, x.n + m.y + h(x, y)), \qquad (22)$$

for all homogeneous (x, m), (y, n) in $\mathcal{E} = A \oplus P$.
Conversely, let $h : A \times A \to P$ be a bilinear homogeneous map of degree 0.

For homogeneous (x,m), (y,n) in $\mathcal{E} = A \oplus P$ we define multiplication by Equation 22. For homogeneous (x,m), (y,n) and (z,p) in $\mathcal{E}$ we have

$$\begin{aligned}((x,m).(y,n)).(z,p) &= ((x.y).z, (x.y).p + (x.n).z + (m.y).z \\ &\quad + h(x,y).z + h(x.y,z) \qquad (23)\\ (x,m).((y,n).(z,p)) &= (x.(y.z), x.(y.p) + x.(n.z) + m.(y.z) \\ &\quad + x.h(y,z) + h(x,y.z) \qquad (24)\end{aligned}$$

From Equations 23, 24, we conclude that $\mathcal{E} = A \oplus P$ is an associative superalgebra with product given by Equation 22 if and only if $\delta^2 h = 0$. We denote the associative superalgebra given by Equation 22 using notation $\mathcal{E}_h$. Thus for every cocycle h in $C_0^2(A;P)$ there exists an extension

$$E_h : 0 \longrightarrow P \xrightarrow{i} \mathcal{E}_h \xrightarrow{\pi} A \longrightarrow 0$$

of A by P of degree 0, where i and π are inclusion and projection maps, that is, $i(m) = (0,m)$, $\pi(x,m) = x$. We say that two extensions

$$0 \longrightarrow P \longrightarrow \mathcal{E}^i \longrightarrow A \longrightarrow 0 \ \ (i = 1,2)$$

of A by P of degree 0 are equivalent if there is an associative superalgebra isomorphism $\psi : \mathcal{E}^1 \to \mathcal{E}^2$ of degree 0 such that following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & \mathcal{E}^1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Id_P} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle Id_A} & & \\ 0 & \longrightarrow & P & \longrightarrow & \mathcal{E}^2 & \longrightarrow & A & \longrightarrow & 0 \end{array} \qquad (**)$$

We use $F(A,P)$ to denote the set of all equivalence classes of extensions of A by P of degree 0. Equation 22 defines a mapping of $Z_0^2(A;P)$ onto $F(A,P)$. If for $h, h' \in Z_0^2(A;P)$ E_h is equivalent to $E_{h'}$, then commutativity of diagram $(**)$ is equivalent to

$$\psi(x,m) = (x, m + f(x)),$$

for some $f \in C_0^1(A;P)$. We have

$$\begin{aligned}\psi((x_1,m_1).(x_2,m_2)) &= \psi(x_1.x_2, x_1.m_2 + m_1.x_2 + h(x_1,x_2)) \\ &= (x_1.x_2, x_1.m_2 + m_1.x_2 + h(x_1,x_2) + f(x_1.x_2)), \\ \psi(x_1,m_1).\psi(x_2,m_2) &= (x_1, m_1 + f(x_1)).(x_2, m_2 + f(x_2)) \\ &= (x_1.x_2, x_1.(m_2 + f(x_2)) + (m_1 + f(x_1)).x_2 + h'(x_1,x_2)).\end{aligned}$$

Since $\psi((x_1, m_1).(x_2, m_2)) = \psi(x_1, m_1).\psi(x_2, m_2)$, we have

$$\begin{aligned} h(x_1, x_2) - h'(x_1, x_2) &= -f(x_1.x_2) + x_1.f(x_2) + f(x_1).x_2 \\ &= \delta^1(f)(x_1, x_2) \end{aligned} \tag{25}$$

Thus two extensions E_h and $E_{h'}$ of degree 0 are equivalent if and only if there exists some $f \in C_0^1(A; P)$ such that $\delta^1 f = h - h'$. We thus have following theorem:

Theorem 7.1. *The set $F(A, P)$ of all equivalence classes of extensions of A by P of degree 0 is in one to one correspondence with the cohomology group $H_0^2(A; P)$. This correspondence $\omega : H_0^2(A; P) \to F(A, P)$ is obtained by assigning to each cocycle $h \in Z_0^2(A; P)$, the extension given by multiplication 22.*

8. Cup product for $H^*(A; A)$

Definition 8.1. Let P be a (two sided) supermodule over an associative superalgebra A. Also, assume that P has an structure associative superalgebra, in particular we may take $P = A$. We define a multiplication $\cup$ on $C^*(A; P) = \bigoplus C^n(A; P)$ by defining $\cup : C^m(A; P) \times C^n(A; P) \to C^{m+n}(A; P)$ by

$$\begin{aligned} &f \cup g(a_1, \cdots, a_m, b_1, \cdots, b_n) \\ &\qquad = (-1)^{\tilde{g}(a_1 + \cdots + a_m)} f(a_1, \cdots, a_m) g(b_1, \cdots, b_n), \end{aligned} \tag{26}$$

for all $f \in C_{\tilde{f}}^m(A; P)$, $g \in C_{\tilde{g}}^n(A; P)$, for all $m, n \geqslant 0$. If we put $C^n(A; P) = 0$ for $n < 0$, then $C^*(A; P)$ is a $\mathbb{Z}$-graded associative superalgebra with respect to the multiplication $\cup$.

As a direct consequence of definitions of coboundary map δ 15 and $\cup$ we have following proposition:

Proposition 8.1. *For $f \in C_{\tilde{f}}^m(A; P)$, $g \in C_{\tilde{g}}^n(A; P)$,*

$$\delta(f \cup g) = \delta f \cup g + (-1)^m f \cup \delta g. \tag{27}$$

From Proposition 8.1, we conclude that δ is a derivation of $\{C^*(A; P), \cup\}$ of degree $(1, 0)$. Hence the cochain complex $C^*(A; A)$ of an associative super algebra is a differential graded associative superalgebra with respect to the multiplication $\cup$. Next, we have some observations as following theorem.

Theorem 8.1. (i) *$\{Z^*(A;P),\cup\}$ is a subsuperalgebra of $\{C^*(A;P),\cup\}$, where $Z^*(A;P)=\bigoplus Z^n(A;P)$;*

(ii) *If one of f or g is in $B^m(A;P)$ and other in $Z^n(A;P)$, then $f\cup g\in B^{m+n}(A;P)$, that is, $B^*(A;P)=\bigoplus B^n(A;P)$ is a (two sided) ideal of $Z^*(A;P)$;*

(iii) *$H^*(A;P)=\bigoplus H^n(A;P)$ is a $\mathbb{Z}$-graded associative superalgebra with respect to multiplication $\cup$ (called as cup product) defined by*

$$(f+B^m(A;P))\cup(g+B^n(A;P))=(f\cup g)+B^{m+n}(A;P),$$

for all $f\in Z^m(A;P)$, $g\in Z^n(A;P)$.

Remark 8.1. *If A^e is the enveloping algebra of A, then $H^n(A,P)=Ext_{A^e}(A,P)$, and that using this equality, the cup product is the Yoneda product [14], [15], [16].*

Let A be an associative superalgebra and P be a (two sided) supermodule over A. If we define two actions of $C^*(A;A)=\oplus C^n(A;A)$ on $C^*(A;P)=\oplus C^n(A;P)$ (we use same symbol $\cup$ for both the actions and differentiate them from context) by

$$\begin{aligned} &f\cup g(a_1,\cdots,a_m,b_1,\cdots,b_n)\\ &\quad=(-1)^{\tilde{g}(a_1+\cdots+a_m)}f(a_1,\cdots,a_m)g(b_1,\cdots,b_n), \end{aligned}$$

for all $f\in C^m_{\tilde{f}}(A;A)$, $g\in C^n_{\tilde{g}}(A;P)$ or $f\in C^m_{\tilde{f}}(A;P)$, $g\in C^n_{\tilde{g}}(A;A)$. With these two actions $C^*(A;P)$ is a (two sided) supermodule over $C^*(A;A)$

It is clear that Equation 27 holds also when either $f\in C^m(A;P)$, $g\in C^n(A;A)$ or $f\in C^m(A;A)$, $g\in C^n(A;P)$. This implies that $Z^m(A;A)\cup Z^n(A;P)$ and $Z^m(A;P)\cup Z^n(A;A)$ are contained in $Z^{m+n}(A;P)$, and $Z^m(A;A)\cup B^n(A;P)$ and $B^m(A;A)\cup Z^n(A;P)$ are contained in $B^{m+n}(A;P)$. Hence we conclude that $H^*(A;P)$ is a (two sided) supermodule over $\{H^*(A;A),\cup\}$.

9. Bracket product for $H^*(A;A)$

Definition 9.1. A right pre-Lie super system $\{V_m,\circ_i\}$ is defined as a sequence

$$\cdots,V_{-1},V_0,V_1,V_2\cdots,V_n,\cdots$$

of $\mathbb{Z}_2$-graded K-vector spaces $V_m=V_0^m\oplus V_1^m$ together with an assignment for every triple of integers $m,n,i\geqslant 0$ with $i\leqslant m$, of a homogeneous K-bilinear map $\circ_i=\circ_i(m,n)$ of $V_m\times V_n$ into V_{m+n} of degree 0 such that following conditions are satisfied:

$$(f \circ_i g) \circ_j h = \begin{cases} (-1)^{\tilde{g}\tilde{h}} (f \circ_j h) \circ_{i+p} g, & \text{if } 0 \leqslant j \leqslant i-1 \\ f \circ_i (g \circ_{j-i} h), & \text{if } i \leqslant j \leqslant n+i \end{cases} \tag{28}$$

for all $f \in V^m_{\tilde{f}}$, $g \in V^n_{\tilde{g}}$ and $h \in V^p_{\tilde{h}}$. Here $\circ_i(f,g) = f \circ_i g$. From the first case of Equation 28, we have

$$(f \circ_j h) \circ_{i+p} g = (-1)^{\tilde{g}\tilde{h}} (f \circ_{i+p} g) \circ_{j+n} h, \text{ if } 0 \leqslant i+p \leqslant j-1 \tag{29}$$

In Equation 29, replacing $i+p$ by i and $j+n$ by j we have

$$(f \circ_i g) \circ_j h = (-1)^{\tilde{g}\tilde{h}} (f \circ_{j-n} h) \circ_i g, \text{ if} n+i+1 \leqslant j \leqslant m+n, \tag{30}$$

Example 9.1. Let $A = A_0 \oplus A_1$ be an associative superalgebra. Put $V_0 = A$, and $V_m = 0$, for all $m \in \mathbb{Z}$ with $m \neq 0$. We define $\circ_i = \circ_i(m,n) : V_m \times V_n \to V_{m+n}$ by

$$a \circ_i b = \begin{cases} ab, & \text{if } i = m = n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\{V_m, \circ_i\}$ is a right pre-Lie supersystem.

Example 9.2. Let $U = U_0 \oplus V_1$, $W = W_0 \oplus W_1$ $\mathbb{Z}_2$-graded K-vector spaces. Let $\phi : W \to U$ be a homogeneous K-linear map of degree 0. Put

$$V_m = \begin{cases} C^{m+1}(U; W), & \text{if } m \in \mathbb{Z}, m \geqslant -1 \\ 0, & \text{otherwise.} \end{cases}$$

For integers $m, n, i \geqslant 0$, $i \leqslant m$ we define $\circ_i : V_m \times V_n \to V_{m+n}$ by

$$\begin{aligned} & f \circ_i g(x_1, \cdots, x_i, y_1, \cdots, y_{n+1}, x_{i+2}, \cdots, x_{m+1}) \\ & \quad = (-1)^{\tilde{g}(x_1 + \cdots + x_i)} f(x_1, \cdots, x_i, \phi g(y_1, \cdots, y_{n+1}), x_{i+2}, \cdots, x_{m+1}), \end{aligned} \tag{31}$$

for all $f \in V^m_{\tilde{f}} = C^{m+1}_{\tilde{f}}(A; P)$, $g \in V^n_{\tilde{g}} = C^{n+1}_{\tilde{g}}(A; P)$. The definition of $\circ_i$ can be extended to the case when $n = -1$ as follows

$$\begin{aligned} & f \circ_i g(x_1, \cdots, x_i, x_{i+2}, \cdots, x_{m+1}) \\ & \quad = (-1)^{\tilde{g}(x_1 + \cdots + x_i)} f(x_1, \cdots, x_i, \phi g, x_{i+2}, \cdots, x_{m+1}) \end{aligned} \tag{32}$$

One can readily verify that the condition 28 holds and $\{V_m, \circ_i\}$ is a right pre-Lie supersystem.

Definition 9.2. Let $(V_m, \circ_i)$ be a pre-Lie supersystem. For every m and n we define a homogeneous K-bilinear map $\circ : V_m \times V_n \to V_{m+n}$ of degree $(0,0)$ by

$$f \circ g = \begin{cases} \sum_{i=0}^{m} (-1)^{ni} f \circ_i g, & \text{if } m \geqslant 0 \\ 0, & \text{if } m < 0, \end{cases} \tag{33}$$

for all $f \in V_m$, $g \in V_n$.

Theorem 9.1. *Let $(V_m, \circ_i)$ be a right pre-Lie supersystem. Then for $f \in V_{\tilde{f}}^m$, $g \in V_{\tilde{g}}^n$, $h \in V_{\tilde{h}}^p$*

(a)

$$(f \circ g) \circ h - f \circ (g \circ h) = \sum_{\substack{0 \leqslant j \leqslant i-1 \\ n+i+1 \leqslant j \leqslant m+n}} (-1)^{ni+pj} (f \circ_i g) \circ_j h$$

(b) $(f \circ g) \circ h - f \circ (g \circ h) = (-1)^{np+\tilde{g}\tilde{h}}((f \circ h) \circ g - f \circ (h \circ g))$.

Proof. By using Definitions 33, 28

$$\begin{aligned}
&(f \circ g) \circ h - f \circ (g \circ h) \\
&\quad = \sum_{\substack{0 \leqslant j \leqslant m+n \\ 0 \leqslant i \leqslant m}} (-1)^{pj+ni} (f \circ_i g) \circ_j h - \sum_{\substack{0 \leqslant \lambda \leqslant m \\ 0 \leqslant \mu \leqslant n}} (-1)^{\lambda(n+p)+\mu p} f \circ_\lambda (g \circ_\mu h) \\
&\quad = \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant i-1 \\ n+i+1 \leqslant j \leqslant m+n}} (-1)^{pj+ni} (f \circ_i g) \circ_j h + \sum_{\substack{0 \leqslant i \leqslant m \\ i \leqslant j \leqslant n+i}} (-1)^{pj+ni} (f \circ_i g) \circ_j h \\
&\qquad - \sum_{\substack{0 \leqslant \lambda \leqslant m \\ 0 \leqslant \mu \leqslant n}} (-1)^{\lambda(n+p)+\mu p} f \circ_\lambda (g \circ_\mu h) \\
&\quad = \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant i-1 \\ n+i+1 \leqslant j \leqslant m+n}} (-1)^{pj+ni} (f \circ_i g) \circ_j h + \sum_{\substack{0 \leqslant i \leqslant m \\ i \leqslant j \leqslant n+i}} (-1)^{pj+ni} f \circ_i (g \circ_{j-i} h) \\
&\qquad - \sum_{\substack{0 \leqslant \lambda \leqslant m \\ 0 \leqslant \mu \leqslant n}} (-1)^{\lambda(n+p)+\mu p} f \circ_\lambda (g \circ_\mu h) \\
&\quad = \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant i-1 \\ n+i+1 \leqslant j \leqslant m+n}} (-1)^{pj+ni} (f \circ_i g) \circ_j h
\end{aligned} \tag{34}$$

From Equation 34,

$$
\begin{aligned}
&(f \circ g) \circ h - f \circ (g \circ h) \\
&\quad = \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant i-1 \\ n+i+1 \leqslant j \leqslant m+n}} (-1)^{pj+ni} (f \circ_i g) \circ_j h \\
&\quad = (-1)^{pj+ni+\tilde{g}\tilde{h}} \left\{ \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant i-1}} (f \circ_j h) \circ_{i+p} g + \sum_{\substack{0 \leqslant i \leqslant m \\ n+i+1 \leqslant j \leqslant m+n}} (f \circ_{j-n} h) \circ_i g \right\}
\end{aligned}
\tag{35}
$$

Putting $j = \lambda$, $i + p = \mu$ in the first sum and $j - n = \lambda$, $i = \mu$ in the second sum of Equation 35, we get

$$
\begin{aligned}
&(f \circ g) \circ h - f \circ (g \circ h) \\
&\quad = \sum_{\substack{0 \leqslant \lambda \leqslant m \\ \lambda+p+1 \leqslant \mu \leqslant m+p}} (-1)^{p\lambda+n\mu+np+\tilde{g}\tilde{h}} (f \circ_\lambda h) \circ_\mu g \\
&\qquad + \sum_{\substack{0 \leqslant \lambda \leqslant m \\ 0 \leqslant \mu \leqslant \lambda-1}} (-1)^{p\lambda+pn+n\mu+\tilde{g}\tilde{h}} (f \circ_\lambda h) \circ_\mu g \\
&\quad = (-1)^{np+gh} ((f \circ h) \circ g - f \circ (h \circ g))
\end{aligned}
$$

□

Corollary 9.1. *Let $\{V_m, \circ_i\}$ be a right pre-Lie supersystem and let $V = \bigoplus V_m$ be the direct sum of the $\mathbb{Z}_2$-graded K-vector spaces $V_m = V_0^m \oplus V_1^m$. Then with respect to multiplication given by $\circ$ in Definition 9.2 V is a $\mathbb{Z}$-graded right pre-Lie superalgebra.*

Definition 9.3. Let $\{V_m, \circ_i\}$ be a right pre-Lie supersystem and $W = \bigoplus_{(m,n)} W_n^m$ be a $\mathbb{Z} \times \mathbb{Z}_2$-graded K-vector space. We write $W_m = W_0^m \oplus W_1^m$ for all $m \in \mathbb{Z}$. We say that W has the structure of a right supermodule over the right pre-Lie supersystem $\{V_m, \circ_i\}$ if there exist homogeneous linear maps (for which we use the same notation $\circ_i$) from $W_m \times V_n$ to W_{m+n} such that Equation 28 holds for all $f \in W_m$, $g \in V_n$ and $h \in V_p$. Now define $\circ : W \times V \to W$ by

$$
f \circ g = \begin{cases} \sum_{i=0}^{m} (-1)^{ni} f \circ_i g, & \text{if } m \geqslant 0 \\ 0, & \text{if } m < 0, \end{cases} \tag{36}
$$

for all $f \in W_m$, $g \in V_n$. Let $V = \bigoplus V_n$ be the $\mathbb{Z}$-graded right pre-Lie superalgebra given by the right pre-Lie supersystem $\{V_m, \circ_i\}$. Using similar arguments as in the proof of Theorem 9.1 we conclude that W is a right supermodule over the $\mathbb{Z}$-graded right pre-Lie superalgebra V with respect to the action $\circ$ of V on W given by Equation 36. Consider the $\mathbb{Z}$-graded Lie superalgebra structure given by the $\mathbb{Z}$-graded right pre-Lie superalgebra using Theorem 3.1. If we define a right action of the $\mathbb{Z}$-graded Lie superalgebra V on W by $[X, f] = X \circ f$, $X \in W$, $f \in V$, then W is a right supermodule over the $\mathbb{Z}$-graded Lie superalgebra V.

As in Example 9.2, if we consider only $\mathbb{Z}_2$-graded K-vector space structure of A and put $U = W = A$, $\phi = Id_A$, then $\{C^m(A;A), \circ_i\}$ is a right pre-Lie supersystem where elements of $C^m(A;A)$ has degree $m-1$ and dimension m. For $f \in C^m(A;P)$, $g \in C^n(A;A)$ we define $f \circ_i g \in C^{m+n-1}(A;P)$ by

$$\begin{aligned} &f \circ_i g(x_1, \cdots, x_i, y_1, \cdots, y_{n+1}, x_{i+2}, \cdots, x_{m+1}) \\ &\quad = (-1)^{\tilde{g}(x_1+\cdots+x_i)} f(x_1, \cdots, x_i, g(y_1, \cdots, y_{n+1}), x_{i+2}, \cdots, x_{m+1}), \end{aligned} \tag{37}$$

The definition of $\circ_i$ can be extended to the case when $n = -1$ as follows

$$\begin{aligned} &f \circ_i g(x_1, \cdots, x_i, x_{i+2}, \cdots, x_{m+1}) \\ &\quad = (-1)^{\tilde{g}(x_1+\cdots+x_i)} f(x_1, \cdots, x_i, g, x_{i+2}, \cdots, x_{m+1}) \end{aligned} \tag{38}$$

We can easily verify that $C^*(A;P)$ is a right supermodule over the right pre-Lie supersystem $\{C^m(A;A), \circ_i\}$. From Corollary 9.1, $\{C^*(A;A), \circ\}$ is right pre-Lie superalgebra. By Theorem 3.1, $\{C^*(A;A), [-,-]\}$ is a $\mathbb{Z}$-graded Lie superalgebra. Also, from Definition 9.3, we can see that $C^*(A;P)$ is a right supermodule over the right pre-Lie superalgebra $\{C^*(A;A), \circ\}$. For $f \in C^m_{\tilde{f}}(A;P)$, $g \in C^n_{\tilde{g}}(A;A)$, if we define $[f, g] = f \circ g$, and $[g, f] = -(-1)^{mn+\tilde{f}\tilde{g}}[f, g]$, then $C^*(A;P)$ is (two-sided) supermodule over the $\mathbb{Z}$-graded Lie superalgebra $\{C^*(A;A), [-,-]\}$.

We define $\pi \in C^2_0(A;A)$ by $\pi(a, b) = ab$. We observe that $\delta\pi = 0$ and $\delta Id_A = \pi$, that is, $\pi \in B^2_0(A;A)$. By using direct definitions we have

$$f \cup g = (\pi \circ_0 f) \circ_m g, \tag{39}$$

for all $f \in C^m(A;A)$ and $g \in C^n(A;A)$. Also, one can easily verify using definitions of π, δ and $\circ$ that

$$\begin{aligned} \delta f &= -f \circ \pi + (-1)^{m-1} \pi \circ f \\ &= (-1)^{m-1}(\pi \circ f - (-1)^{m-1} f \circ \pi), \end{aligned} \tag{40}$$

for all $f \in C^m_{\tilde{f}}(A;A)$. By using Equation 3, we have

$$\delta f = [f, -\pi] = (-1)^{m-1}[\pi, f]. \tag{41}$$

From Equation 40 it is clear that δ is a right inner derivation of degree $(1,0)$ of the $\mathbb{Z}$-graded Lie superalgebra $\{C^*(A;A), [-,-]\}$ induced by $-\pi$. Hence $\{C^*(A;A), [-,-]\}$ is a differential $\mathbb{Z}$-graded Lie superalgebra. Next, we have following result:

Theorem 9.2. *Let A be a associative superalgebra. Then*

$$\begin{aligned} &f \circ \delta g - \delta(f \circ g) + (-1)^{n-1}\delta f \circ g \\ &\qquad = (-1)^{n-1}\{(-1)^{\tilde{f}\tilde{g}} g \cup f - (-1)^{mn} f \cup g\}, \end{aligned} \tag{42}$$

for all $f \in C^m_{\tilde{f}}(A;A)$ and $g \in C^n_{\tilde{g}}(A;A)$.

Proof. Using the Equation 40 and Definition 3.2, we have

$$\begin{aligned} f &\circ \delta g - \delta(f \circ g) + (-1)^{n-1}\delta f \circ g \\ &= (-1)^{n-1} f \circ (\pi \circ g) - f \circ (g \circ \pi) - (-1)^{m+n-2}\pi \circ (f \circ g) \\ &\qquad + (f \circ g) \circ \pi + (-1)^{n-1+m-1}(\pi \circ f) \circ g - (-1)^{n-1}(f \circ \pi) \circ g \\ &= (-1)^{m+n}\{(\pi \circ f) \circ g - \pi \circ (f \circ g)\} \end{aligned} \tag{43}$$

Using Theorem 9.1 (a) and Equations 39, 30 we have

$$\begin{aligned} (\pi \circ f) \circ g - \pi \circ (f \circ g) &= \sum_{\substack{0 \leqslant j \leqslant i-1 \\ m+i \leqslant j \leqslant m}} (-1)^{(m-1)i+(n-1)j}(\pi \circ_i f) \circ_j g \\ &= (-1)^{(n-1)m}(\pi \circ_0 f) \circ_m g + (-1)^{m-1}(\pi \circ_1 f) \circ_0 g \\ &= (-1)^{(n-1)m}(\pi \circ_0 f) \circ_m g + (-1)^{m-1+\tilde{f}\tilde{g}}(\pi \circ_0 g) \circ_n f \\ &= (-1)^{(n-1)m} f \cup g + (-1)^{m-1+\tilde{f}\tilde{g}} g \cup f \\ &= (-1)^{m-1}\{(-1)^{\tilde{f}\tilde{g}} g \cup f - (-1)^{mn} f \cup g\}. \end{aligned} \tag{44}$$

Using Equations 43,44 we conclude Equation 42. This completes proof of the theorem. □

Corollary 9.2. *If A is an associative superalgebra, then the $\mathbb{Z}$-graded associative superalgebra $\{H^*(A;A), \cup\}$ is commutative, that is, if $u \in H^m_{\tilde{u}}(A;A)$, $v \in H^n_{\tilde{v}}(A;A)$, then*

$$u \cup v = (-1)^{mn+\tilde{u}\tilde{v}} v \cup u.$$

Proof. By using Theorem 9.2, for $f \in Z^m_{\tilde{f}}(A;A)$, $g \in Z^n_{\tilde{g}}(A;A)$ we have

$$\{(-1)^{\tilde{f}\tilde{g}} g \cup f - (-1)^{mn} f \cup g = (-1)^n \delta(f \circ g).$$

Hence we conclude that $u \cup v = (-1)^{mn+\tilde{u}\tilde{v}} v \cup u$, for all $u \in H^m_{\tilde{u}}(A;A)$, $v \in H^n_{\tilde{v}}(A;A)$. □

If A is an associative superalgebra and P is a (two sided) supermodule over A, then $A \oplus P$ has the structure of an associative superalgebra with respect to the multiplication given by

$$(a,x).(b,y) = (a.b, a.y + x.b),$$

for all $a, b \in A$, $x, y \in P$. Consider the natural inclusions

$$C^n(A;A), C^n(A;P) \subset C^n(A \oplus P; A \oplus P)$$

defined by $f(a_1, \cdots, a_n) = 0$, if some a_i is in P,for all f in $C^n(A;A)$ or $C^n(A;P)$. Now, Theorem 9.2 holds for the associative superalgebra $A \oplus P$. Observe that if $f \in C^m(A;A)$, $g \in C^n(A;P)$ are cocyles, then $g \in C^n(A \oplus P; A \oplus P)$ is a cocycle but $f \in C^n(A \oplus P; A \oplus P)$ need not be a cocycle. Also, observe that if $f \in Z^m(A;A)$, $g \in Z^n(A;P)$, then in the Equation 42 in the Theorem 9.2 for the associative superalgebra $A \oplus P$, $(\delta f) \circ g = 0$, $f \circ (\delta g) = 0$. This implies that if $f \in Z^m(A;A)$, $g \in Z^n(A;P)$, then

$$\delta(f \circ g) = -(-1)^{n-1}\{(-1)^{\tilde{f}\tilde{g}} g \cup f - (-1)^{mn} f \cup g\}.$$

Hence we have following result:

Corollary 9.3. *Let A be a $\mathbb{Z}$-graded associative superalgebra and P be a supermodule over A. Then*

$$u \cup v = (-1)^{mn+\tilde{u}\tilde{v}} v \cup u,$$

if $u \in H^m_{\tilde{u}}(A;A)$, $v \in H^n_{\tilde{v}}(A;P)$.

Theorem 9.3. *Let A be a associative superalgebra and P be a (two sided) supermodule over A. Then $C^*(A;P)$ is a (two sided) supermodule over the $\mathbb{Z}$-graded Lie superalgebra $\{C^*(A;A), [-,-]\}$. Also, we have*

$$[Z^{m_1}_{n_1}(A;P), Z^{m_2}_{n_2}(A;A)] \subset Z^{m_1+m_2}_{n_1+n_2}(A;P);$$

$$[B^{m_1}_{n_1}(A;P), Z^{m_2}_{n_2}(A;A)], [Z^{m_1}_{n_1}(A;P), Z^{m_2}_{n_2}(A;A)] \subset B^{m_1+m_2}_{n_1+n_2}(A;P),$$

$\{H^*(A;A), [-,-]\}$ is a $\mathbb{Z}$-graded Lie superalgebra and $H^*(A;P)$ is a (two sided) supermodule over $\{H^*(A;A), [-,-]\}$.

Proof. From our previous discussion we already know that $C^*(A;P)$ is a supermodule over the $\mathbb{Z}$-graded Lie superalgebra $\{C^*(A;A),[-,-]\}$. Consider the associative superalgebra $A\oplus P$ and $\mathbb{Z}$-graded Lie superalgebra $\{C^*(A\oplus P;A\oplus P),[-,-]\}$. Since coboundary map δ is an inner right derivation of $\{C^*(A\oplus P;A\oplus P),[-,-]\}$ induced by $-\pi$, we have

$$\delta[f,g]=[f,\delta g]+(-1)^{n-1}[\delta f,g] \tag{45}$$

for all $f\in C^m(A\oplus P;A\oplus P)$, $g\in C^n(A\oplus P;A\oplus P)$. In particular, Equation 45 holds for f in $C^m(A;A)$ or $C^m(A;P)$, g in $C^n(A;P)$ or $C^n(A;A)$; and for δ as a coboundary map on $C^*(A;A)$ or $C^*(A;P)$. From this we conclude the theorem. □

10. Formal Deformation of Associative Superalgebras

Given an associative superalgebra $A=A_0\oplus A_1$, we denote the ring of all formal power series with coefficients in A by $A[[t]]$. Thus $A[[t]]=A_0[[t]]\oplus A_1[[t]]$. If $a_t\in A[[t]]$, then $a_t=a_{t_0}\oplus a_{t_1}$, where $a_{t_0}\in A_0[[t]]$ and $a_{t_1}\in A_1[[t]]$. $K[[t]]$ denotes the ring of all formal power series with coefficients in K.

Definition 10.1. A formal one-parameter deformation of an associative superalgebra $A=A_0\oplus A_1$ is a $K[[t]]$-bilinear map

$$\mu_t : A[[t]]\times A[[t]]\to A[[t]]$$

satisfying the following properties:

(a) $\mu_t(a,b)=\sum_{i=0}^{\infty}\mu_i(a,b)t^i$, for all $a,b\in A$, where $\mu_i : A\times A\to A$, $i\geqslant 0$ are bilinear homogeneous mappings of degree zero and $\mu_0(a,b)=\mu(a,b)$ is the original product on A.

(b)
$$\mu_t(\mu_t(a,b),c)=\mu_t(a,\mu_t(b,c)), \tag{46}$$
for all homogeneous $a,b,c\in A$.

The Equation 46 is equivalent to following equation:

$$\sum_{i+j=r}\mu_i(\mu_j(a,b),c)=\sum_{i+j=r}\{\mu_i(a,\mu_j(b,c)), \tag{47}$$

for all homogeneous $a,b,c\in A$.

Next we give definition of a formal deformation of finite order of an associative superalgebra A.

Definition 10.2. A formal one-parameter deformation of order n of an associative superalgebra $A = A_0 \oplus A_1$ is a $K[[t]]$-bilinear map

$$\mu_t : A[[t]] \times A[[t]] \to A[[t]]$$

satisfying the following properties:

(a) $\mu_t(a,b) = \sum_{i=0}^{n} \mu_i(a,b)t^i$, $\forall a,b,c \in A$, where $\mu_i : A \times A \to A$, $0 \leqslant i \leqslant n$, are K-bilinear homogeneous maps of degree 0, and μ_0 is the original product on A.

(b)
$$\mu_t(\mu_t(a,b),c) = \mu_t(a,\mu_t(b,c)), \tag{48}$$

for all homogeneous $a,b,c \in A$.

Remark 10.1. 1) *For $r = 0$, conditions 47 is equivalent to the fact that A is an associative superalgebra.*

2) *For $r = 1$, conditions 47 is equivalent to*

$$0 = -\mu_1(\mu_0(a,b),c) - \mu_0(\mu_1(a,b),c) \tag{49}$$
$$+ \mu_1(a,\mu_0(b,c)) + \mu_0(a,\mu_1(b,c)) \tag{50}$$
$$= \delta^2\mu_1(a,b,c); \text{ for all homogeneous } a,b,c \in A. \tag{51}$$

Thus for $r = 1$, 47 is equivalent to saying that $\mu_1 \in C_0^2(A;A)$ is a cocycle. In general, for $r \geqslant 0$, μ_r is just a 2-cochain, that is, $\mu_r \in C_0^2(A;A)$.

Definition 10.3. We call the cochain $\mu_1 \in C_0^2(A;A)$ infinitesimal of the deformation μ_t. In general, if $\mu_i = 0$, for $1 \leqslant i \leqslant n-1$, and μ_n is a nonzero cochain in $C_0^2(A;A)$, then we call μ_n n-infinitesimal of the deformation μ_t.

Proposition 10.1. *The infinitesimal $\mu_1 \in C_0^2(A;A)$ of the deformation μ_t is a cocycle. In general, n-infinitesimal μ_n is a cocycle in $C_0^2(A;A)$.*

Proof. For n=1, proof is obvious from the Remark 10.1. For $n > 1$, proof is similar. □

From Equation 47, we have

$$\sum_{\substack{i+j=r\\ i,j>0}} \mu_i(\mu_j(a,b),c) - \sum_{\substack{i+j=r\\ i,j>0}} \{\mu_i(a,\mu_j(b,c))$$
$$= \mu_r(a,\mu_0(b,c)) + \mu_0(a,\mu_r(b,c)) - \mu_0(\mu_r(a,b),c) - \mu_r(\mu_0(a,b),c)$$
$$= \delta^2\mu_r(a,b,c) \tag{52}$$

Definition 10.4. Given a formal deformation μ_t of order n of an associative superalgebra $A = A_0 \oplus A_1$, we define a 3-cochain $Ob_{n+1}(A)$ by

$$Ob_{n+1}(A) = \sum_{\substack{i+j=r\\ i,j>0}} \mu_i(\mu_j(a,b),c) - \sum_{\substack{i+j=r\\ i,j>0}} \{\mu_i(a,\mu_j(b,c)).$$

We call $Ob_{n+1}(A)$ $(n+1)$th obstruction cochain for extending μ_t to a deformation of A of order $n+1$.

As an application of Theorem 9.1 we conclude following Lemma:

Lemma 10.1. *For all $\mu_\alpha, \mu_\beta, \mu_\gamma \in C_0^n(A;A)$ with β,γ an even integer, we have*

(a) $\mu_\alpha \circ (\mu_\beta \circ \mu_\beta) = (\mu_\alpha \circ \mu_\beta) \circ \mu_\beta$,
(b) $(\mu_\alpha \circ \mu_\beta) \circ \mu_\gamma - \mu_\alpha \circ (\mu_\beta \circ \mu_\gamma) = -(\mu_\alpha \circ \mu_\gamma) \circ \mu_\beta + \mu_\alpha \circ (\mu_\gamma \circ \mu_\beta)$.

Theorem 10.1. *The $(n+1)$th obstruction cochain is a 3-cocycle.*

Proof. Consider the right pre-Lie superalgebra $\{c*(A,A),\circ\}$. By definition of $\circ$, we have

$$Ob_{n+1}(A)(a,b,c) = \sum_{\substack{i+j=n+1\\ i,j>0}} \mu_i \circ \mu_j(a,b,c). \tag{53}$$

By using Theorem 9.2 we have

$$\mu_i \circ \delta\mu_j - \delta(\mu_i \circ \mu_j) - \delta\mu_i \circ \mu_j = -\mu_j \cup \mu_i + \mu_i \cup \mu_j, \tag{54}$$

for all $\mu_i, \mu_j \in C_0^2(A;A)$. From relations 53 and 54 we have

$$\begin{aligned}
\delta Ob_{n+1}(A) &= \sum_{\substack{i+j=n+1\\ i,j>0}} \delta(\mu_i \circ \mu_j)\\
&= \sum_{\substack{i+j=n+1\\ i,j>0}} \{\mu_i \circ \delta\mu_j - \delta\mu_i \circ \mu_j + \mu_j \cup \mu_i - \mu_i \cup \mu_j\}\\
&= \sum_{\substack{i+j=n+1\\ i,j>0}} \{\mu_i \circ \delta\mu_j - \delta\mu_i \circ \mu_j\} \qquad (55)
\end{aligned}$$

Observe that if $\mu_t = \sum_{i=0}^n \mu_i t^i$ is a deformation of A of order n, then

$$\delta\mu_\gamma(a,b,c) = \sum_{\substack{\alpha+\beta=\gamma\\ \alpha,\beta>0}} \mu_\alpha \circ \mu_\beta(a,b,c), \tag{56}$$

$\forall\gamma \leqslant n$. Using Equations 55 and 56 we have

$$\delta Ob_{n+1}(A) = \sum_{\substack{i+j+k=n+1\\ i,j,k>0}} \{\mu_i \circ (\mu_j \circ \mu_k) - (\mu_i \circ mu_j) \circ \mu_k\}$$

$$= \sum_{\substack{i+j+k=n+1\\ i,j,k>0, j<k}} \{\mu_i \circ (\mu_j \circ \mu_k + \mu_k \circ \mu_j) - (\mu_i \circ \mu_j) \circ \mu_k + (\mu_i \circ \mu_k) \circ \mu_j\}$$

(Using Lemma 10.1 Part (a))

$= 0$ (Using Lemma 10.1 Part (b)) □

As a consequence of above theorem we conclude following corollary

Corollary 10.1. *If $H^3(A;A) = 0$, then every 2-cocycle in $C_0^2(A;A)$ is an infinitesimal of some deformation of A.*

Definition 10.5. Let μ_t and $\tilde{\mu_t}$ be two formal deformations of an associative superalgebra $A = A_0 \oplus A_1$. A formal isomorphism from the deformation μ_t to $\tilde{\mu_t}$ is a $K[[t]]$-linear automorphism $\Psi_t : A[[t]] \to A[[t]]$ given by

$$\Psi_t = \sum_{i=0}^{\infty} \psi_i t^i,$$

where each ψ_i is a homogeneous K-linear map $A \to A$ of degree 0, $\psi_0(a) = a$, for all $a \in A$ and

$$\tilde{\mu_t}(\Psi_t(a), \Psi_t(b)) = \Psi_t \circ \mu_t(a, b),$$

for all $a, b \in A$.

We call two deformations μ_t and $\tilde{\mu_t}$ of an associative superalgebra A to be equivalent if there exists a formal isomorphism Ψ_t from μ_t to $\tilde{\mu_t}$. Observe that Formal isomorphism on the collection of all formal deformations of an associative superalgebra A is an equivalence relation. We call a formal deformation of A that is equivalent to the deformation μ_0 a trivial deformation.

Theorem 10.2. *The cohomology class of the infinitesimal of a deformation μ_t of an associative superalgebra A is same for each member of equivalence class of μ_t.*

Proof. If Ψ_t is a formal isomorphism from μ_t to $\tilde{\mu_t}$, the we have, for all $a, b \in A$, $\tilde{\mu_t}(\Psi_t a, \Psi_t b) = \Psi_t \circ \mu_t(a, b)$. In particular, we have

$$(\mu_1 - \tilde{\mu_1})(a, b) = \mu_0(\psi_1 a, b) + \mu_0(a, \psi_1 b) - \psi_1(\mu_0(a, b)) = \delta^1\psi_1(a, b).$$

Thus we have $\mu_1 - \tilde{\mu_1} = \delta^1\psi_1$. This completes the proof. □

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Received by the editors: 31.08.2022
and in final form 09.03.2023.