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RESEARCH ARTICLE

Transformations of $(0, 1]$ preserving tails of Δ^{μ}-representation of numbers

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Abstract. Let $\mu \in (0, 1)$ be a given parameter, $\nu \equiv 1-\mu$. We consider Δ^{μ}-representation of numbers $x = \Delta^{\mu}_{a_1 a_2 \dots a_n \dots}$ belonging to $(0, 1]$ based on their expansion in alternating series or finite sum in the form:

$$x = \sum_n (B_n - B'_n) \equiv \Delta^{\mu}_{a_1 a_2 \dots a_n \dots},$$

where $B_n = \nu^{a_1+a_3+\dots+a_{2n-1}-1}\mu^{a_2+a_4+\dots+a_{2n-2}}$,
$B'_n = \nu^{a_1+a_3+\dots+a_{2n-1}-1}\mu^{a_2+a_4+\dots+a_{2n}}$, $a_i \in \mathbb{N}$.
This representation has an infinite alphabet $\{1, 2, \dots\}$, zero redundancy and N-self-similar geometry.

In the paper, classes of continuous strictly increasing functions preserving "tails" of Δ^{μ}-representation of numbers are constructed. Using these functions we construct also continuous transformations of $(0, 1]$. We prove that the set of all such transformations is infinite and forms non-commutative group together with an composition operation.

Introduction

We consider representation of real numbers belonging to half-interval $(0, 1]$. It depends on real parameter $\mu \in (0, 1)$ and has an infinite alphabet $\mathbb{N} = \{1, 2, 3, \dots\}$. This representation is based on the following theorem.

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Theorem 1 ([19]). *Let $(0,1)\ni\mu$ be a fixed real number, $\nu\equiv 1-\mu$. For any $x\in(0,1]$, there exists a finite tuple of positive integers $(a_1,a_2,\dots,a_m)$ or a sequence of positive integers (a_n) such that*

$$x=\nu^{a_1-1}-\nu^{a_1-1}\mu^{a_2}+\nu^{a_1+a_3-1}\mu^{a_2}-\nu^{a_1+a_3-1}\mu^{a_2+a_4}+\ldots=$$

$$=\sum_n (B_n-B'_n), \tag{1}$$

where $B_n=\nu^{a_1+a_3+\ldots+a_{2n-1}-1}\mu^{a_2+a_4+\ldots+a_{2n-2}},\ \ B'_n=B_n\cdot\mu^{a_{2n}}.$

We call expansion of the number x in the form of alternating series (1) the Δ^μ*-expansion* and its symbolic notation $\Delta^\mu_{a_1a_2\ldots a_m(\varnothing)}$ for finite expansion of number x or $\Delta^\mu_{a_1a_2\ldots a_n\ldots}$ for infinite sum the Δ^μ*-representation.*

Remark that expansion of a number in the form of alternating series (1) first appeared in papers [23, 24] in an expression of strictly increasing singular function φ_μ being an unique continuous solution of a system of functional equations:

$$\begin{cases}\varphi_\mu\left(\dfrac{x}{1+x}\right)=(1-\mu)\varphi_\mu(x),\\ \varphi_\mu(1-x)=1-\varphi_{1-\mu}(x)\,.\end{cases}$$

This function generalizes the well-known singular Minkowski function [1–8, 10–16, 25] and coincides with it for $\mu=1/2$. In this case the Δ^μ-representation is the $\Delta^\sharp$-representation studied in papers [20, 21].

There exists a countable everywhere dense in $[0,1]$ set of numbers having two Δ^μ-representation. These numbers have a form: $\Delta^\mu_{a_1\ldots[a_m+1](\varnothing)}=$ $=\Delta^\mu_{a_1\ldots a_m1(\varnothing)}$. We call these numbers Δ^μ*-finite*. Other numbers belonging to $(0,1]$ have a unique Δ^μ-representation, their expansions are infinite, so we call them Δ^μ*-infinite* numbers. That is, Δ^μ-representation has a zero redundancy. We denote the set of all Δ^μ-infinite numbers by H and the set of Δ^μ-finite numbers by S.

The Δ^μ-representation of number is called the *rational* Δ^μ*-representation* if $\mu\in(0,1)$ is rational. In this case irrational numbers belonging to $(0,1]$ have infinite non-periodic Δ^μ-representation and rational numbers have either finite or infinite periodic or infinite non-periodic Δ^μ-representation [19]. So the set H contains all irrational numbers and everywhere dense in $[0,1]$ subset of rational numbers.

Remark that Δ^μ-representation has much in common with encoding of real numbers by regular continued fraction [9, 17], namely, they have the same topology, rules for comparing numbers etc. However, Δ^μ-representation generates other metric relations, that is, it has own original metric theory [19].

In the paper, we construct an infinite non-commutative group of continuous strictly increasing piecewise linear transformations of $(0,1]$ preserving tails of Δ^μ-representation of numbers. Analogous objects for E-representation based on expansions of numbers in the form of positive Engel series are discussed in paper [18]. This representation has fundamental distinctions from E-representation in topological as well as metric aspects.

1. Geometry of Δ^μ-representation of numbers

Geometric meaning of digits of Δ^μ-representation of numbers and essence of related positional and metric problems are disclosed by the following important notion.

Definition 1. Let $(c_1, c_2, \ldots, c_m)$ be a tuple of positive integers. *Cylinder of rank m with base $c_1c_2\ldots c_m$* is a set $\Delta^\mu_{c_1c_2\ldots c_m}$ of numbers $x \in (0,1]$ having Δ^μ-representation such that $a_i(x) = c_i$, $i = \overline{1,m}$.

Cylinders have the following properties.

1. $\bigcup\limits_{a_1\in\mathbb{N}} \bigcup\limits_{a_2\in\mathbb{N}} \ldots \bigcup\limits_{a_m\in\mathbb{N}} \Delta^\mu_{a_1a_2\ldots a_m} = (0,1]$;   2. $\Delta^\mu_{c_1c_2\ldots c_m} = \bigcup\limits_{i=1}^{\infty} \Delta^\mu_{c_1c_2\ldots c_m i}$;
3. Cylinder $\Delta^\mu_{c_1c_2\ldots c_m}$ is a closed interval, moreover, if m is odd, then $\Delta^\mu_{c_1c_2\ldots c_{2k-1}} = [a-\delta, a]$, where

$$\delta = \nu^{c_1+c_3+\ldots+c_{2k-1}-1} \cdot \mu^{c_2+c_4+\ldots+c_{2k-2}+1};$$

$$a = \nu^{c_1-1} - \nu^{c_1-1}\mu^{c_2} + \ldots + \nu^{c_1+c_3+\ldots+c_{2k-1}-1}\mu^{c_2+c_4+\ldots+c_{2k-2}},$$

if m is even, then $\Delta^\mu_{c_1c_2\ldots c_{2k}} = [a, a+\delta]$, where

$$\delta = \nu^{c_1+c_3+\ldots+c_{2k-1}} \cdot \mu^{c_2+c_4+\ldots+c_{2k}}.$$

$$a = \nu^{c_1-1} - \nu^{c_1-1}\mu^{c_2} + \ldots +$$

$$+\nu^{c_1+c_3+\ldots+c_{2k-1}-1}\mu^{c_2+c_4+\ldots+c_{2k-2}} - \nu^{c_1+c_3+\ldots+c_{2k-1}-1}\mu^{c_2+c_4+\ldots+c_{2k}},$$

4. The length of cylinder of rank m is calculated by the formulae:

$$|\Delta^\mu_{c_1\ldots c_m}| = \begin{cases} \nu^{c_1+c_3+\ldots+c_{2k-1}-1} \cdot \mu^{c_2+c_4+\ldots+c_{2k-2}+1} & \text{if } m=2k-1, \\ \nu^{c_1+c_3+\ldots+c_{2k-1}} \cdot \mu^{c_2+c_4+\ldots+c_{2k}} & \text{if } m=2k. \end{cases}$$

5. If $\Delta^{\mu}_{c_1c_2\ldots c_m}$ is a fixed cylinder, then the following equality (basic metric relation) holds:

$$\frac{|\Delta^{\mu}_{c_1c_2\ldots c_m i}|}{|\Delta^{\mu}_{c_1c_2\ldots c_m}|} = \begin{cases} \nu\mu^{i-1} & \text{if} \quad m = 2k-1, \\ \mu\nu^{i-1} & \text{if} \quad m = 2k. \end{cases}$$

6. $\min\Delta^{\mu}_{c_1\ldots c_{2k-1}i} = \max\Delta^{\mu}_{c_1\ldots c_{2k-1}(i+1)}$; $\max\Delta^{\mu}_{c_1\ldots c_{2k}} = \min\Delta^{\mu}_{c_1\ldots c_{2k}(i+1)}$;
7. Cylinders of the same rank do not intersect or coincide. Moreover,

$$\Delta^{\mu}_{c_1c_2\ldots c_m} = \Delta^{\mu}_{c'_1c'_2\ldots c'_m} \iff c_i = c'_i \quad i = \overline{1,m};$$

8. For any sequence (c_m), $c_m \in \mathbb{N}$, intersection

$$\bigcap_{m=1}^{\infty} \Delta^{\mu}_{c_1c_2\ldots c_m} = x \equiv \Delta^{\mu}_{c_1c_2\ldots c_m\ldots}$$

is a point belonging to half-interval $(0,1]$.

In paper [19], it is proved that geometry of Δ^{μ}-representation of numbers is N-self-similar and foundations of metric theory are laid. In paper [22], functions with fractal properties defined in terms of Δ^{μ}-representation are considered. Geometry plays an essential role in studies of such functions.

2. Tail sets and functions preserving tails of $\boldsymbol{\Delta^{\mu}}$-representation of numbers

Let $\mathcal{Z}^{\mu}_H$ be the set of all Δ^{μ}-representations of numbers belonging to set H. We introduce binary relation "has the same tail" (symbolically: $\sim$) on the set $\mathcal{Z}^{\mu}_H$.

Two Δ^{μ}-representations $\Delta^{\mu}_{a_1a_2\ldots a_n\ldots}$ and $\Delta^{\mu}_{b_1b_2\ldots b_n\ldots}$ are said *to have the same tail* (or they are *$\sim$-related*) if there exist positive integers k and m such that $a_{k+j} = b_{m+j}$ for any $j \in \mathbb{N}$.

It is evident that binary relation $\sim$ is an equivalence relation (i.e., it is reflexive, symmetric and transitive) and provides a partition of the set $\mathcal{Z}^{\mu}_H$ into equivalence classes. Any equivalence class is said to be a *tail set*. Any tail set is uniquely determined by its arbitrary element (representative).

We say that two numbers x and y belonging to set H have the same tail of Δ^{μ}-representation (or they are $\sim$-related) if their Δ^{μ}-representations are $\sim$-related. We denote this symbolically as $x \sim y$.

Theorem 2. *Any tail set is countable and dense in* $(0,1]$*; quotient set* $F \equiv (0,1]/\sim$ *is a continuum set.*

Proof. Suppose K is an arbitrary equivalence class, $x_0 = \Delta^{\mu}_{c_1c_2\ldots c_n\ldots}$ is its representative. Then it is evident that, for any $m \in \mathbb{Z}_0$, there exists set $K_m = \left\{x : \ x = \Delta^{\mu}_{a_1\ldots a_k c_{m+1}c_{m+2}\ldots}, \quad a_i \in \mathbb{N}, \ k = 0, 1, 2, \ldots\right\}$ of numbers x such that for some $k \in \mathbb{Z}_0$

$$a_{k+j}(x) = c_{m+j} \quad \text{for any } j \in \mathbb{N} \quad \text{and} \quad K = \bigcup_{m\in\mathbb{Z}_0} K_m \,.$$

The set K is countable because it is a countable union of countable sets.

Now we prove that K is a dense in $(0, 1]$ set. Since number x belongs or does not belong to the set K irrespective of any finite amount of first digits of its Δ^{μ}-representation, we have that any cylinder of arbitrary rank m contains point belonging to K. Thus K is an everywhere dense in half-interval $(0, 1]$ set.

To prove that quotient set $F \equiv (0, 1]/\sim$ is continuum set, we assume the converse. Suppose that F is a countable set. Then half-interval $(0, 1]$ is a countable set as a countable union of countable sets (equivalence classes of quotient set F). This contradiction proves the theorem. □

Remark that it is easy to introduce a distance function (metric) in the quotient set F.

Definition 2. Suppose function f is defined on the set H and takes values from this set. We say that function f *preserves tails* of Δ^{μ}-representations of numbers if for any $x \in (0, 1]$ there exist positive integers $k = k(x)$ and $m = m(x)$ such that

$$a_{k+n}(x) = a_{m+n}\left(f(x)\right) \quad \text{for all } n \in \mathbb{N}.$$

It is clear that functions preserving tails of Δ^{μ}-representations of numbers form an infinite set. However, only continuous functions are interested for us. Identity transformation $y = e(x)$ is a simplest example of such function.

By X we denote the set of all functions satisfying Definition 2. In the sequel, we consider some representatives of this class.

3. Function $\sigma_1(x)$

We consider function defined on the set H by equality

$$y = \sigma_1(x) = \sigma_1\left(\Delta^{\mu}_{a_1(x)a_2(x)a_3(x)a_4(x)\ldots a_n(x)\ldots}\right) = \Delta^{\mu}_{[a_1+a_2+a_3]a_4\ldots a_n\ldots} \,.$$

This function is well-defined due to uniqueness of Δ^{μ}-representation of numbers belonging to the set H. It is evident that it preserves tails of Δ^{μ}-representation of numbers.

Lemma 1. *Analytic expression for function $y = \sigma_1(x)$ is given by formula*

$$\sigma_1(x) = \left(\frac{\nu}{\mu}\right)^{a_2(x)} \cdot x + \nu^{a_1(x)+a_2(x)-1}\left(1 - \frac{1}{\mu^{a_2(x)}}\right), \tag{2}$$

this function is linear on every cylinder of rank 2 and has the following properties:

1) it is continuous strictly increasing function;

2) $\sup\limits_{x\in\Delta^{\mu}_{ij}} \sigma_1(x) = \nu^{i+j}$, $\inf\limits_{x\in\Delta^{\mu}_{ij}} \sigma_1(x) = 0$;

3) $\int\limits_{\Delta^{\mu}_{ij}} \sigma_1(x)dx = \frac{1}{2}\nu^{2i+j}\mu^j$; *4)* $\int\limits_0^1 \sigma_1(x)dx = \frac{1}{2} \cdot \frac{\nu^3}{1+\nu^3}$.

Proof. 1. Indeed, if $x = \Delta^{\mu}_{a_1a_2a_3a_4a_5\ldots a_n\ldots}$, then

$$x = \nu^{a_1-1} - \nu^{a_1-1}\mu^{a_2} + \nu^{a_1+a_3-1}\mu^{a_2} - \nu^{a_1+a_3-1}\mu^{a_2+a_4} + \ldots =$$

$$= \nu^{a_1-1} - \nu^{a_1-1}\mu^{a_2} + \frac{\mu^{a_2}}{\nu^{a_2}} \cdot \sigma_1(x).$$

Whence it follows that

$$\sigma_1(x) = \left(\frac{\nu}{\mu}\right)^{a_2(x)} \cdot x + \nu^{a_1(x)+a_2(x)-1}\left(1 - \frac{1}{\mu^{a_2(x)}}\right).$$

It is evident that function $\sigma_1(x)$ is linear. Therefore it is continuous strictly increasing on the set $H \cap \Delta^{\mu}_{a_1a_2}$. Extending by continuity in Δ^{μ}-finite points we obtain continuous function on the whole cylinder $\Delta^{\mu}_{a_1a_2}$.

2. Boundary values of function $\sigma_1(x)$ on cylinder Δ^{μ}_{ij} can be calculated by formulae:

$$\sup_{x\in\Delta^{\mu}_{ij}} \sigma_1(x) = \lim_{k\to\infty} \sigma_1\left(\Delta^{\mu}_{ij1(k)}\right) = \Delta^{\mu}_{[i+j+1](\varnothing)} = \nu^{i+j}.$$

$$\inf_{x\in\Delta^{\mu}_{ij}} \sigma_1(x) = \lim_{k\to\infty} \sigma_1\left(\Delta^{\mu}_{ij(k)}\right) = \lim_{k\to\infty} \Delta^{\mu}_{[i+j+k](k)} = 0.$$

3. Calculate integral on cylinder Δ^{μ}_{ij}:

$$\int\limits_{\Delta^{\mu}_{ij}} \sigma_1(x)dx = \int\limits_{\Delta^{\mu}_{ij(\varnothing)}}^{\Delta^{\mu}_{i[j+1](\varnothing)}} \sigma_1(x)dx = \int\limits_{\nu^{i-1}(1-\mu^j)}^{\nu^{i-1}\left(1-\mu^{j+1}\right)} \sigma_1(x)dx = \frac{1}{2}\nu^{2i+j}\mu^j.$$

4. Calculate integral on the unit interval:

$$\int_0^1 \sigma_1(x)dx = \frac{1}{2}\sum_{i=1}^{\infty}\nu^{2i}\sum_{j=1}^{\infty}\nu^j\mu^j = \frac{1}{2}\cdot\frac{\nu^2}{1-\nu^2}\cdot\frac{\nu\mu}{1-\nu\mu} = \frac{1}{2}\cdot\frac{\nu^3}{1+\nu^3}. \quad \square$$

4. Function $d_s(x)$

Let s be a fixed positive integer. We consider function depending on parameter s, well-defined on half-interval $(0,1]$ by equality

$$y = d_s(x) = d_s\left(\Delta^{\mu}_{a_1(x)a_2(x)a_3(x)\ldots}\right) = \Delta^{\mu}_{[s+a_1]a_2a_3\ldots}.$$

Since s is an arbitrary positive integer, we have a countable class of functions $y = d_s(x)$.

Theorem 3. *Function d_s is analytically expressed by formula:*

$$d_s(x) = \nu^s \cdot x$$

and has the following properties:

1) it is linear strictly increasing, 2) $\inf\limits_{x\in(0,1]} d_s(x)=0$, $\sup\limits_{x\in(0,1]} d_s(x)=\nu^s$.

Moreover, equation $\sigma_1(x) = d_s(x)$ does not have solutions if $a_2 \geqslant s$, and has a countable set of solutions:

$$E = \left\{x : x = \Delta^{\mu}_{a_1(a_2[s-a_2])}, \quad \text{where } a_1 \in \mathbb{N},\ a_2 \in \{1,2,\ldots,s-1\}\right\}$$

if $a_2 < s$.

Proof. By definition of function d_s, we have

$$d_s(x) = \Delta^{\mu}_{[s+a_1]a_2a_3\ldots} = \nu^{s+a_1-1} - \nu^{s+a_1-1}\mu^{a_2} + \ldots = \nu^s \cdot x,$$

Thus $d_s(x) = \nu^s \cdot x$. It is evident that function d_s is linear strictly increasing on half-interval $(0,1]$. Moreover,

$$\inf_{x\in(0,1]} d_s(x) = \lim_{x\to 0+0} d_s(x) = \lim_{k\to\infty} d_s\left(\Delta^{\mu}_{(k)}\right) = \lim_{k\to\infty}\Delta^{\mu}_{[s+k](k)} = 0;$$

$$\sup_{x\in(0,1]} d_s(x) = \lim_{x\to 1-0} d_s(x) = \lim_{k\to\infty} d_s\left(\Delta^{\mu}_{1(k)}\right) = \Delta^{\mu}_{[s+1](\varnothing)} = \nu^s.$$

We can write equation $\sigma_1(x) = d_s(x)$ in the form

$$\Delta^{\mu}_{[a_1(x)+a_2(x)+a_3(x)]a_4(x)\ldots} = \Delta^{\mu}_{[s+a_1(x)]a_2(x)a_3(x)a_4(x)\ldots}.$$

From uniqueness of Δ^{μ}-representation of numbers belonging to set H it follows that following equalities hold simultaneously:

$$\begin{aligned}&a_1(x)+a_2(x)+a_3(x)=s+a_1(x), \quad a_4(x)=a_2(x),\\ &a_5(x)=a_3(x)=s-a_2(x), \quad \ldots \quad a_{2k}(x)=a_2(x),\\ &a_{2k+1}(x)=s-a_2(x), \quad k\in\mathbb{N}.\end{aligned}$$

It is evident that this system is inconsistent if $a_2 \geqslant s$. However, for $a_2 < s$, equation has a countable set of solutions $x=\Delta^{\mu}_{a_1(a_2[s-a_2])}$, where a_1, a_2 are independent positive integer parameters. □

5. Left shift operator on digits of Δ^{μ}-representation of number

Let $\mathcal{Z}^{\mu}_{H}$ be the set of all Δ^{μ}-representations of numbers belonging to set H. We consider shift operator ω_2 on digits defined by equality

$$\omega_2\left(\Delta^{\mu}_{a_1a_2a_3a_4\ldots a_n\ldots}\right)=\Delta^{\mu}_{a_3a_4\ldots a_n\ldots}.$$

This operator generates function $y=\omega_2(x)=\Delta^{\mu}_{a_3(x)a_4(x)\ldots a_n(x)\ldots}$ on the set H. It is evident that operator ω_2 is surjective but not injective.

Any point $\Delta^{\mu}_{(ij)}=\dfrac{\nu^{i-1}(1-\mu^j)}{1-\nu^i\mu^j}$, where (i,j) is any pair of positive integers, is an invariant point of the mapping ω_2.

Lemma 2. *Function $y=\omega_2(x)$ is analytically expressed by formula*

$$\omega_2(x)=\frac{x}{\nu^{a_1(x)}\mu^{a_2(x)}}-\frac{1-\mu^{a_2(x)}}{\nu\mu^{a_2(x)}} \tag{3}$$

and is continuous monotonically increasing on any cylinder of rank 2.

Proof. Let $x\in\Delta^{\mu}_{ij}$. Then $x=\Delta^{\mu}_{ija_3a_4\ldots}$ and

$$\begin{aligned}x=\nu^{i-1}-\nu^{i-1}\mu^j+\nu^{i+a_3-1}\mu^j-\nu^{i+a_3-1}\mu^{j+a_4}+\ldots=\\ =\nu^{i-1}-\nu^{i-1}\mu^j+\nu^i\mu^j\cdot\omega_2(x).\end{aligned}$$

Whence, $\omega_2(x)=\dfrac{x}{\nu^i\mu^j}-\dfrac{1-\mu^j}{\nu\mu^j}$.

Since function ω_2 is linear, we have that this function is continuous strictly increasing on the set $H\cap\Delta^{\mu}_{a_1a_2}$. Extending by continuity in the points of the set S we obtain continuous function on the whole cylinder $\Delta^{\mu}_{a_1a_2}$. □

Lemma 3. *Equation* $d_s(x) = \omega_2(x)$ *has a countable set of solutions having the form* $x = \Delta^{\mu}_{a_1(a_2[s+a_1])}$, *where* a_1, a_2 *are arbitrary positive integers.*

Proof. We can write equation $d_s(x) = \omega_2(x)$ in the form

$$\Delta^{\mu}_{[s+a_1(x)]a_2(x)a_3(x)a_4(x)\ldots} = \Delta^{\mu}_{a_3(x)a_4(x)\ldots}.$$

From uniqueness of Δ^{μ}-representation of numbers belonging to set H it follows that the following equalities hold simultaneously:

$$s + a_1(x) = a_3(x), \quad a_2(x) = a_4(x), \quad a_3(x) = a_5(x) = s + a_1(x),$$
$$a_4(x) = a_6(x) = a_2(x), \quad \ldots, \quad a_{2k+1}(x) = s + a_1(x),$$
$$a_{2k}(x) = a_2(x), \quad k \in \mathbb{N}.$$

Then solutions of equation are numbers having the form $x=\Delta^{\mu}_{a_1(a_2[s+a_1])}$, where a_1, $a_2 \in \mathbb{N}$. □

6. Right shift operator on digits of Δ^{μ}-representation of number

Let i, j be fixed positive integers. We consider operator depending on parameters i, j, well-defined on half-interval $(0, 1]$ by equality

$$\delta_{ij}(x) = \delta_{ij}\left(\Delta^{\mu}_{a_1(x)a_2(x)\ldots}\right) = \Delta^{\mu}_{ija_1a_2\ldots}.$$

This operator defines a countable set of functions $y = \delta_{ij}(x)$, $i \in \mathbb{N}$, $j \in \mathbb{N}$.

Lemma 4. *Function* $y = \delta_{ij}(x)$ *is analytically expressed by formula*

$$y = \delta_{ij}(x) = \nu^i \mu^j \cdot x + \nu^{i-1}\left(1 - \mu^j\right)$$

and is linear strictly increasing on half-interval $(0, 1]$, *moreover,*

$$\inf_{x\in(0,1]} \delta_{ij}(x) = \Delta^{\mu}_{ij(\varnothing)} = \nu^{i-1}\left(1 - \mu^j\right),$$

$$\sup_{x\in(0,1]} \delta_{ij}(x) = \Delta^{\mu}_{ij1(\varnothing)} = \nu^{i-1}\left(1 - \mu^{j+1}\right).$$

Proof. In fact, by definition of function δ_{ij}, we have:

$$y=\delta_{ij}(\Delta^{\mu}_{a_1a_2\ldots})=\Delta^{\mu}_{ija_1a_2\ldots}=\nu^{i-1}-\nu^{i-1}\mu^j+\nu^{i+a_1-1}\mu^j-\nu^{i+a_1-1}\mu^{j+a_2}+\ldots=$$

$$= \nu^{i-1} - \nu^{i-1}\mu^j + \nu^i\mu^j \underbrace{\left(\nu^{a_1-1} - \nu^{a_1-1}\mu^{a_2} + \ldots\right)}_{x} = \nu^{i-1} - \nu^{i-1}\mu^j + \nu^i\mu^j \cdot x.$$

Therefore, $y = \delta_{ij}(x) = \nu^i\mu^j \cdot x + \nu^{i-1}\left(1 - \mu^j\right)$.

From linearity of function δ_{ij} it follows that it is a continuous strictly increasing function on $(0, 1]$ for any pair of positive integers (i, j). Moreover,

$$\inf_{x\in(0,1]} \delta_{ij}(x) = \lim_{x\to 0+0} \delta_{ij}(x) = \lim_{k\to\infty} \delta_{ij}\left(\Delta^{\mu}_{(k)}\right) = \lim_{k\to\infty} \Delta^{\mu}_{ij(k)} =$$

$$= \Delta^{\mu}_{ij(\varnothing)} = \nu^{i-1}\left(1 - \mu^j\right);$$

$$\sup_{x\in(0,1]} \delta_{ij}(x) = \lim_{x\to 1-0} \delta_{ij}(x) = \lim_{k\to\infty} \delta_{ij}\left(\Delta^{\mu}_{1(k)}\right) =$$

$$= \Delta^{\mu}_{ij1(\varnothing)} = \nu^{i-1}\left(1 - \mu^{j+1}\right).$$ □

For functions ω_2 and δ_{ij}, the following equalities are obvious:

$$\omega_2\left(\delta_{ij}\right) = x, \qquad \delta_{a_1(x)a_2(x)}\left(\omega_2(x)\right) = x.$$

Theorem 4. *For function δ_{ij}, the following propositions are true.*
1. *Equation $\sigma_1(x) = \delta_{ij}(x)$ does not have any solution if $a_1 + a_2 \geqslant i$ and has a countable set of solutions*

$$E = \left\{x : \ x = \Delta^{\mu}_{(a_1a_2[i-a_1-a_2]j)},\ a_1 \in \mathbb{N},\ a_2 \in \mathbb{N},\ a_1 + a_2 \in \{1, 2, \ldots, i-1\}\right\}$$

if $a_1 + a_2 < i$.

2. *Equation $d_s(x) = \delta_{ij}(x)$ does not have any solution if $s \geqslant i$ and has a countable set of solutions*

$$E = \left\{x : \ x = \Delta^{\mu}_{([i-s]j)},\ s \in \mathbb{N},\ s \in \{1, 2, \ldots, i-1\}\right\}$$

if $s < i$.

3. *Equation $\omega_2(x) = \delta_{ij}(x)$ has infinitely many solutions having a general form*

$$x = \Delta^{\mu}_{(a_1a_2ij)}, \quad \text{where } (a_1, a_2) \text{ is an arbitrary pair of positive integers.}$$

Proof. 1. We can write equation $\sigma_1(x) = \delta_{ij}(x)$ in the form

$$\Delta^{\mu}_{[a_1(x)+a_2(x)+a_3(x)]a_4(x)a_5(x)\ldots} = \Delta^{\mu}_{ija_1(x)a_2(x)a_3(x)a_4(x)\ldots}.$$

From uniqueness of Δ^{μ}-representation of numbers belonging to H it follows that following equalities holds simultaneously:

$$\begin{aligned}&a_1(x)+a_2(x)+a_3(x)=i,\ \ a_4(x)=j,\ \ a_5(x)=a_1(x),\ \ a_6(x)=a_2(x),\\&a_7(x){=}a_3{=}i-(a_1+a_2),\ a_8(x){=}a_4{=}j,\ \dots,\ a_{4k-1}(x){=}i-(a_1+a_2),\\&a_{4k}(x)=j,\ a_{4k+1}(x)=a_1,\ a_{4k+2}(x)=a_2,\quad k\in\mathbb{N}.\end{aligned}$$

Then this system does not have any solution if $a_1+a_2\geqslant i$ and have a countable set of solutions $E=\left\{x:\ x=\Delta^{\mu}_{(a_1a_2[i-a_1-a_2]j)}\right\}$, where a_1, a_2 are independent positive integer parameters, if $a_1+a_2<i$.

Similarly, we can prove statements 2 and 3 of the theorem. □

7. Transformations preserving tails of Δ^{μ}-representation of numbers

Recall that *transformation* of non-empty set E is any bijective (i.e., both injective and surjective) mapping of this set onto itself.

It is clear that continuous transformations of $[0,1]$ are strictly monotonic (increasing or decreasing) functions such that $f(0){=}0$ and $f(1){=}1$ or $f(0)=1$ and $f(1)=0$.

If f is a transformation of $[0,1]$, then $\varphi(x)=1-f(x)$ is also transformation of this set. Therefore, to study continuous transformations of $[0,1]$, we can consider only strictly increasing functions, i.e., continuous probability distribution functions.

Simple examples of continuous strictly increasing transformations preserving tails of Δ^{μ}-representation of numbers are the following functions:

$$\varphi_\tau(x)=\begin{cases}d_i(x) & \text{if}\quad 0<x\leqslant x_1\equiv\Delta^{\mu}_{a_1(a_2[i+a_1])},\\ \omega_2(x) & \text{if}\quad x_1<x\leqslant x_2\equiv\Delta^{\mu}_{(a_1a_2)},\\ e(x) & \text{if}\quad x_2<x\leqslant 1,\end{cases}$$

where $\tau=(i,a_1,a_2)$ is an arbitrary triplet of positive integers;

$$\psi(x)=\begin{cases}d_1(x) & \text{if}\quad 0<x\leqslant x_1\equiv\Delta^{\mu}_{1(12)},\\ \omega_2(x) & \text{if}\quad x_1<x\leqslant x_2\equiv\Delta^{\mu}_{(1112)},\\ \delta_{12}(x) & \text{if}\quad x_2<x\leqslant x_3\equiv\Delta^{\mu}_{(12)},\\ e(x) & \text{if}\quad x_3<x\leqslant 1;\end{cases}$$

$$\gamma(x) = \begin{cases} d_3(x) & \text{if} \quad 0 < x \leqslant x_1 \equiv \Delta^{\mu}_{1(12)}, \\ \sigma_1(x) & \text{if} \quad x_1 < x \leqslant x_2 \equiv \Delta^{\mu}_{(1111)}, \\ \delta_{31}(x) & \text{if} \quad x_2 < x \leqslant x_3 \equiv \Delta^{\mu}_{(1231)}, \\ \omega_2(x) & \text{if} \quad x_3 < x \leqslant x_4 \equiv \Delta^{\mu}_{(1212)}, \\ e(x) & \text{if} \quad x_4 < x \leqslant 1. \end{cases}$$

Theorem 5. *The set G of all continuous strictly increasing transformations of half-interval $(0,1]$ preserving tails of Δ^{μ}-representation of numbers together with an operation $\circ$ (function composition) form an infinite non-commutative group.*

Proof. The set of continuous transformations of $(0,1]$ is a subset of all transformations of $(0,1]$ forming a group. Thus we use a subgroup test. It is evident that set G is closed under the composition operation. For continuous strictly increasing function, inverse function is continuous and strictly increasing too. If transformation f preserves "tails" of Δ^{μ}-representations, then inverse transformation preserves them too. Therefore, for transformation $f \in G$, inverse transformation belongs to G too.

Since set of transformations φ_{τ}, $\tau \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, is countable, we see that set G is infinite.

To prove that group $(G, \circ)$ is non-commutative, we provide an example of two transformations f_1 and f_2 such that they are not commute, i.e., $f_2 \circ f_1 \neq f_1 \circ f_2$. Consider two transformations $\varphi_{\tau_1}(x)$ and $\varphi_{\tau_2}(x)$, where $\tau_1 = (1,2,3)$, $\tau_2 = (1,1,2)$, i.e.,

$$\varphi_{\tau_1}(x) = \begin{cases} d_1(x) & \text{if} \quad 0 < x \leqslant x_1 \equiv \Delta^{\mu}_{2(33)}, \\ \omega_2(x) & \text{if} \quad x_1 < x \leqslant x_2 \equiv \Delta^{\mu}_{(23)}, \\ e(x) & \text{if} \quad x_2 < x \leqslant 1; \end{cases}$$

$$\varphi_{\tau_2}(x) = \begin{cases} d_1(x) & \text{if} \quad 0 < x \leqslant x_3 \equiv \Delta^{\mu}_{1(22)}, \\ \omega_2(x) & \text{if} \quad x_3 < x \leqslant x_4 \equiv \Delta^{\mu}_{(12)}, \\ e(x) & \text{if} \quad x_4 < x \leqslant 1. \end{cases}$$

Then, for $x_0 = \Delta^{\mu}_{12(3)}$, tacking into account inequalities $x_0 > x_2 = \Delta^{\mu}_{(23)}$ but $\varphi_{\tau_1}(x_0) < x_3 = \Delta^{\mu}_{1(22)}$ and $x_0 < x_3 = \Delta^{\mu}_{1(22)}$ but $\varphi_{\tau_2}(x_0) < x_1 = \Delta^{\mu}_{2(33)}$, we obtain

$$\varphi_{\tau_2}\left(\varphi_{\tau_1}\left(\Delta^{\mu}_{12(3)}\right)\right) = \varphi_{\tau_2}\left(\Delta^{\mu}_{12(3)}\right) = \Delta^{\mu}_{22(3)};$$

$$\varphi_{\tau_1}\left(\varphi_{\tau_2}\left(\Delta^{\mu}_{12(3)}\right)\right) = \varphi_{\tau_1}\left(\Delta^{\mu}_{22(3)}\right) = \Delta^{\mu}_{32(3)} \neq \Delta^{\mu}_{22(3)}.$$

Therefore $\varphi_{\tau_2} \circ \varphi_{\tau_1} \neq \varphi_{\tau_1} \circ \varphi_{\tau_2}$ and $(G, \circ)$ is a non-commutative group. □

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