© Algebra and Discrete Mathematics
Volume **36** (2023). Number 2, pp. 202–216
DOI:10.12958/adm2013

RESEARCH ARTICLE

Algebras of generalized tree languages with fixed variables

S. Phuapong, N. Chansuriya, and T. Kumduang*

Communicated by A. V. Zhuchok

Dedicated to the memory of Professor V. A. Artamonov
(1946–2021)

Abstract. Tree languages which are sets of terms always play a prominent role in the first-order languages and theoretical computer science. In this paper, tree languages induced by terms with fixed variables are considered. Under the applications of an operation on tree languages, we construct the algebra of such languages having many properties of abstract clones. A strong connection with theory of general functions is given through a representation theorem. Additionally, the semigroup of mappings of which their images are tree languages with fixed variables is given.

1. Introduction and basic properties

Following the paper [15], a memory to Professor V. A. Artamonov who is one of the outstanding algebraists was mentioned. Faithful representations of Hopf algebras originally given by his papers in [1,2] motivate us to consider some representations by functions of tree languages. Hence, this paper is dedicated to his scientific works by providing the study of tree languages in sense of algebra.

*Corresponding author.
2020 Mathematics Subject Classification: *03B45, 08A30, 68Q70.*
Key words and phrases: *tree language, superposition, representation.*

Actually, the wide use of terms or trees as a natural structure in computer science allows us to consider its theoretical basics [3]. In the study of logic, terms can be regarded as one of important instruments in both the first and the second-order languages. Basically, a term of type τ is a formal expression which combined from the following two components: variables and compositions of operation symbols in a sequence τ of arities. Let $X_n = \{x_1, \ldots, x_n\}$, for n in $\mathbb{N}^+ := \{1, 2, \ldots\}$, be a set which elements are called *variables* and $X = \{x_1, \ldots, x_n, \ldots\}$. To define terms, we use a set $\{f_i \mid i \in I\}$ of operation symbols, indexed by the set I. The type is the sequence $\tau = (n_i)_{i\in I}$ of the natural number arities of each symbol f_i. Formally, an *n-ary term of type τ* is inductively defined by the following: (1) every variable $x_i \in X_n$ is an n-ary term of type τ and (2) if $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i has the arity n_i, then the composition $f_i(t_1, \ldots, t_{n_i})$ is also an n-ary term of type τ. For the set X_n, by $W_\tau(X_n)$ we mean the set of all n-ary terms of type τ. On the other hand, $W_\tau(X)$ denotes the set of all terms of type τ. The set of all variables that appear in t is denoted by $var(t)$. More background and current trends in the investigation of terms may be found in [5, 13, 18].

We now illustrate some examples of terms. Let us consider the type $\tau = (3, 2)$ with one ternary operation symbol g and one binary operation symbol f. Then we have

$$x_1, x_2, f(x_2, x_1), g(x_1, f(x_2, x_1), x_1) \in W_{(3,2)}(X_2),$$

$$x_1, x_2, x_3, g(x_3, x_3, x_2), g(f(x_3, x_2), x_1, x_3) \in W_{(3,2)}(X_3).$$

There are many types of specific terms, for examples, linear terms [24], k terms [7], full terms with restricted range [23], terms induced by order-decreasing transformations [28]. In this paper, we are interested in a special class of terms of type τ, called *terms of a fixed variable*, which was introduced by K. Wattanatripop and T. Changphas in [27]. We now recall the concept of n-ary terms of a fixed variable of type $\tau = (n_i)_{i\in I}$ as follows:

(1) every $x_j \in X_n$ is an n-ary term of a fixed variable of type τ and

(2) if $t_1, \ldots, t_{n_i}$ are n-ary terms of a fixed variable of type τ with $var(t_l) = var(t_k)$ for every $1 \leq l < k \leq n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of a fixed variable of type τ.

The symbol $W_\tau^{fv}(X_n)$ stands for the set of all n-ary terms of a fixed variable of type τ over alphabet X_n.

For examples, let us consider type $\tau = (2)$ with one binary operation symbol f. Then

$$x_1, x_2, f(x_1, x_1), f(f(x_2, x_2), f(x_2, x_2)) \in W_{(2)}^{fv}(X_2),$$
$$f(x_3, x_3), f(f(x_2, x_2), x_2), f(f(x_3, x_3), f(x_3, x_3)) \in W_{(2)}^{fv}(X_3).$$

On the other hand,

$$f(x_1, x_2), f(x_2, f(x_1, x_1)), f(f(x_1, x_2), x_2) \notin W_{(2)}^{fv}(X_2),$$
$$f(x_1, x_3), f(f(x_2, x_1), x_1), f(f(x_3, x_1), f(x_2, x_2)) \notin W_{(2)}^{fv}(X_3).$$

One of the outstanding structures that connect with terms and related concepts is a Menger algebra. In fact, it is a pair of a nonempty set G and an $(n+1)$-ary operation o on G, where n is a natural number, which satisfies the following equation, also called the *superassociative law*

$$o(o(a, b_1, \ldots, b_n), c_1, \ldots, c_n) =$$
$$= o(a, o(b_1, c_1, \ldots, c_n), \ldots, o(b_n, c_1, \ldots, c_n)).$$

Theoretical and applicable results of Menger algebras can be found in [9, 10, 16].

Actually, the power set of all terms of type τ is naturally denoted by $P(W_\tau(X))$. Every element of $P(W_\tau(X))$ is a set of terms, always called *tree languages*. For example, we provide some subsets of $W_{(3)}(X)$:

$$\emptyset, \{x_1\}, \{x_3, x_5\}, \{h(x_1, x_7, x_2)\}, \{x_4, h(x_{10}, h(x_3, x_3, x_6), x_{25})\}.$$

These sets are examples of tree languages over the terms from $W_{(3)}(X)$. In 2021, tree languages with fixed variables which can be considered as one of particular classes of tree languages were presented in [17]. We mentioned that tree languages generalize formal languages, i.e., sets of words over a given alphabet. Normally, tree languages and tree automata were widely studied in various areas, for example, see [11, 12, 19]. In particular, the variety theorems of binary tree languages and finite tree algebras were proved in the paper [25]. Another important development in tree languages is the state complexity problem of regular tree languages for tree matching problem which is the problem of finding subtree occurrences of a tree in L from a set of trees T. For more details, see [14].

In views of the algebraic construction of operation for tree languages, the superposition operation on $P(W_\tau(X))$ was first presented in [6]. Let n be a natural number, $B, B_1, \ldots, B_n$ are arbitrary subsets of $W_\tau(X)$. Then an $(n+1)$-ary generalized superposition operation

$$\widehat{S}^n : P(W_\tau(X))^{n+1} \to P(W_\tau(X))$$

is defined inductively by

(1) $\widehat{S}^n(B, B_1, \ldots, B_n) := B_j$ if $B = \{x_j\}$ and $1 \le j \le n$.

(2) $\widehat{S}^n(B, B_1, \ldots, B_n) := \{x_j\}$ if $B = \{x_j\}$ and $n < j$.

(3) If $B = \{f_i(t_1, \ldots, t_{n_i})\}$, and suppose that $\widehat{S}^n(\{t_k\}, B_1, \ldots, B_n)$ for all $k = 1, \ldots, n_i$ are already defined, then

$$\widehat{S}^n(\{f_i(t_1, \ldots, t_{n_i})\}, B_1, \ldots, B_n)$$

$$:= \{f_i(r_1, \ldots, r_{n_i}) \mid r_k \in \widehat{S}^n(\{t_k\}, B_1, \ldots, B_n), 1 \le k \le n_i\}.$$

(4) If $|B| > 1$, then $\widehat{S}^n(B, B_1, \ldots, B_n) := \bigcup_{b \in B} \widehat{S}^n(\{b\}, B_1, \ldots, B_n)$.

(5) If $B = \emptyset$ or $B_j = \emptyset$ for some $1 \le j \le n$, then

$$\widehat{S}^n(B, B_1, \ldots, B_n) := \emptyset.$$

Applying this operation, the algebra $(P(W_\tau(X)), \widehat{S}^n)$ of type $(n+1)$ forms a Menger algebra and called a power Menger algebra with infinitely many nullary operations. Other development in tree languages were appeared in [8, 22, 26].

In this paper, we further develop the investigation of tree languages, in particular, tree languages generated by terms of fixed variables and their corresponding operations. The paper is organized as follows: the first result of our study in Section 2 consists of proposing a novel concept of tree languages which are induced by terms of a fixed variable and presenting a generalized operation for them. Applying these two preparations, some structures are constructed and their properties are provided. We also prove a representation theorem of such structure by giving a class of functions that generated by each element of our obtained structures. We continue in Section 3 with giving an idea of mappings which takes from the set of operation symbols to the set of tree languages which are induced by terms of a fixed variable. Due to the importance of the original idea of this mapping related in different areas, especially hyperidentities, a binary composition of such mappings is defined.

2. The generalized power clone of tree languages with fixed variables

On the set $W_\tau^{fv}(X)$ of all terms of a fixed variable of type τ, the power set $P(W_\tau^{fv}(X))$ can be described in a natural way. Each element of

$P(W_\tau^{fv}(X))$ is called *generalized tree languages of terms with fixed variables.* To see some concrete examples of them, let us consider the type (2) with one binary operation symbol g. Then the following sets are subsets of $W_{(2)}^{fv}(X)$, i.e., some elements of $P(W_{(2)}^{fv}(X))$:

$$\emptyset, \{x_1\}, \{x_5\}, \{x_4, g(x_3, x_3)\}, \{x_7, g(g(x_1, x_1), g(x_1, x_1))\}.$$

The following theorem shows that the power set $\mathcal{P}(W_\tau^{fv}(X))$ is closed under the generalized superposition operation $\widehat{S}^n$ for every $n \geq 1$.

Theorem 1. *For any natural number $n \geq 1$ and $A, B_1, \ldots, B_n \subseteq W_\tau^{fv}(X)$, we have*

$$\widehat{S}^n(A, B_1, \ldots, B_n) \in \mathcal{P}(W_\tau^{fv}(X)).$$

Proof. Let $A, B_1, \ldots, B_n \subseteq W_\tau^{fv}(X)$. If one of sets $A, B_1, \ldots, B_n$ that contain in the domain of the generalized superposition operation $\widehat{S}^n$ is empty, then we have an empty set and thus the proof is finished. Suppose now that all of sets $A, B_1, \ldots, B_n$ are non-empty. We give a proof on the characteristic of a set A. If A is a singleton set of the term of a fixed variable s, then we consider in the following three cases: s is a variable $x_i \in X_n$, s is a variable $x_j \in X \setminus X_n$ and $s = f_i(s_1, \ldots, s_{n_i}) \in W_\tau^{fv}(X)$. In the first case, we get $\widehat{S}^n(A, B_1, \ldots, B_n) = \widehat{S}^n(\{x_i\}, B_1, \ldots, B_n) = B_i \in P(W_\tau^{fv}(X))$. In the second case, we have $\widehat{S}^n(A, B_1, \ldots, B_n) = \widehat{S}^n(\{x_j\}, B_1, \ldots, B_n) = \{x_j\} \in P(W_\tau^{fv}(X))$. In the third case, we prove that the set $\widehat{S}^n_m(\{f_i(s_1, \ldots, s_{n_i})\}, B_1, \ldots, B_n)$ belong to $P(W_\tau^{fv}(X))$. Following the definition of $\widehat{S}^n$, $\widehat{S}^n(\{f_i(s_1, \ldots, s_{n_i})\}, B_1, \ldots, B_n)$ equals to the set $\{f_i(r_1, \ldots, r_{n_i}) \mid r_k \in \widehat{S}^n(\{s_k\}, B_1, \ldots, B_n), 1 \leq k \leq n_i\}$. Thus, for every $1 \leq k \leq n_i$, we have to show that $\widehat{S}^n(\{s_k\}, B_1, \ldots, B_n)$ are generalized tree languages of terms of fixed variables. From the assumption, we get $\{f_i(s_1, \ldots, s_{n_i})\} \subseteq W_\tau^{fv}(X)$. This means $var(\{f_i(s_1, \ldots, s_{n_i})\}) = \{x_k\}$ for some $k \geq 1$, which implies $var(\widehat{S}^n(\{t_s\}, B_1, \ldots, B_n)) = \{x_k\}$ for some $k \geq 1$. That means, this language contains only one variable from X. It follows directly that

$$|var(\widehat{S}^n_m(\{f_i(s_1, \ldots, s_{n_i})\}, B_1, \ldots, B_n))| = 1.$$

This case is completed. If A is an arbitrary non-singleton set, i.e., $|A| > 1$, we have $\widehat{S}^n(A, B_1, \ldots, B_n) = \bigcup\limits_{s \in A} \widehat{S}^n(\{s\}, B_1, \ldots, B_n)$. As we known from the previous case, then the arbitrary union of them so is. $\square$

Applying Theorem 1, we have an $(n+1)$-ary operation on the set of tree languages of fixed variables

$$\widehat{S}^n : P(W_\tau^{fv}(X))^{n+1} \to P(W_\tau^{fv}(X))$$

for $n \geq 1$. Furthermore, the algebra

$$P - clone_G^{fv}(\tau) = (P(W_\tau^{fv}(X)), \widehat{S}^n),$$

which is called the *generalized power clone of tree languages with fixed variables*, is obtained.

The next theorem follows in a straightforward way from Theorem 1 and the fact that $P(W_\tau^{fv}(X))$ is a subset of $P(W_\tau(X))$.

Theorem 2. *The many-sort algebra $P - clone_G^{fv}(\tau)$ satisfies (C1)-(C4), i.e., for every natural number $n \geq 1$:*

(C1) $\widehat{S}^n(\widehat{S}^n(A, B_1, \ldots, B_n), C_1, \ldots, C_n)$
$= \widehat{S}^n(A, \widehat{S}^n(B_1, C_1, \ldots, C_n), \ldots, \widehat{S}^n(B_n, C_1, \ldots, C_n))$
whenever $A, B_1, \ldots, B_n, C_1, \ldots, C_n \subseteq W_\tau^{fv}(X)$.

(C2) $\widehat{S}^n(\{x_i\}, B_1, \ldots, B_n) = B_i$ *whenever* $B_1, \ldots, B_n \subseteq W_\tau^{fv}(X)$ *and* $1 \leq i \leq n$.

(C3) $\widehat{S}^n(\{x_i\}, B_1, \ldots, B_n) = \{x_i\}$ *whenever* $B_1, \ldots, B_n \subseteq W_\tau^{fv}(X)$ *and* $i > n$.

(C4) $\widehat{S}^n(A, \{x_1\}, \ldots, \{x_n\}) = A$.

Proof. Applying the result of Theorem 1, the proof of this theorem is obtained. □

We now illustrate the algebra $P - clone_G^{fv}(\tau)$ in a specific type.

Example 1. Consider type $\tau = (2)$ with one binary operation symbol f and a subset

$$B = \{\{x_1\}, \{x_2, x_7\}, \{f(x_5, x_5)\}, \{f(f(x_4, x_4), x_4)\}\}$$

of $P(W_{(2)}^{fv}(X))$ with respect to a binary operation $\widehat{S}^1$ which is defined by the following table.

$\widehat{S}^1$	$\{x_1\}$	$\{x_2, x_7\}$	$\{f(x_5, x_5)\}$	$\{f(f(x_4, x_4), x_4)\}$
$\{x_1\}$	$\{x_1\}$	$\{x_2, x_7\}$	$\{f(x_5, x_5)\}$	$\{f(f(x_4, x_4), x_4)\}$
$\{x_2, x_7\}$	$\{x_2, x_7\}$	$\{x_2, x_7\}$	$\{x_2, x_7\}$	$\{x_2, x_7\}$
$\{f(x_5, x_5)\}$	$\{f(x_5, x_5)\}$	$\{f(x_5, x_5)\}$	$\{f(x_5, x_5)\}$	$\{f(x_5, x_5)\}$
$\{f(f(x_4, x_4), x_4)\}$	$\{f(f(x_4, x_4), x_4)\}$	$\{f(f(x_4, x_4), x_4)\}$	$\{f(f(x_4, x_4), x_4)\}$	$\{f(f(x_4, x_4), x_4)\}$

It is not difficult to show that the operation $\widehat{S}^1$ defined on $P(W_{(2)}^{fv}(X))$ is associative. Consequently, $(B, \widehat{S}^1)$ forms a semigroup. Furthermore, it is also a subsemigroup of $(P(W_{(2)}^{fv}(X)), \widehat{S}^1)$.

Our next aim is to propose a representation theorem of the generalized power clone $P - clone_G^{fv}(\tau)$. Representation in other structures can be found, for instance, in [29–32]. Let us start with recalling some elementary tools which were collected from [9,18]. On the set A^n of n-th Cartesian product of a nonempty set A, a full n-ary function or an n-ary operation is a mapping from A^n to A. The symbol $T(A^n, A)$ stands for the set of all such mappings. On the set $T(A^n, A)$, one can define the *Menger's composition* $\mathcal{O} : T(A^n, A)^{n+1} \to T(A^n, A)$ by

$$\mathcal{O}(f, g_1, \ldots, g_n)(a_1, \ldots, a_n) = f(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)),$$

where $f, g_1, \ldots, g_n \in T(A^n, A), a_1, \ldots, a_n \in A$. The set together with the Menger's composition is said to be an *algebra of full functions* or *algebra of operations*. If the composition of $(n + 1)$ functions from $T(A^n, A)$ is also in this set, then this pair is called *algebra of full functions*. In general, the Menger's composition generalizes the usual composition of functions.

We construct a mapping generated by each element of $P(W_\tau^{fv}(X))$. For each set A of the algebra $(P(W_\tau^{fv}(X)), \widehat{S}^n)$, the full n-ary function $\lambda_A : P(W_\tau^{fv}(X))^n \to P(W_\tau^{fv}(X))$ can be defined by

$$\lambda_A(B_1, \ldots, B_n) = \widehat{S}^n(A, B_1, \ldots, B_n)$$

for all $B_1, \ldots, B_n \in P(W_\tau^{fv}(X))$, where $\widehat{S}^n$ is an $(n + 1)$-ary generalized superposition operation defined on $P(W_\tau^{fv}(X))$.

It is clear that the full n-ary function λ_A is an element of the set

$$T^{Pfv} := T(P(W_\tau^{fv}(X))^n, P(W_\tau^{fv}(X)))$$

which is called an *inner left translation* of $P(W_\tau^{fv}(X))$ which corresponds to a tree language A of $P(W_\tau^{fv}(X))$.

The following lemmas are essential tools for proving the main theorem.

Lemma 1. *On the algebra* $(P(W_\tau^{fv}(X)), \widehat{S}^n)$*, the equation*

$$\lambda_{\widehat{S}^n(A,B_1,\ldots,B_n)} = \mathcal{O}(\lambda_A, \lambda_{B_1}, \ldots, \lambda_{B_n})$$

is satisfied for every $A, B_1, \ldots, B_n \in P(W_\tau^{fv}(X))$ *where* $\widehat{S}^n$ *and* $\mathcal{O}$ *is a generalized superposition operation of tree languages with fixed variables and Menger's composition, respectively.*

Proof. Let $A, B_1, \ldots, B_n, D_1, \ldots, D_n$ be arbitrary subsets of $W_\tau^{fv}(X)$. Then we have

$$\begin{aligned}
&\lambda_{\widehat{S}^n(A,B_1,\ldots,B_n)}(D_1, \ldots, D_n)\\
&\quad= \widehat{S}^n(\widehat{S}^n(A, B_1, \ldots, B_n), D_1, \ldots, D_n)\\
&\quad= \widehat{S}^n(A, \widehat{S}^n(B_1, D_1, \ldots, D_n), \ldots, \widehat{S}^n(B_n, D_1, \ldots, D_n))\\
&\quad= \lambda_A(\widehat{S}^n(B_1, D_1, \ldots, D_n), \ldots, \widehat{S}^n(B_n, D_1, \ldots, D_n))\\
&\quad= \lambda_A(\lambda_{B_1}(D_1, \ldots, D_n), \ldots, \lambda_{B_n}(D_1, \ldots, D_n))\\
&\quad= \mathcal{O}(\lambda_A, \lambda_{B_1}, \ldots, \lambda_{B_n})(D_1, \ldots, D_n).
\end{aligned}$$

The proof is finished. □

By Λ', we denote the set of all λ_A where $A \in P(W_\tau^{fv}(X))$, i.e.,

$$\Lambda' = \{\lambda_A \mid A \in P(W_\tau^{fv}(X))\}.$$

Lemma 2. *The set* Λ' *forms a subalgebra of* $(T^{Pfv}, \mathcal{O})$ *and thus* $(\Lambda', \mathcal{O})$ *is a Menger algebra of full* n*-ary functions.*

Proof. Obviously, $\emptyset \neq \Lambda' \subseteq T^{Pfv}$. Let $\lambda_A, \lambda_{B_1}, \ldots, \lambda_{B_n}$ be arbitrary full n-ary functions in T^{Pfv}. It follows immediately from Lemma 1 that the composition of such mappings again a full n-ary function. □

We now state and prove a representation theorem for the generalized power clone of tree languages with fixed variables as follows:

Theorem 3. *Let* $(P(W_\tau^{fv}(X)), \widehat{S}^n)$ *be an algebra of type* $(n+1)$*. Define a mapping* $\psi : P(W_\tau^{fv}(X)) \to \Lambda'$ *by* $\psi(A) = \lambda_A$ *for all* $A \in P(W_\tau^{fv}(X))$*. Then* ψ *is an isomorphism and so* $P(W_\tau^{fv}(X)) \cong \Lambda'$*.*

Proof. Clearly, ψ is surjective. By Lemma 1, we have

$$\psi(\widehat{S}^n(A, B_1, \ldots, B_n)) = \lambda_{\widehat{S}^n(A,B_1,\ldots,B_n)}$$

$$= \mathcal{O}(\lambda_A, \lambda_{B_1}, \ldots, \lambda_{B_n}) = \mathcal{O}(\psi(A), \psi(B_1), \ldots, \psi(B_n))$$

and thus ψ is a homomorphism. Suppose that $\lambda_{A_1} = \lambda_{A_2}$. Then we obtain $\lambda_{A_1}(B_1, \ldots, B_n) = \lambda_{A_2}(B_1, \ldots, B_n)$. By the definition of a generalized superposition $\widehat{S}^n$, we conclude that A_1 and A_2 coincide. Hence ψ is injective. Therefore, ψ is an isomorphism from $P(W_\tau^{fv}(X))$ to Λ'. □

3. Non-deterministic generalized hypersubstitutions with fixed variables

One of the concepts which are closely related to tree homomorphism is a non-deterministic generalized hypersubstitution. It was mentioned in [3,11] that for a tree homomorhosim h and a recognizable tree language L the image of L under h is also recognizable. For this reason, we begin this section with collecting some elementary concepts of non-deterministic generalized hypersubstitutions of type τ which were extensively applied in the theory of hyperidentity [4,20,21,23]. A mapping

$$\sigma_{nd} : \{f_i \mid i \in I\} \to P(W_\tau(X))$$

is called a *non-deterministic generalized hypersubstitution of type* τ. The set of all such mappings is denoted by $Hyp_G^{nd}(\tau)$. It is well-known that every σ_{nd} generates a mapping $\widehat{\sigma}_{nd} : P(W_\tau(X)) \to P(W_\tau(X))$ which is defined by the following inductive way:

(1) $\widehat{\sigma}_{nd}[\emptyset] := \emptyset$,

(2) $\widehat{\sigma}_{nd}[\{x_i\}] := \{x_i\}$ where x_i is a variable from X,

(3) $\widehat{\sigma}_{nd}[\{f_i(s_1, \ldots, s_{n_i})\}] := \widehat{S}^{n_i}(\sigma_{nd}(f_i), \widehat{\sigma}_{nd}[\{s_1\}], \ldots, \widehat{\sigma}_{nd}[\{s_{n_i}\}])$ if $\widehat{\sigma}_{nd}[\{s_k\}], 1 \leq k \leq n_i$ are already defined,

(4) $\widehat{\sigma}_{nd}[B] := \bigcup_{b \in B} \widehat{\sigma}_{nd}[\{b\}]$ if B is an arbitrary non-singleton subset of $W_\tau(X)$.

Under a binary operation $\circ_G^{nd}$ on $Hyp_G^{nd}(\tau)$ given by $\sigma_{nd} \circ_G^{nd} \alpha_{nd} := \widehat{\sigma}_{nd} \circ \alpha_{nd}$ where $\circ$ is a usual composition, it was proved that the triple $(Hyp_G^{nd}(\tau), \circ_G^{nd}, \sigma_{id})$ forms a monoid where σ_{id} was defined to be an identity element where $\sigma_{id}(f_i) := \{f_i(x_1, \ldots, x_{n_i})\}$ for all $i \in I$.

It is possible to study the situation when the images of σ_{nd} are sets of terms with fixed variables and necessarily preserve the arity. This leads us to introduce the following concept.

A non-deterministic generalized hypersubstitution σ_{nd} of type τ is said to be *non-deterministic generalized hypersubstitution with fixed variables of type* τ if $\sigma_{nd}(f_i) \in P(W_\tau^{fv}(X))$ where a mapping σ_{nd} does not necessarily preserve the arity. By $Hyp_{nd-G}^{fv}(\tau)$, we denote the set of all non-deterministic generalized hypersubstitutions with fixed variables of type τ

Now, more examples of non-deterministic hypersubstitutions with fixed variables are provided.

Example 2. Let $\tau = (4,2)$ be a type with a quaternary operation symbol h and a binary operation symbol g. Let σ_{nd} be a non-deterministic hypersubstitution with fixed variables which maps a quaternary operation symbol h to a set $\{x_3, h(x_8, x_8, x_8, x_8)\}$ and g to a tree language $\{x_2, g(x_4, x_4), g(x_1, g(x_1, x_1))\}$. Then $\sigma_{nd} \in Hyp^{fv}_{nd-G}(4,2)$.

Example 3. Let $\tau = (3,2,1)$ be a type with a ternary operation symbol f, a binary operation symbol g, and a unary operation symbol h. Define the following: a mapping β takes f to $\{x_1, f(g(x_5, x_5), x_5, x_5), h(x_9)\}$, takes g to $\{f(x_1, x_1, x_1), g(x_6, x_6)\}$, and takes h to $\{h(h(h(h(x_2))))\}$. Clearly, $\beta \in Hyp^{fv}_{nd-G}(3,2,1)$. Nevertheless, if a mapping γ is defined by $\gamma(f) = \{x_3, g(x_3, x_4)\}, \gamma(g) = \{f(x_1, x_4, x_7)\}$, and $\gamma(h) = \{h(h(x_5))\}$, then $\gamma \notin Hyp^{fv}_{nd-G}(3,2,1)$.

To prove that a binary operation $\circ^{nd}_G$ on $Hyp^{nd}_G(\tau)$ can be applied to $Hyp^{fv}_{nd-G}(\tau)$, the following lemma is needed.

Lemma 3. *The extended mapping $\widehat{\sigma}_{nd}$ of a non-deterministic hypersubstitution with fixed variables σ_{nd} is a mapping on $P(W^{fv}_\tau(X))$.*

Proof. Our aim is to prove that $\widehat{\sigma}_{nd}$ is a mapping from the set $P(W^{fv}_\tau(X))$ to itself, i.e., $\widehat{\sigma}_{nd} : P(W^{fv}_\tau(X)) \to P(W^{fv}_\tau(X))$. To do this, let σ_{nd} be a mapping in $Hyp^{fv}_{nd-G}(\tau)$ and let A be an arbitrary subset of $W^{fv}_\tau(X)$. If A is empty, the lemma is clear. Suppose now that A is non-empty. If $A = \{x_i\}$, $x_i \in X$, we have $\widehat{\sigma}_{nd}[\{x_i\}] = \{x_i\} \in P(W^{fv}_\tau(X))$. Assume that $A = \{f_i(s_1, \ldots, s_{n_i})\}$ where $f_i(s_1, \ldots, s_{n_i})$ is a term with fixed variable. Furthermore, we inductively assume that $\widehat{\sigma}_{nd}[\{s_k\}] \in P(W^{fv}_\tau(X))$ for all $1 \leq k \leq n_i$. Because $\sigma_{nd}(f_i)$ belongs to the set $P(W^{fv}_\tau(X))$, it follows immediately from Theorem 1 that $\widehat{\sigma}_{nd}[\{f_i(s_1, \ldots, s_{n_i})\}] = \widehat{S}^{n_i}(\sigma_{nd}(f_i), \widehat{\sigma}_{nd}[\{s_1\}], \ldots, \widehat{\sigma}_{nd}[\{s_{n_i}\}])$ is a generalized tree language with fixed variables. Finally, if A is a non-singleton subset of $W^{fv}_\tau(X)$, i.e., $|A| > 1$, the proof is obtained from the property of the union of sets in the natural way. □

We now give the example that describes the fact of Lemma 3.

Example 4. Let $\tau = (3)$ with a ternary operation symbol g and $\sigma_{nd} \in Hyp^{fv}_{nd-G}(3)$ which is given by $\sigma_{nd}(g) = \{g(x_3, x_3, x_3)\}$. If a generalized tree language with fixed variables $A = \{x_5, g(x_2, x_2, x_2)\} \in P(W^{fv}_{(3)}(X))$, then we have

$$\widehat{\sigma}_{nd}[A] = \{x_5\} \cup \widehat{S}^3(\sigma_{nd}(g), \widehat{\sigma}_{nd}[\{x_2\}], \widehat{\sigma}_{nd}[\{x_2\}], \widehat{\sigma}_{nd}[\{x_2\}]).$$

According to the definition of generalized superposition and the defining of $\sigma_{nd}(g)$, we have

$$\widehat{S}^3(\sigma_{nd}(g), \widehat{\sigma}_{nd}[\{x_2\}], \widehat{\sigma}_{nd}[\{x_2\}], \widehat{\sigma}_{nd}[\{x_2\}]) = \{g(x_3, x_3, x_3)\}.$$

As a result, $\widehat{\sigma}_{nd}[A] = \{x_5, g(x_3, x_3, x_3)\}$.

Consequently, we prove

Theorem 4. *$(Hyp^{fv}_{nd-G}(\tau), \circ^{nd}_G)$ is a subsemigroup of $(Hyp^{nd}_G(\tau), \circ^{nd}_G)$.*

Proof. Let $\sigma_{nd}, \alpha_{nd} \in Hyp^{fv}_{nd}(\tau)$. We have to show that $\sigma_{nd} \circ^{nd}_G \alpha_{nd} \in Hyp^{fv}_{nd-G}(\tau)$. In fact, we have $(\sigma_{nd} \circ^{nd}_G \alpha_{nd})(f_i) = \widehat{\sigma}_{nd}[\alpha_{nd}(f_i)]$. Since $\alpha_{nd}(f_i) \in P(W^{fv}_\tau(X))$ and $\widehat{\sigma}_{nd}$ is an extension of a non-deterministic hypersubstitution with fixed variables of type τ, then $\widehat{\sigma}_{nd}[\alpha_{nd}(f_i)]$ is a tree language with fixed variables by Lemma 3. □

Finally, we provide the following property that gives a close connection between $\widehat{\sigma}_{nd}$ and the generalized power clone $P - clone^{fv}_G(\tau)$.

Theorem 5. *For every non-deterministic hypersubstitution with fixed variables σ_{nd}, its extension $\widehat{\sigma}_{nd}$ is an endomorphism of the generalized power clone $P - clone^{fv}_G(\tau)$.*

Proof. To prove that $\widehat{\sigma}_{nd}$ is an endomorphism of $P - clone^{fv}_G(\tau)$, we have to show that the equation

$$\widehat{\sigma}_{nd}[\widehat{S}^n(A, B_1, \ldots, B_n)] = \widehat{S}^n(\widehat{\sigma}_{nd}[A], \widehat{\sigma}_{nd}[B_1], \ldots, \widehat{\sigma}_{nd}[B_n])$$

holds for all $A, B_1, \ldots, B_n \in P(W^{fv}_\tau(X))$. If $A = \emptyset$ or $B_j = \emptyset$ for some $j \in \{1, \ldots, n\}$, by the definitions of $\widehat{S}^n$ and $\widehat{\sigma}_{nd}$ then both sides are empty and the equation is satisfied. Now we give a proof on the characteristic of a set A. If $A = \{t\}$ where t is a term with fixed variable, then three cases are considered. For $t = x_i$, $1 \le i \le n$, we have

$$\begin{aligned}
&\widehat{\sigma}_{nd}[\widehat{S}^n(A, B_1, \ldots, B_n)] \\
&\quad = \widehat{\sigma}_{nd}[\widehat{S}^n(\{x_i\}, B_1, \ldots, B_n)] \\
&\quad = \widehat{\sigma}_{nd}[B_i] \\
&\quad = \widehat{S}^n(\{x_i\}, \widehat{\sigma}_{nd}[B_1], \ldots, \widehat{\sigma}_{nd}[B_n]) \\
&\quad = \widehat{S}^n(\widehat{\sigma}_{nd}[\{x_i\}], \widehat{\sigma}_{nd}[B_1], \ldots, \widehat{\sigma}_{nd}[B_n]).
\end{aligned}$$

If $t = x_j$, for $j > n$, then we have

$$\begin{aligned}
&\widehat{\sigma}_{nd}[\widehat{S}^n(A, B_1, \ldots, B_n)] \\
&\quad = \widehat{\sigma}_{nd}[\widehat{S}^n(\{x_j\}, B_1, \ldots, B_n)]
\end{aligned}$$

$= \widehat{\sigma}_{nd}[\{x_j\}]$
$= \{x_j\}$
$= \widehat{S}^n(\{x_j\}, \widehat{\sigma}_{nd}[B_1], \ldots_n, \widehat{\sigma}_{nd}[B_n])$
$= \widehat{S}^n(\widehat{\sigma}_{nd}[\{x_j\}], \widehat{\sigma}_{nd}[B_1], \ldots_n, \widehat{\sigma}_{nd}[B_n])$.
Assume now that $t = f_i(s_1, \ldots, s_{n_i})$ and

$$\widehat{\sigma}_{nd}[\widehat{S}^n(\{s_k\}, B_1, \ldots, B_n)] = \widehat{S}^n(\widehat{\sigma}_{nd}[\{s_k\}], \widehat{\sigma}_{nd}[B_1], \ldots, \widehat{\sigma}_{nd}[B_n])$$

is satisfied for all $1 \leq k \leq n_i$. By Theorem 2, we obtain that
$\widehat{\sigma}_{nd}[\widehat{S}^n(\{f_i(s_1, \ldots, s_{n_i})\}, B_1, \ldots, B_n)]$
$= \widehat{\sigma}_{nd}[\{f_i(r_1, \ldots, r_{n_i}) \mid r_k \in \widehat{S}^n(\{s_k\}, B_1, \ldots, B_n), 1 \leq k, \leq n_i\}]$
$= \bigcup \widehat{\sigma}_{nd}[\{f_i(r_1 \in \widehat{S}^n(\{s_1\}, B_1, \ldots, B_n), \ldots,$
$r_{n_i} \in \widehat{S}^n(\{s_{n_i}\}, B_1, \ldots, B_n)\}]$
$= \bigcup \widehat{S}^{n_i}(\sigma_{nd}(f_i), \widehat{\sigma}_{nd}[\{r_1 \in \widehat{S}^n(\{s_1\}, B_1, \ldots, B_n)\}], \ldots,$
$\widehat{\sigma}_{nd}[\{r_{n_i} \in \widehat{S}^n(\{s_{n_i}\}, B_1, \ldots, B_n)\}])$
$= \widehat{S}^{n_i}(\sigma_{nd}(f_i), \widehat{\sigma}_{nd}[\widehat{S}^n(\{s_1\}, B_1, \ldots, B_n)], \ldots,$
$\widehat{\sigma}_{nd}[\widehat{S}^n(\{s_{n_i}\}, B_1, \ldots, B_n)])$
$= \widehat{S}^{n_i}(\sigma_{nd}(f_i), \widehat{S}^n(\widehat{\sigma}_{nd}[\{s_1\}], \widehat{\sigma}_{nd}[B_1], \ldots, \widehat{\sigma}_{nd}[B_n]), \ldots,$
$\widehat{S}^n(\widehat{\sigma}_{nd}[\{s_{n_i}\}], \widehat{\sigma}_{nd}[B_1], \ldots, \widehat{\sigma}_{nd}[B_n]))$
$= \widehat{S}^n(\widehat{S}^{n_i}(\widehat{\sigma}_{n_i}(f_i), \widehat{\sigma}_{nd}[s_1], \ldots, \widehat{\sigma}_{nd}[s_{n_i}]), \widehat{\sigma}_{nd}[B_1], \ldots, \widehat{\sigma}_{nd}[B_n])$
$= \widehat{S}^n(\widehat{\sigma}_{nd}[\{f_i(s_1, \ldots, s_{n_i})\}], \widehat{\sigma}_{nd}[B_1], \ldots, \widehat{\sigma}_{nd}[B_n])$.
Let A be arbitrary non-singleton subset of $W_\tau^{fv}(X)$. Then
$\widehat{\sigma}_{nd}[\widehat{S}^n(A, B_1, \ldots, B_n)]$

$$\begin{aligned}
&= \widehat{\sigma}_{nd}[\bigcup_{a \in A} \widehat{S}^n(\{a\}, B_1, \ldots, B_n)] \\
&= \bigcup_{a \in A} \widehat{\sigma}_{nd}[\widehat{S}^n(\{a\}, B_1, \ldots, B_n)] \\
&= \bigcup_{a \in A} \widehat{S}^n(\widehat{\sigma}_{nd}[\{a\}], \widehat{\sigma}_{nd}[B_1], \ldots, \widehat{\sigma}_{nd}[B_n]) \\
&= \bigcup_{a \in A} (\bigcup_{s \in \widehat{\sigma}_{nd}[\{a\}]} \widehat{S}^n(\{s\}, \widehat{\sigma}_{nd}[B_1], \ldots, \widehat{\sigma}_{nd}[B_n])) \\
&= \bigcup_{s \in \widehat{\sigma}_{nd}[A]} \widehat{S}^n(\{s\}, \widehat{\sigma}_{nd}[B_1], \ldots, \widehat{\sigma}_{nd}[B_n]) \\
&= \widehat{S}^n(\widehat{\sigma}_{nd}[A], \widehat{\sigma}_{nd}[B_1], \ldots, \widehat{\sigma}_{nd}[B_n]).
\end{aligned}$$

The proof is finished. □

4. Conclusions

This paper is contributed to the investigation of the first and the second-order languages, especially tree languages generated by terms of a fixed variable. There are many real-word examples of such lauguages, for example, the equation of idempotency. Applying the generalized superposition on tree languages, the generalized algebra consisting of the set of obtained languages and the operation of type $(n+1)$ satisfying the clone axioms were presented. Furthermore, if $n=1$, then we restricted our intention to a semigroup. In a connection with the theory of functions, a representation theorem of this algebra was stated via a construction of n-ary functions induced by tree languages with fixed variables. Finally, based on the theory of hypersubstitutions, the semigroup of mappings which take any operation symbol to the sets of terms with fixed variables was given under a binary associative operation for these mappings. The work may be considered as tools for constructing M-solid non-deterministic varieties with fixed variables. Characterization theorems of bands and other other classes of algebras may be proved by our results. However, it is possible to extend our study in the near future by continuing in other kinds of tree languages and examine a lattice of all languages.

Acknowledgements

The authors are grateful to the referees whose valuable remarks helped to increase the quality of the paper. This work was supported by Rajamangala University of Technology Rattanakosin, Thailand.

References

[1] V.A. Artamonov, *On semisimple Hopf algebras with few representations of dimension greater than one*, Rev. Unión Mat. Argent., **51** (2), 2010, pp. 91–105.

[2] V.A. Artamonov, *Semi-simple Hopf algebras with restrictions on irreducible modules of dimension exceeding 1*, St. Petersbg. Math. J., **26** (2), 2015, pp. 207–223.

[3] S. Bozapalidis, Z. Fulop, G. Rahonis, *Equational tree transformations*, Theoret. Comput. Sci., **412** (29), 2011, pp. 3676–3692.

[4] N. Chansuriya, *All maximal idempotent submonoids of generalized cohypersubstitutions of type* $\tau = (2)$, Discuss. Math. Gen. Algebra Appl., **41** (1), 2021, pp. 45–54.

[5] K. Denecke, *Partial clones*, Asian-Eur. J. Math., **13** (8), 2020, 2050161.

[6] K. Denecke, P. Glubudom, J. Koppitz, *Power clones and nondeterministic hypersubstitutions*, Asian-Eur. J. Math., **1** (2), 2008, pp. 177–188.

[7] K. Denecke, H. Hounnon, *Partial Menger algebras of terms*, Asian-Eur. J. Math., **14** (6), 2021, 2150092.

[8] K. Denecke, N. Sarasit, *Semigroups of tree languages*, Asian-Eur. J. Math., **1** (4), 2008, pp. 489–507.

[9] W.A. Dudek, V.S. Trokhimenko, *Menger algebras of associative and self-distributive n-ary operations*, Quasigroups Relat. Syst., **26**, 2018, pp. 45–52.

[10] W.A. Dudek, V.S. Trokhimenko, *Menger algebras of k-commutative n-place functions*, Georgian Math. J., **28** (3), 2021, pp. 355–361.

[11] F. Gecseg, M. Steinby, *Tree Languages*, Handbook of Formal Languages, vol. **3**, Chapter 1, Tree Languages, pp. 1–68, Springer-Verlag, 1997.

[12] Y. Guellouma, H. Cherroun, *From tree automata to rational tree expressions*, Int. J. Found. Comput. Sci., **29** (6), 2018, pp. 1045–1062

[13] P. Kitpratyakul, B. Pibaljommee, *On substructures of semigroups of inductive terms*, AIMS Mathematics, **7** (6), 2022, pp. 9835–9845.

[14] S.-K. Ko, H.-R. Lee, Y.-S. Han, *State Complexity of Regular Tree Languages for Tree Matching*. Int. J. Found. Comput. Sci., **27** (08), 2016, pp. 965–979.

[15] P. Kolesnikov, *In memoriam: Vyacheslav Aleksandrovich Artamonov (1946–2021)*, Commun. Algebra, **50** (1), 2022, pp. 1–2.

[16] T. Kumduang, S. Leeratanavalee, *Left translations and isomorphism theroems of Menger algebras*, Kyungpook Math. J., **61** (2), 2021, pp. 223–237.

[17] T. Kumduang, S. Leeratanavalee, *Semigroups of terms, tree languages, Menger algebra of n-ary functions and their embedding theorems*, Symmetry, **13** (4), 2021, p. 558.

[18] T. Kumduang, K. Wattanatripop, T. Changphas, *Tree languages with fixed variables and their algebraic structures*, Int. J. Math. Comput. Sci., **16** (4), 2021, pp. 1683–1696.

[19] D. Kuske, I. Meinecke, *Construction of tree automata from regular expressions*, RAIRO – Theor. Inf. and Applic., **45** (3), 2011, pp. 347–370.

[20] S. Leeratanavalee, *Outermost-strongly solid variety of commutative semigroups*, Thai. J. Math., **14** (2), 2016, pp. 305–313.

[21] Yu.M. Movsisyan, *Hyperidentities and related concepts, I*, Armen. J. Math., **9** (2), 2017, pp. 146–222.

[22] T. Petkovic, S. Salehi, *Positive varieties of tree languages*, Theoret. Comput. Sci., **347**, 2005, pp. 1–35.

[23] S. Phuapong, T. Kumduang, *Menger algebras of terms induced by transformations with restricted range*, Quasigroups Relat. Syst., **29**, 2021, pp. 255–268.

[24] D. Phusanga, J. Koppitz, *The semigroup of linear terms*, Asian-Eur. J. Math., **13** (1), 2020, 2050005.

[25] S. Salehi, M. Steinby, *Tree algebras and varieties of tree languages*, Theoret. Comput. Sci., **377**, 2007, pp. 1–24.

[26] M. Steinby, *General varieties of tree languages*, Theoret. Comput. Sci., **205**, 1998, pp. 1–43.

[27] K. Wattanatripop and T. Changphas, *Clones of terms of a fixed variable*, Mathematics, **8** (2), 2020, p. 260.

[28] K. Wattanatripop and T. Changphas, *The Menger algebra of terms induced by order-decreasing transformations*, Commun. Algebra, **49** (7), 2021, pp. 3114–3123.

[29] A.V. Zhuchok, *Dimonoids*, Algebra Logic, **50** (4), 2011, pp. 323–340. DOI: https://doi.org/10.1007/s10469-011-9144-7.

[30] A.V. Zhuchok, *Structure of free strong doppelsemigroups*, Commun. Algebra, **46** (8), 2018, pp. 3262–3279. DOI: https://doi.org/10.1080/00927872.2017.1407422.

[31] A.V. Zhuchok, *Structure of relatively free trioids*, Algebra Discrete Math., **31** (1), 2021, pp. 152–166. DOI: http://dx.doi.org/10.12958/adm1732.

[32] A.V. Zhuchok, Y.V. Zhuchok, J. Koppitz, *Free rectangular doppelsemigroups*, J. Algebra Appl., **19** (11), 2020, 2050205. DOI: http://dx.doi.org/10.1142/S0219498820502059.

CONTACT INFORMATION

Sarawut Phuapong
Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna, Chiang Mai 50300, Thailand
E-Mail: phuapong.s@gmail.com
URL:

Nagornchat Chansuriya
Faculty of Science, Energy and Environment, King Mongkut's University of Technology North Bangkok (Rayong Campus), Rayong 21120, Thailand
E-Mail: nagornchat.c@sciee.kmutnb.ac.th
URL:

Thodsaporn Kumduang
Department of Mathematics, Faculty of Science and Technology, Rajamangala University of Technology Rattanakosin, Nakhon Pathom 73170, Thailand
E-Mail: thodsaporn.kum@rmutr.ac.th
URL:

Received by the editors: 06.08.2022
and in final form 20.12.2023.