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On the group of automorphisms of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ with the family $\mathscr{F}$ of inductive nonempty subsets of $\boldsymbol{\omega}$

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Abstract. We study automorphisms of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ with the family $\mathscr{F}$ of inductive nonempty subsets of ω and prove that the group $\mathbf{Aut}(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ of automorphisms of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is isomorphic to the additive group of integers.

1. Introduction, motivation and main definitions

We shall follow the terminology of [1, 2, 11, 12, 15]. By ω we denote the set of all non-negative integers and by $\mathbb{Z}$ the set of all integers.

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F \in \mathscr{P}(\omega)$ and $n, m \in \omega$ we put $n-m+F = \{n-m+k \colon k \in F\}$ if $F \neq \varnothing$ and $n-m+\varnothing = \varnothing$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called ω-*closed* if $F_1 \cap (-n+F_2) \in \mathscr{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathscr{F}$. For any $a \in \omega$ we denote $[a) = \{x \in \omega \colon x \geqslant a\}$.

A subset A of ω is said to be *inductive*, if $i \in A$ implies $i+1 \in A$. Obvious, that $\varnothing$ is an inductive subset of ω.

Remark 1 ([8])**.** 1) By Lemma 6 from [7] nonempty subset $F \subseteq \omega$ is inductive in ω if and only $(-1+F) \cap F = F$.

2) Since the set ω with the usual order is well-ordered, for any nonempty inductive subset F in ω there exists nonnegative integer $n_F \in \omega$ such that $[n_F) = F$.

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3) Statement (2)) implies that the intersection of an arbitrary finite family of nonempty inductive subsets in ω is a nonempty inductive subset of ω.

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

A *partially ordered set* (or shortly a *poset*) $(X, \leqq)$ is the set X with the reflexive, antisymmetric and transitive relation $\leqq$. In this case relation $\leqq$ is called a partial order on X. A partially ordered set $(X, \leqq)$ is *linearly ordered* or is a *chain* if $x \leqq y$ or $y \leqq x$ for any $x, y \in X$. A map f from a poset $(X, \leqq)$ onto a poset $(Y, \leqslant)$ is said to be an order isomorphism if f is bijective and $x \leqq y$ if and only if $f(x) \leqslant f(y)$.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents.

If S is an inverse semigroup then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the *natural partial order* on S [16].

The bicyclic monoid $\mathscr{C}(p, q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The semigroup operation on $\mathscr{C}(p, q)$ is determined as follows:

$$q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p, q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p, q)$ is a group congruence [1].

On the set $\boldsymbol{B}_\omega = \omega \times \omega$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2) & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2) & \text{if } j_1 \geqslant i_2. \end{cases} \qquad (1)$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ to the semigroup $\boldsymbol{B}_\omega$ is isomorphic by the mapping $\mathfrak{h}\colon \mathscr{C}(p,q) \to \boldsymbol{B}_\omega$, $q^k p^l \mapsto (k,l)$ (see: [1, Section 1.12] or [14, Exercise IV.1.11(ii)]).

Next we shall describe the construction which is introduced in [7].

Let $\boldsymbol{B}_\omega$ be the bicyclic monoid and $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_\omega \times \mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1,j_1,F_1)\cdot(i_2,j_2,F_2)=\begin{cases} (i_1-j_1+i_2,j_2,(j_1-i_2+F_1)\cap F_2) & \text{if } j_1\leqslant i_2;\\ (i_1,j_1-i_2+j_2,F_1\cap(i_2-j_1+F_2)) & \text{if } j_1\geqslant i_2. \end{cases} \tag{2}$$

In [7] is proved that if the family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is ω-closed then $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I} = \{(i,j,\varnothing)\colon i,j \in \omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$. For any ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_\omega^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)/\boldsymbol{I} & \text{if } \varnothing \in \mathscr{F};\\ (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot) & \text{if } \varnothing \notin \mathscr{F} \end{cases}$$

is defined in [7]. The semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in [7] that $\boldsymbol{B}_\omega^{\mathscr{F}}$ is a combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_\omega^{\mathscr{F}}$ and its set of idempotents are described. Here, the criteria when the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is simple, 0-simple, bisimple, 0-bisimple, or it has the identity, are given. In particular in [7] it is proved that the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega\times\omega$-matrix units if and only if $\mathscr{F}$ consists of a singleton set and the empty set, and $\boldsymbol{B}_\omega^{\mathscr{F}}$ is isomorphic to the bicyclic monoid if and only if $\mathscr{F}$ consists of a non-empty inductive subset of ω.

Group congruences on the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ and its homomorphic retracts in the case when an ω-closed family $\mathscr{F}$ consists of inductive non-empty subsets of ω are studied in [8]. It is proven that a congruence $\mathfrak{C}$ on $\boldsymbol{B}_\omega^{\mathscr{F}}$ is a group congruence if and only if its restriction on a subsemigroup of $\boldsymbol{B}_\omega^{\mathscr{F}}$, which is isomorphic to the bicyclic semigroup, is not the identity relation. Also in [8], all non-trivial homomorphic retracts and isomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ are described.

In [5, 13] the algebraic structure of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is established in the case when ω-closed family $\mathscr{F}$ consists of atomic subsets of ω.

The set $\boldsymbol{B}_{\mathbb{Z}} = \mathbb{Z} \times \mathbb{Z}$ with the semigroup operation defined by formula (1) is called the *extended bicyclic semigroup* [17]. On the set $\boldsymbol{B}_{\mathbb{Z}} \times \mathscr{F}$, where $\mathscr{F}$ is an ω-closed subfamily of $\mathscr{P}(\omega)$, we define the semigroup operation "$\cdot$" by formula (2). In [9] it is proved that $(\boldsymbol{B}_{\mathbb{Z}} \times \mathscr{F}, \cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I} = \{(i, j, \varnothing) \colon i, j \in \mathbb{Z}\}$ is an ideal of the semigroup $(\boldsymbol{B}_{\mathbb{Z}} \times \mathscr{F}, \cdot)$. For any ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_{\mathbb{Z}} \times \mathscr{F}, \cdot)/\boldsymbol{I} & \text{if } \varnothing \in \mathscr{F}; \\ (\boldsymbol{B}_{\mathbb{Z}} \times \mathscr{F}, \cdot) & \text{if } \varnothing \notin \mathscr{F} \end{cases}$$

is defined in [9] similarly as in [7]. In [9] it is proven that $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is a combinatorial inverse semigroup. Green's relations, the natural partial order on the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ and its set of idempotents are described. Here, the criteria when the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is simple, 0-simple, bisimple, 0-bisimple, is isomorphic to the extended bicyclic semigroup, are derived. In particularly in [9] it is proved that the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega{\times}\omega$-matrix units if and only if $\mathscr{F}$ consists of a singleton set and the empty set, and $\boldsymbol{B}_{Z}^{\mathscr{F}}$ is isomorphic to the extended bicyclic semigroup if and only if $\mathscr{F}$ consists of a non-empty inductive subset of ω. Also, in [9] it is proved that in the case when the family $\mathscr{F}$ consists of all singletons of ω and the empty set, the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is isomorphic to the Brandt λ-extension of the semilattice $(\omega, \min)$, where $(\omega, \min)$ is the set ω with the semilattice operation $x \cdot y = \min\{x, y\}$.

It is well-known that every automorphism of the bicyclic monoid $\boldsymbol{B}_{\omega}$ is the identity self-map of $\boldsymbol{B}_{\omega}$ [1], and hence the group $\mathbf{Aut}(\boldsymbol{B}_{\omega})$ of automorphisms of $\boldsymbol{B}_{\omega}$ is trivial. The group $\mathbf{Aut}(\boldsymbol{B}_{\mathbb{Z}})$ of automorphisms of the extended bicyclic semigroup $\boldsymbol{B}_{\mathbb{Z}}$ is established in [6] and there it is proved that $\mathbf{Aut}(\boldsymbol{B}_{\mathbb{Z}})$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$. Also in [10] the semigroups of endomorphisms of the bicyclic semigroup and the extended bicyclic semigroup are described.

Later we assume that an ω-closed family $\mathscr{F}$ consists of inductive nonempty subsets of ω.

In this paper we study automorphisms of the semigroup $\boldsymbol{B}_{Z}^{\mathscr{F}}$ with the family $\mathscr{F}$ of inductive nonempty subsets of ω and prove that the group $\mathbf{Aut}(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ of automorphisms of the semigroup $\boldsymbol{B}_{Z}^{\mathscr{F}}$ is isomorphic to the additive group of integers.

2. Algebraic properties of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$

Proposition 1. *Let $\mathscr{F}$ be an arbitrary nonempty ω-closed family of subsets of ω and let $n_0 = \min\{\bigcup \mathscr{F}\}$. Then the following statements hold:*

1) $\mathscr{F}_0 = \{-n_0 + F : F \in \mathscr{F}\}$ *is an* ω*-closed family of subsets of* ω*;*
2) *the semigroups* $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ *and* $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}_0}$ *are isomorphic by the mapping*

$$(i, j, F) \mapsto (i, j, -n_0 + F), \qquad i, j \in \mathbb{Z};$$

Proof. Statement 1) is proved in [8, Proposition 1(1)]. The proof of 2) is similar to the one of Proposition 1(2) from [8]. □

Suppose that $\mathscr{F}$ is an ω-closed family of inductive subsets of ω. Fix an arbitrary $k \in \mathbb{Z}$. If $[0) \in \mathscr{F}$ and $[p) \in \mathscr{F}$ for some $p \in \omega$ then for any $i, j \in \mathbb{Z}$ and we have that

$$(k, k, [0)) \cdot (i, j, [p)) = \begin{cases} (k - k + i, j, (k - i + [0)) \cap [p)) & \text{if } k < i; \\ (k, j, [0) \cap [p)) & \text{if } k = i; \\ (k, k - i + j, [0) \cap (i - k + [p))) & \text{if } k > i \end{cases}$$
$$= \begin{cases} (i, j, [p)) & \text{if } k < i; \\ (k, j, [p)) & \text{if } k = i; \\ (k, k - i + j, [0) \cap [i - k + p)) & \text{if } k > i \end{cases}$$

and

$$(i, j, [p)) \cdot (k, k, [0)) = \begin{cases} (i - j + k, k, (j - k + [p)) \cap [0)) & \text{if } j < k; \\ (i, k, [p) \cap [0)) & \text{if } j = k; \\ (i, j - k + k, [p) \cap (k - j + [0))) & \text{if } j > k \end{cases}$$
$$= \begin{cases} (i - j + k, k, [j - k + p) \cap [0)) & \text{if } j < k; \\ (i, k, [p)) & \text{if } j = k; \\ (i, j, [p) & \text{if } j > k. \end{cases}$$

Therefore the above equalities imply that

$$\begin{aligned}(k, k, [0)) \cdot \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} \cdot (k, k, [0)) &= (k, k, [0)) \cdot \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} \cap \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} \cdot (k, k, [0)) \\ &= \{(i, j, [p)) : i, j \geqslant k,\ [p) \in \mathscr{F}\}\end{aligned}$$

for an arbitrary $k \in \mathbb{Z}$. We define

$$\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}[k, k, 0) = (k, k, [0)) \cdot \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} \cap \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} \cdot (k, k, [0)).$$

It is obvious that $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}[k, k, 0)$ is a subsemigroup of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$.

Proposition 2. *Let* $\mathscr{F}$ *be an arbitrary nonempty* ω*-closed family of inductive nonempty subsets of* ω *such that* $[0) \in \mathscr{F}$*. Then the subsemigroup* $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}[k, k, 0)$ *of* $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ *is isomorphic to* $\boldsymbol{B}_{\omega}^{\mathscr{F}}$*.*

Proof. Since the family $\mathscr{F}$ does not contain the empty set, $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} = (\boldsymbol{B}_{\mathbb{Z}} \times \mathscr{F}, \cdot)$. We define a map $\mathfrak{I}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}} \to \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}[k,k,0)$ in the following way $(i,j,[p)) \mapsto (i+k, j+k, [p))$. It is obvious that $\mathfrak{I}$ is a bijection. Then for any $i_1, i_2, j_1, j_2 \in \mathbb{Z}$ and $F_1, F_2 \in \mathscr{F}$ we have that

$$\begin{aligned}
&\mathfrak{I}((i_1,j_1,F_1)\cdot(i_2,j_2,F_2)) \\
&\quad= \begin{cases} \mathfrak{I}(i_1-j_1+i_2, j_2, (j_1-i_2+F_1)\cap F_2) & \text{if } j_1<i_2; \\ \mathfrak{I}(i_1, j_2, F_1\cap F_2) & \text{if } j_1=i_2; \\ \mathfrak{I}(i_1, j_1-i_2+j_2, F_1\cap(i_2-j_1+F_2)) & \text{if } j_1>i_2 \end{cases} \\
&\quad= \begin{cases} (i_1-j_1+i_2+k, j_2+k, ((j_1-i_2+F_1)\cap F_2)) & \text{if } j_1<i_2; \\ (i_1+k, j_2+k, F_1\cap F_2) & \text{if } j_1=i_2; \\ (i_1+k, j_1-i_2+j_2+k, (F_1\cap(i_2-j_1+F_2))) & \text{if } j_1>i_2 \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{I}(i_1,j_1,F_1)\cdot\mathfrak{I}(i_2,j_2,F_2) = (i_1+k, j_1+k, F_1)\cdot(i_2+k, j_2+k, F_2) \\
&= \begin{cases} (i_1-j_1+i_2+k, j_2+k, (j_1-i_2+F_1)\cap F_2) & \text{if } j_1+k<i_2+k; \\ (i_1+k, j_2+k, F_1\cap F_2) & \text{if } j_1+k=i_2+k; \\ (i_1+k, j_1-i_2+j_2+k, F_1\cap(i_2-j_1+F_2)) & \text{if } j_1+k>i_2+k \end{cases} \\
&= \begin{cases} (i_1-j_1+i_2+k, j_2+k, (j_1-i_2+F_1)\cap F_2) & \text{if } j_1<i_2; \\ (i_1+k, j_2+k, F_1\cap F_2) & \text{if } j_1=i_2; \\ (i_1+k, j_1-i_2+j_2+k, F_1\cap(i_2-j_1+F_2)) & \text{if } j_1>i_2 \end{cases}
\end{aligned}$$

and hence $\mathfrak{I}$ is a homomorphism which implies the statement of the proposition. □

By Remarks 1(2)) and 1(3)) every nonempty subset $F \in \mathscr{F}$ contains the least element, and hence later for every nonempty set $F \in \mathscr{F}$ we denote $n_F = \min F$.

Below we need the following lemma from [8].

Lemma 1 ([8])**.** *Let $\mathscr{F}$ be an ω-closed family of inductive subsets of ω. Let F_1 and F_2 be elements of $\mathcal{F}$ such that $n_{F_1} < n_{F_2}$. Then for any positive integer $k \in \{n_{F_1}+1, \ldots, n_{F_2}-1\}$ there exists $F \in \mathscr{F}$ such that $F = [k)$.*

Proposition 1 implies that without loss of generality later we may assume that $[0) \in \mathscr{F}$ for any ω-closed family $\mathscr{F}$ of inductive subsets of ω. Hence these arguments and Lemma 5 of [7] imply the following proposition.

Proposition 3. *Let $\mathscr{F}$ be an infinite ω-closed family of inductive nonempty subsets of ω. Then the diagram in Fig. 1 describes the natural partial order on the band of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$.*

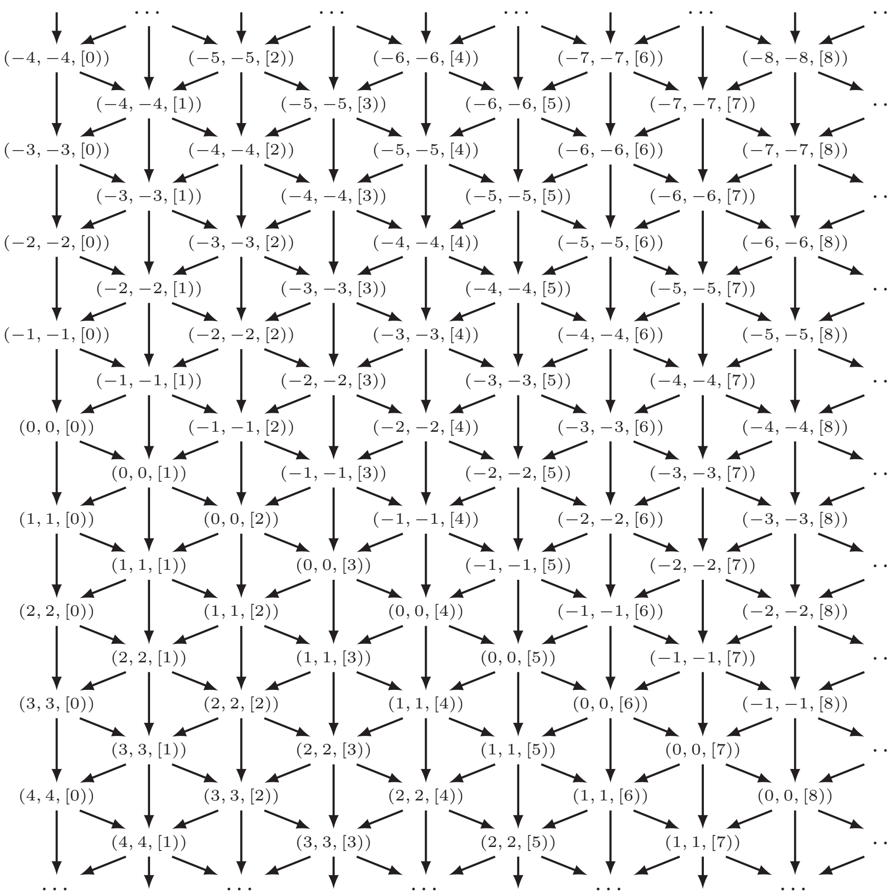

Figure 1. The natural partial order on the band $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$

By the similar way for a finite ω-closed family of inductive nonempty subsets of ω we obtain the following

Proposition 4. *Let* $\mathscr{F} = \{[0), \ldots, [k)\}$. *Then the diagram on Fig. 1 without elements of the form* $(i, j, [p))$ *and their arrows,* $i, j \in \mathbb{Z}$, $p > k$, *describes the natural partial order on the band of* $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$.

The definition of the semigroup operation in $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ implies that in the case when $\mathscr{F}$ is an ω-closed family subsets of ω and $F \in \mathscr{F}$ is a nonempty inductive subset in ω then the set

$$\boldsymbol{B}_{\mathbb{Z}}^{\{F\}} = \{(i, j, F) \colon i, j \in \mathbb{Z}\}$$

with the induced semigroup operation from $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is a subsemigroup of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ which by Proposition 5 from [9] is isomorphic to the extended bicyclic semigroup $\boldsymbol{B}_{\mathbb{Z}}$.

Proposition 5. *Let $\mathscr{F}$ be an arbitrary ω-closed family of inductive subsets of ω and S be a subsemigroup of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ which is isomorphic to the extended bicyclic semigroup $\boldsymbol{B}_{\mathbb{Z}}$. Then there exists a subset $F\in\mathscr{F}$ such that S is a subsemigroup in $\boldsymbol{B}_{\mathbb{Z}}^{\{F\}}$.*

Proof. Suppose that $\mathfrak{I}\colon \boldsymbol{B}_{\mathbb{Z}} \to S \subseteq \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is an isomorphism. Proposition 21(2) of [12, Section 1.4] implies that the image $\mathfrak{I}(0,0)$ is an idempotent of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$, and hence by Lemma 1(2) from [9], $\mathfrak{I}(0,0)=(i,i,F)$ for some $i\in\mathbb{Z}$ and $F\in\mathscr{F}$. By Proposition 2.1($viii$) of [3] the subset $(0,0)\boldsymbol{B}_{\mathbb{Z}}(0,0)$ of $\boldsymbol{B}_{\mathbb{Z}}$ is isomorphic to the bicyclic semigroup, and hence the image $\mathfrak{I}\left((0,0)\boldsymbol{B}_{\mathbb{Z}}(0,0)\right)$ is isomorphic to the bicyclic semigroup $\boldsymbol{B}_{\omega}$. Then the definition of the natural partial order on $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ and Corollary 1 from [9] imply that there exists an integer k such that $(i,i,F)\preccurlyeq(k,k,[0))$. By Proposition 2 the subsemigroup

$$\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}[k,k,0)=(k,k,[0))\cdot\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}\cdot(k,k,[0))$$

of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is isomorphic to $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. Since $(i,i,F)\preccurlyeq(k,k,[0))$ we have that $\mathfrak{I}\left((0,0)\boldsymbol{B}_{\mathbb{Z}}(0,0)\right)\subseteq\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}[k,k,0)$, and hence $\mathfrak{I}\left((0,0)\boldsymbol{B}_{\mathbb{Z}}(0,0)\right)\subseteq\boldsymbol{B}_{\mathbb{Z}}^{\{F\}}$ by Proposition 4 of [8].

Next, fix any negative integer n. By Proposition 2.1($viii$) of [3] the subset $(n,n)\boldsymbol{B}_{\mathbb{Z}}(n,n)$ of $\boldsymbol{B}_{\mathbb{Z}}$ is isomorphic to the bicyclic semigroup. Since $(0,0)\boldsymbol{B}_{\mathbb{Z}}(0,0)$ is an inverse subsemigroup of $(n,n)\boldsymbol{B}_{\mathbb{Z}}(n,n)$, the above arguments imply that $\mathfrak{I}\left((n,n)\boldsymbol{B}_{\mathbb{Z}}(n,n)\right)\subseteq\boldsymbol{B}_{\mathbb{Z}}^{\{F\}}$ for any negative integer n. Since

$$\boldsymbol{B}_{\mathbb{Z}}=\bigcup\left\{(k,k)\boldsymbol{B}_{\mathbb{Z}}(k,k)\colon -k\in\omega\right\},$$

we get that $\mathfrak{I}\left(\boldsymbol{B}_{\mathbb{Z}}\right)\subseteq\boldsymbol{B}_{\mathbb{Z}}^{\{F\}}$. □

3. On authomorphisms of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$

Recall [4] define relations $\mathscr{L}$ and $\mathscr{R}$ on an inverse semigroup S by

$$(s,t)\in\mathscr{L}\ \Leftrightarrow\ s^{-1}s=t^{-1}t \qquad \text{and} \qquad (s,t)\in\mathscr{R} \quad\Leftrightarrow\quad ss^{-1}=tt^{-1}.$$

Both $\mathscr{L}$ and $\mathscr{R}$ are equivalence relations on S. The relation $\mathscr{D}$ is defined to be the smallest equivalence relation which contains both $\mathscr{L}$ and $\mathscr{R}$, which is equivalent that $\mathscr{D}=\mathscr{L}\circ\mathscr{R}=\mathscr{R}\circ\mathscr{L}$ [12].

Remark 2. It is obvious that every semigroup isomorphism $\mathfrak{i}\colon S \to T$ maps a $\mathscr{D}$-class (resp. $\mathscr{L}$-class, $\mathscr{R}$-class) of S onto a $\mathscr{D}$-class (resp. $\mathscr{L}$-class, $\mathscr{R}$-class) of T.

In this section we assume that $[0) \in \mathscr{F}$ for any ω-closed family $\mathscr{F}$ of inductive subsets of ω.

An automorphism $\mathfrak{a}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is called a $(0,0,[0))$-*automorphism* if $\mathfrak{a}(0,0,[0)) = (0,0,[0))$.

Theorem 1. *Let $\mathscr{F}$ be an ω-closed family of inductive nonempty subsets of ω. Then every* $(0,0,[0))$-automorphism *of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is the identity map.*

Proof. Let $\mathfrak{a}\colon \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} \to \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ be an arbitrary $(0,0,[0))$-automorphism.

By Theorem 4(iv) of [9] the elements (i_1, j_1, F_1) and (i_2, j_2, F_2) of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ are $\mathscr{D}$-equivalent if and only if $F_1 = F_2$. Since every automorphism preserves $\mathscr{D}$-classes, the above argument implies that $\mathfrak{a}(\boldsymbol{B}_{\mathbb{Z}}^{\{F_1\}}) = \boldsymbol{B}_{\mathbb{Z}}^{\{F_2\}}$ if and only if $F_1 = F_2$ for $F_1, F_2 \in \mathscr{F}$. Hence we have that $\mathfrak{a}(\boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}) = \boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$. By Proposition 21(6) of [12, Section 1.4] every automorphism preserves the natural partial order on the semilattice $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ and since $\mathfrak{a}$ is a $(0,0,[0))$-automorphism of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ we get that $\mathfrak{a}(i,i,[0)) = (i,i,[0))$ for any integer i.

Fix arbitrary $k, l \in \mathbb{Z}$. Suppose that $\mathfrak{a}(k,l,[0)) = (p,q,[0))$ for some integers p and q. Since the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is inverse, Proposition 21(1) of [12, Section 1.4] and Lemma 1(4) of [9] imply that

$$\big(\mathfrak{a}(k,l,[0))\big)^{-1} = (p,q,[0))^{-1} = (q,p,[0)).$$

Again by Proposition 21(1) of [12, Section 1.4] we have that

$$\begin{aligned}(k,k,[0)) &= \mathfrak{a}(k,k,[0)) = \mathfrak{a}((k,l,[0)) \cdot (l,k,[0)))\\ &= \mathfrak{a}(k,l,[0)) \cdot \mathfrak{a}(l,k,[0)) = \mathfrak{a}(k,l,[0)) \cdot \mathfrak{a}\big((k,l,[0))^{-1}\big)\\ &= (p,q,[0)) \cdot (q,p,[0)) = (p,p,[0)),\end{aligned}$$

and hence $p = k$. By similar way we get that $l = q$. Therefore, $\mathfrak{a}(k,l,[0)) = (k,l,[0))$ for any integers k and l.

If $\mathscr{F} \neq \{[0)\}$ then by Lemma 1, $[1) \in \mathscr{F}$. The definition of the natural partial order on the semilattice $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ (also, see Proposition 3) and Corollary 5 of [9] imply that $(0,0,[1))$ is the unique idempotent $\boldsymbol{\varepsilon}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ with the property

$$(1,1,[0)) \preccurlyeq \boldsymbol{\varepsilon} \preccurlyeq (0,0,[0)).$$

Since by Proposition 21(6) of [12, Section 1.4] the automorphism $\mathfrak{a}$ preserves the natural partial order on the semilattice $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$, we get that $\mathfrak{a}(0,0,[1)) = (0,0,[1))$. Similar arguments as in the above paragraph imply that $\mathfrak{a}(k,l,[1)) = (k,l,[1))$ for any integers k and l.

Next, by induction we obtain that $\mathfrak{a}(k,l,[p)) = (k,l,[p))$ for any $k,l \in \mathbb{Z}$ and $[p) \in \mathscr{F}$. □

Proposition 6. *Let $\mathscr{F}$ be an ω-closed family of inductive nonempty subsets of ω. Then for every integer k the map $\mathfrak{h}_k \colon \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} \to \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$, $(i,j,[p)) \mapsto (i+k,j+k,[p))$ is an automorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$.*

The proof of Proposition 6 is similar to Proposition 2.

For a partially ordered set $(P,\leqq)$, a subset X of P is called *order-convex*, if $x \leqq z \leqq y$ and $x,y \subset X$ implies that $z \in X$, for all $x,y,z \in P$ [11].

Lemma 2. *If $\mathscr{F}$ is an infinite ω-closed family of inductive nonempty subsets of ω then*

$$\{(0,0,[k)) \colon k \in \omega\}$$

is an order-convex linearly ordered subset of $(E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}),\preccurlyeq)$.

Proof. Fix arbitrary $(0,0,[m)),(0,0,[n)),(0,0,[p)) \in E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$. If

$$(0,0,[m)) \preccurlyeq (0,0,[n)) \preccurlyeq (0,0,[p))$$

then Corollary 1 of [9] implies that $[m) \subseteq [n) \subseteq [p)$. Hence we have that $m \geqslant n \geqslant p$, which implies the statement of the lemma. □

Proposition 7. *Let $\mathscr{F}$ be an infinite ω-closed family of inductive nonempty subsets of ω. Then*

$$\mathfrak{a}(0,0,[0)) \in \boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$$

for any automorphism $\mathfrak{a}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$.

Proof. Suppose to the contrary that there exists an automorphism $\mathfrak{a}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ such that $\mathfrak{a}(0,0,[0)) \notin \boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$. Then $\mathfrak{a}(0,0,[0))$ is an idempotent of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$. Lemma 1(2) of [9] implies that $\mathfrak{a}(0,0,[0)) = (i,i,[p))$ for some integer i and some positive integer p. Since the automorphism $\mathfrak{a}$ maps a $\mathscr{D}$-class of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ onto its $\mathscr{D}$-class there exists an element $(0,0,[s))$ of the chain

$$\cdots \preccurlyeq (0,0,[k)) \preccurlyeq (0,0,[k-1)) \preccurlyeq \cdots \preccurlyeq (0,0,[2)) \preccurlyeq (0,0,[1)) \preccurlyeq (0,0,[0)) \quad (3)$$

such that $\mathfrak{a}(0,0,[s))=(m,m,[0))\in \boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$ for some integer m. By Proposition 21(6) of [12, Section 1.4] every automorphism preserves the natural partial order on the semilattice $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$, and hence the inequality $(0,0,[s))\preccurlyeq(0,0,[0))$ implies that

$$\mathfrak{a}(0,0,[s))=(m,m,[0))\preccurlyeq(i,i,[p))=\mathfrak{a}(0,0,[0)).$$

By Corollary 1 of [9] we have that $m\geqslant i$ and $[0)\subseteq i-m+[p)$. The last inclusion implies that $m\geqslant i+p$. Since the chain (3) is infinite and any its two distinct elements belong to distinct two $\mathscr{D}$-classes of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$, Proposition 21(6) of [12, Section 1.4] and Remark 2 imply that there exists a positive integer $q>s$ such that $\mathfrak{a}(0,0,[q))=(t,t,[x))$ for some positive integer $x>p$ and some integer t. Then

$$\mathfrak{a}(0,0,[q))=(t,t,[x))\preccurlyeq(m,m,[0))=\mathfrak{a}(0,0,[s))$$

and by Corollary 1 of [9] we have that $t\geqslant m$ and $[x)\subseteq t-m+[0)$, and hence $x\geqslant t-m$.

Next we consider the idempotent $(i+1,i+1,[p))$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$. By Corollary 1 of [9] we get that $(i+1,i+1,[p))\preccurlyeq(i,i,[p))$ in $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$. Since $x>p$ we have that $x\geqslant p+1$. The inequalities $t\geqslant m\geqslant i+p$ and $p\geqslant 1$ imply that $t\geqslant i+1$. Also, the inequalities $t\geqslant m\geqslant i$ and $x\geqslant p+1$ imply that $t+x\geqslant i+1+p$, and hence we obtain the inclusion $[x)\subseteq i+1-t+[p)$. By Corollary 1 of [9] we have that $(t,t,[x))\preccurlyeq(i+1,i+1,[p))$. Since $\mathfrak{a}$ is an automorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$, its restriction $\mathfrak{a}|_{E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})}\colon E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})\to E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ onto the band $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ is an order automorphism of the partially ordered set $(E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}),\preccurlyeq)$, and hence the map $\mathfrak{a}|_{E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})}$ preserves order-convex subsets of $(E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}),\preccurlyeq)$. By Lemma 2 chain (3) is order-convex in the partially ordered set $(E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}),\preccurlyeq)$. The inequalities $(t,t,[x))\preccurlyeq(i+1,i+1,[p))\preccurlyeq(i,i,[p))$ in $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ imply that the image of order-convex chain (3) under the order automorphism $\mathfrak{a}|_{E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})}$ is not an order-convex subset of $(E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}),\preccurlyeq)$, a contradiction. The obtained contradiction implies the statement of the proposition. □

Later for any integer k we assume that $\mathfrak{h}_k\colon \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}\to\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is an automorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ defined in Proposition 6.

Theorem 2. *Let $\mathscr{F}$ be an infinite ω-closed family of inductive nonempty subsets of ω. Then for any automorphism $\mathfrak{a}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ there exists an integer p such that $\mathfrak{a}=\mathfrak{h}_p$.*

Proof. By Proposition 7 there exists an integer p such that $\mathfrak{a}(0,0,[0)) = (-p,-p,[0))$. Then the composition $\mathfrak{h}_p \circ \mathfrak{a}$ is a $(0,0,[0))$-automorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$, i.e., $(\mathfrak{h}_p \circ \mathfrak{a})(0,0,[0)) = (0,0,[0))$, and hence by Theorem 1 the composition $\mathfrak{h}_p \circ \mathfrak{a}$ is the identity map of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$. Since $\mathfrak{h}_p$ and $\mathfrak{a}$ are bijections of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ the above arguments imply that $\mathfrak{a} = \mathfrak{h}_p$. □

Since $\mathfrak{h}_{k_1} \circ \mathfrak{h}_{k_2} = \mathfrak{h}_{k_1+k_2}$ and $\mathfrak{h}_{k_1}^{-1} = \mathfrak{h}_{-k_1}$, $k_1, k_2 \in \mathbb{Z}$, for any automorphisms $\mathfrak{h}_{k_1}$ and $\mathfrak{h}_{k_2}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$, Theorem 2 implies the following.

Corollary 1. *Let $\mathscr{F}$ be an infinite ω-closed family of inductive nonempty subsets of ω. Then the group of automorphisms $\mathbf{Aut}(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is isomorphic to the additive group of integers $(\mathbb{Z},+)$.*

The following example shows that for an arbitrary nonnegative integer k and the finite family $\mathscr{F} = \{[0),[1),\ldots,[k)\}$ there exists an automorphism $\widetilde{\mathfrak{a}}\colon \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} \to \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ which is distinct from the form $\mathfrak{h}_p$.

Example 1. Fix an arbitrary nonnegative integer k. Put

$$\widetilde{\mathfrak{a}}(i,j,[s)) = (i+s, j+s, [k-s))$$

for any $s = 0,1,\ldots,k$ and all $i,j \in \mathbb{Z}$.

Lemma 3. *Let k be an arbitrary nonnegative integer and $\mathscr{F} = \{[0),[1),\ldots,[k)\}$. Then $\widetilde{\mathfrak{a}}\colon \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} \to \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is an automorphism.*

Proof. Fix arbitrary $i,j,m,n \in \mathbb{Z}$. Without loss of generality we may assume that $s,t \in \{0,1,\ldots,k\}$ with $s<t$. Then we have that

$$\begin{aligned}
&\widetilde{\mathfrak{a}}((i,j,[s))\cdot(m,n,[t))) = \\
&= \begin{cases}
\widetilde{\mathfrak{a}}(i-j+m, n, (j-m+[s))\cap[t)) & \text{if } j<m;\\
\widetilde{\mathfrak{a}}(i,n,[s)\cap[t)) & \text{if } j=m;\\
\widetilde{\mathfrak{a}}(i, j-m+n, [s)\cap(m-j+[t))) & \text{if } j>m
\end{cases}\\
&= \begin{cases}
\widetilde{\mathfrak{a}}(i-j+m,n,[t)) & \text{if } j<m;\\
\widetilde{\mathfrak{a}}(i,n,[1)) & \text{if } j=m;\\
\widetilde{\mathfrak{a}}(i,j-m+n,[s)) & \text{if } j>m \text{ and } m+t<j+s;\\
\widetilde{\mathfrak{a}}(i,j-m+n,[s)) & \text{if } j>m \text{ and } m+t=j+s;\\
\widetilde{\mathfrak{a}}(i,j-m+n,m-j+[t)) & \text{if } j>m \text{ and } m+t>j+s
\end{cases}\\
&= \begin{cases}
(i-j+m+s, n+s, [k-t)) & \text{if } j<m;\\
(i+t, n+t, [k-t)) & \text{if } j=m;\\
(i+s, j-m+n+s, [k-s)) & \text{if } j>m \text{ and } m+t<j+s;\\
(i+s, j-m+n+s, [k-s)) & \text{if } j>m \text{ and } m+t=j+s;\\
(i-j+m+t, n+t, [k-m+j-t)) & \text{if } j>m \text{ and } m+t>j+s,
\end{cases}
\end{aligned}$$

$$\widetilde{\mathfrak{a}}(i,j,[s))\cdot\widetilde{\mathfrak{a}}(m,n,[t)) = (i+s,j+s,[k-s))\cdot(m+t,n+t,[k-t))$$
$$= \begin{cases} (i-j+m+t,n+t, \\ \quad (j+s-m-t+[k-s))\cap[k-t)) & \text{if } j+s<m+t; \\ (i+s,n+t,[k-s)\cap[k-t)) & \text{if } j+s=m+t; \\ (i+s,j+s-m+n, \\ \quad [k-s)\cap(m+t-s-j+[k-t))) & \text{if } j+s>m+t \end{cases}$$
$$= \begin{cases} (i{-}j{+}m{+}t,n{+}t, \\ \quad [k{-}t{+}j{-}m){\cap}[k{-}t)) & \text{if } j<m \text{ and } j+s<m+t; \\ (i+t,n+t, \\ \quad [k-t)\cap[k-t)) & \text{if } j=m \text{ and } j+s<m+t; \\ (i{-}j{+}m{+}t,n{+}t, \\ \quad [k{-}t{+}j{-}m){\cap}[k-t)) & \text{if } j>m \text{ and } j+s<m+t; \\ \text{vagueness} & \text{if } j<m \text{ and } j+s=m+t; \\ \text{vagueness} & \text{if } j=m \text{ and } j+s=m+t; \\ (i+s,n+t,[k-t)) & \text{if } j>m \text{ and } j+s=m+t; \\ \text{vagueness} & \text{if } j<m \text{ and } j+s>m+t; \\ \text{vagueness} & \text{if } j=m \text{ and } j+s>m+t; \\ (i{+}s,j{-}m{+}n{+}s, \\ \quad [k{-}s){\cap}[k{-}s{-}j{+}m)) & \text{if } j>m \text{ and } j+s>m+t \end{cases}$$
$$= \begin{cases} (i-j+m+t,n+t,[k-t)) & \text{if } j<m \text{ and } j+s<m+t; \\ \text{vagueness} & \text{if } j<m \text{ and } j+s=m+t; \\ \text{vagueness} & \text{if } j<m \text{ and } j+s>m+t; \\ (i+t,n+t,[k-t)) & \text{if } j=m \text{ and } j+s<m+t; \\ \text{vagueness} & \text{if } j=m \text{ and } j+s=m+t; \\ \text{vagueness} & \text{if } j=m \text{ and } j+s>m+t; \\ (i-j+m+t,n+t, \\ \quad [k-t+j-m)) & \text{if } j>m \text{ and } j+s<m+t; \\ (i+s,n+t,[k-t)) & \text{if } j>m \text{ and } j+s=m+t; \\ (i+s,j-m+n+s,[k-s)) & \text{if } j>m \text{ and } j+s>m+t, \end{cases}$$

$$\widetilde{\mathfrak{a}}((m,n,[t))\cdot(i,j,[s))$$
$$= \begin{cases} \widetilde{\mathfrak{a}}(m-n+i,j,(n-i+[t))\cap[s)) & \text{if } n<i; \\ \widetilde{\mathfrak{a}}(m,j,[t)\cap[s)) & \text{if } n=i; \\ \widetilde{\mathfrak{a}}(m,n-i+j,[t)\cap(i-n+[s))) & \text{if } n>i \end{cases}$$
$$= \begin{cases} \widetilde{\mathfrak{a}}(m-n+i,j,[s)) & \text{if } n<i \text{ and } n+t<i+s; \\ \widetilde{\mathfrak{a}}(m-n+i,j,[s)) & \text{if } n<i \text{ and } n+t=i+s; \\ \widetilde{\mathfrak{a}}(m-n+i,j,[n-i+t)) & \text{if } n<i \text{ and } n+t>i+s; \\ \widetilde{\mathfrak{a}}(m,j,[t)) & \text{if } n=i; \\ \widetilde{\mathfrak{a}}(m,n-i+j,[t)) & \text{if } n>i \end{cases}$$

$$= \begin{cases} (m-n+i+s, j+s, [k-s)) & \text{if } n<i \text{ and } n+t<i+s;\\ (m-n+i+s, j+s, [k-s)) & \text{if } n<i \text{ and } \quad n+t=i+s;\\ (m+t, j+n-i+t, [k-t+i-n)) & \text{if } n<i \text{ and } \quad n+t>i+s;\\ (m+t, j+t, [k-t)) & \text{if } n=i;\\ (m+t, n-i+j+t, [k-t)) & \text{if } n>i, \end{cases}$$

$$\widetilde{\mathfrak{a}}(m,n,[t))\cdot\widetilde{\mathfrak{a}}(i,j,[s)) = (m+t, n+t, [k-t))\cdot(i+s, j+s, [k-s))$$

$$= \begin{cases} (m-n+i+s, j+s, \\ \quad (n+t-i-s+[k-t))\cap[k-s)) & \text{if } n+t<i+s;\\ (m+t, j+s, [k-t)\cap[k-s)) & \text{if } n+t=i+s;\\ (m+t, n-i+j+t, \\ \quad [k-t)\cap(i+s-n-t+[k-s))) & \text{if } n+t>i+s \end{cases}$$

$$= \begin{cases} (m-n+i+s, j+s, \\ \quad [k-s+n-i))\cap[k-s)) & \text{if } n+t<i+s;\\ (m+t, j+s, [k-s)) & \text{if } n+t=i+s;\\ (m+t, n-i+j+t, [k-t+i-n)\cap[k-t)) & \text{if } n+t>i+s \end{cases}$$

$$= \begin{cases} (m-n+i+s, j+s, [k-s)) & \text{if } n<i \text{ and } n+t<i+s;\\ \text{vagueness} & \text{if } n=i \text{ and } n+t<i+s;\\ \text{vagueness} & \text{if } n>i \text{ and } n+t<i+s;\\ (m+t, j+s, [k-s)) & \text{if } n<i \text{ and } n+t=i+s;\\ \text{vagueness} & \text{if } n=i \text{ and } n+t=i+s;\\ \text{vagueness} & \text{if } n>i \text{ and } n+t=i+s;\\ (m+t, n-i+j+t, [k-t+i-n)) & \text{if } n<i \text{ and } n+t>i+s;\\ (m+t, n-i+j+t, [k-t)) & \text{if } n=i \text{ and } n+t>i+s;\\ (m+t, n-i+j+t, [k-t)) & \text{if } n>i \text{ and } n+t>i+s \end{cases}$$

$$= \begin{cases} (m-n+i+s, j+s, [k-s)) & \text{if } n<i \text{ and } n+t<i+s;\\ (m+t, j+s, [k-s)) & \text{if } n<i \text{ and } n+t=i+s;\\ (m+t, n-i+j+t, [k-t+i-n)) & \text{if } n<i \text{ and } n+t>i+s;\\ \text{vagueness} & \text{if } n=i \text{ and } n+t<i+s;\\ \text{vagueness} & \text{if } n=i \text{ and } n+t=i+s;\\ (m+t, n-i+j+t, [k-t)) & \text{if } n=i \text{ and } n+t>i+s;\\ \text{vagueness} & \text{if } n>i \text{ and } n+t<i+s;\\ \text{vagueness} & \text{if } n>i \text{ and } n+t=i+s;\\ (m+t, n-i+j+t, [k-t)) & \text{if } n>i \text{ and } n+t>i+s, \end{cases}$$

$$\widetilde{\mathfrak{a}}((i,j,[s))\cdot(m,n,[s))$$

$$= \begin{cases} \widetilde{\mathfrak{a}}(i-j+m, n, (j-m+[s))\cap[s)) & \text{if } j<m;\\ \widetilde{\mathfrak{a}}(i,n,[s)\cap[s)) & \text{if } j=m;\\ \widetilde{\mathfrak{a}}(i, j-m+n, [s)\cap(m-j+[s))) & \text{if } j>m \end{cases}$$

$$= \begin{cases} \widetilde{\mathfrak{a}}(i-j+m,n,[s)) & \text{if } j<m; \\ \widetilde{\mathfrak{a}}(i,n,[s)) & \text{if } j=m; \\ \widetilde{\mathfrak{a}}(i,j-m+n,[s)) & \text{if } j>m \end{cases}$$
$$= \begin{cases} (i-j+m+s,n+s,[k-s)) & \text{if } j<m; \\ (i+s,n+s,[k-s)) & \text{if } j=m; \\ (i+s,j-m+n+s,[k-s)) & \text{if } j>m, \end{cases}$$

$$\widetilde{\mathfrak{a}}(i,j,[s))\cdot\widetilde{\mathfrak{a}}(m,n,[s)) = (i+s,j+s,[k-s))\cdot(m+s,n+s,[k-s))$$
$$= \begin{cases} (i-j+m+s,n+s, \\ \quad (j-m+[k-s))\cap[k-s)) & \text{if } j+s<m+s; \\ (i+s,n+s,[k-s)\cap[k-s)) & \text{if } j+s=m+s; \\ (i+s,j-m+n+s, \\ \quad [k-s)\cap(m-j+[k-s))) & \text{if } j+s>m+s \end{cases}$$
$$= \begin{cases} (i-j+m+s,n+s,[k-s)) & \text{if } j<m; \\ (i+s,n+s,[k-s)) & \text{if } j=m; \\ (i+s,j-m+n+s,[k-s) & \text{if } j>m. \end{cases}$$

The above equalities imply that the map $\widetilde{\mathfrak{a}}\colon \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} \to \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is an endomorphism, and since $\widetilde{\mathfrak{a}}$ is bijective, it is an automorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$. □

Proposition 8. *Let k be any positive integer and $\mathscr{F}=\{[0),\ldots,[k)\}$. Then either $\mathfrak{a}(0,0,[0))\in\boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$ or $\mathfrak{a}(0,0,[0))\in\boldsymbol{B}_{\mathbb{Z}}^{\{[k)\}}$ for any automorphism $\mathfrak{a}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$.*

Proof. Suppose to the contrary that there exists a positive integer $m<k$ such that $\mathfrak{a}(0,0,[0))\in\boldsymbol{B}_{\mathbb{Z}}^{\{[m)\}}$. Since $\mathfrak{a}(0,0,[0))$ is an idempotent of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$, by Lemma 1(2) of [9] there exists an integer p such that $\mathfrak{a}(0,0,[0)) = (p,p,[m))$. Then by the order convexity of the subset

$$L_1 = \{(0,0,[0)),(0,0,[1))\}$$

of $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ we obtain that the image $\mathfrak{a}(L_1)$ is an order convex chain in $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ with the respect to the natural partial order. Then Remark 2 and the description of the natural partial order on $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ (see: Proposition 4) imply that either $\mathfrak{a}(0,0,[1))=(p,p,[m+1))$ or $\mathfrak{a}(0,0,[1))=(p+1,p+1,[m-1))$.

Suppose that the equality $\mathfrak{a}(0,0,[1))=(p,p,[m+1))$ holds. If $m+1=k$ then the equalities $0<m<k$ and Remark 2 imply that $\mathfrak{a}(0,0,[2))\in\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}\setminus\boldsymbol{B}_{\mathbb{Z}}^{\{[k-1),[k)\}}$. Since $(0,0,[2))\preccurlyeq(0,0,[1))\preccurlyeq(0,0,[0))$, Proposition 21(6) of [12, Section 1.4] implies that $\mathfrak{a}(0,0,[2))\preccurlyeq\mathfrak{a}(0,0,[1))\preccurlyeq\mathfrak{a}(0,0,[0))$.

Then $\{\mathfrak{a}(0,0,[0)),\mathfrak{a}(0,0,[1)),\mathfrak{a}(0,0,[2))\}$ is not an order convex subset of $(E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}),\preccurlyeq)$, because $\mathfrak{a}(0,0,[2))\in\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}\setminus\boldsymbol{B}_{\mathbb{Z}}^{\{[k-1),[k)\}}$, a contradiction, and hence we obtain that $m+1<k$.

The above arguments and induction imply that there exists a positive integer $n_0<k$ such that $\mathfrak{a}(0,0,[n_0))=(p,p,[k))$. Then $\mathfrak{a}(0,0,[n_0+1))\in\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}\setminus\boldsymbol{B}_{\mathbb{Z}}^{\{[m),\ldots,[k)\}}$ and by the description of the natural partial order on $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ (see Proposition 3) we get that

$$\{\mathfrak{a}(0,0,[0)),\mathfrak{a}(0,0,[1)),\ldots,\mathfrak{a}(0,0,[n_0)),\mathfrak{a}(0,0,[n_0+1))\}$$

is not an order convex subset of $(E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}),\preccurlyeq)$, a contradiction. The obtained contradiction implies that $\mathfrak{a}(0,0,[1))\neq(p,p,[m+1))$.

In the case $\mathfrak{a}(0,0,[1))=(p+1,p+1,[m-1))$ by similar way we get a contradiction. □

Later we assume that $\mathfrak{h}_p$ and $\widetilde{\mathfrak{a}}$ are automorphisms of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ defined in Proposition 6 and Example 1, respectively.

Proposition 9. *Let k be any positive integer and $\mathscr{F}=\{[0),\ldots,[k)\}$. Let $\mathfrak{a}\colon\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}\to\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ be an automorphisms such that $\mathfrak{a}(0,0,[0))\in\boldsymbol{B}_{\mathbb{Z}}^{\{[k)\}}$. Then there exists an integer p such that $\mathfrak{a}=\mathfrak{h}_p\circ\widetilde{\mathfrak{a}}=\widetilde{\mathfrak{a}}\circ\mathfrak{h}_p$.*

Proof. First we remark that for any integer p the automorphisms $\mathfrak{h}_p$ and $\widetilde{\mathfrak{a}}$ commute, i.e., $\mathfrak{h}_p\circ\widetilde{\mathfrak{a}}=\widetilde{\mathfrak{a}}\circ\mathfrak{h}_p$.

Suppose that $\mathfrak{a}(0,0,[0))=(p,p,[k))$ for some integer p. Then $\mathfrak{b}=\mathfrak{a}\circ\mathfrak{h}_{-p}$ is an automorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ such that $\mathfrak{b}(0,0,[0))=(0,0,[k))$. Then the order convexity of the linearly ordered set $L_1=\{(0,0,[0)),(0,0,[1))\}$ implies that the image $\mathfrak{a}(L_1)$ is an order convex chain in $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ with the respect to the natural partial order. Remark 2 and the description of the natural partial order on $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ (see: Proposition 4) imply that $\mathfrak{b}(0,0,[1))=(1,1,[k-1))$. This completes the proof of the base of induction. Fix an arbitrary $s=2,\ldots,k$ and suppose that $\mathfrak{b}(0,0,[j))=(j,j,[k-j))$ for any $j<s$, which is the assumption of induction. Next, since the linearly ordered set $L_s=\{(0,0,[s-1)),(0,0,[s))\}$ is order convex in $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$, the image $\mathfrak{a}(L_s)$ is an order convex chain in $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$, as well. Then the equality $\mathfrak{b}(0,0,[s-1))=(s-1,s-1,[k-s+1))$, Remark 2 and the description of the natural partial order on $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ (Proposition 4) imply that $\mathfrak{b}(0,0,[s))=(s,s,[k-s))$ for all $s=2,\ldots,k$.

Fix an arbitrary $s\in\{0,1,\ldots,k\}$. Since $(1,1,[s))$ is the biggest element of the set of idempotents of $\boldsymbol{B}_{\mathbb{Z}}^{\{[s)\}}$ which are less then $(0,0,[s))$, Remark 2 and the description of the natural partial order on $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ (see: Proposition 4) imply that $\mathfrak{b}(1,1,[s))=(1+s,1+s,[k-s))$. Then by induction and presented above arguments we get that $\mathfrak{b}(i,i,[s))=(i+s,i+s,[k-s))$ for any positive integer i. Also, since $(-1,-1,[s))$ is the smallest element of the set of idempotents of $\boldsymbol{B}_{\mathbb{Z}}^{\{[s)\}}$ which are greater then $(0,0,[s))$, Remark 2 and the description of the natural partial order on $E(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ imply that $\mathfrak{b}(-1,-1,[s))=(-1+s,-1+s,[k-s))$. Similar, by induction and presented above arguments we get that $\mathfrak{b}(-i,-i,[s))=(-i+s,-i+s,[k-s))$ for any positive integer i. This implies that $\mathfrak{b}(i,i,[s))=(i+s,i+s,[k-s))$ for any integer i.

Fix any $i,j\in\mathbb{Z}$ and an arbitrary $s=0,1,\ldots,k$. Remark 2 implies that $\mathfrak{b}(i,j,[s))=(m,n,[k-s))$ for some $m,n\in\mathbb{Z}$. By Proposition 21(1) of [12, Section 1.4] and Lemma 1(4) of [9] we get that $\mathfrak{b}(j,i,[s))=(n,m,[k-s))$. This implies that

$$\begin{aligned}\mathfrak{b}(i,i,[s))&=\mathfrak{b}((i,j,[s))\cdot(j,i,[s)))=\mathfrak{b}(i,j,[s))\cdot\mathfrak{b}(j,i,[s))\\&=(m,n,[k-s))\cdot(n,m,[k-s))=(m,m,[k-s))\end{aligned}$$

and

$$\begin{aligned}\mathfrak{b}(j,j,[s))&=\mathfrak{b}((j,i,[s))\cdot(i,j,[s)))=\mathfrak{b}(j,i,[s))\cdot\mathfrak{b}(i,j,[s))\\&=(n,m,[k-s))\cdot(m,n,[k-s))=(n,n,[k-s)),\end{aligned}$$

and hence we have that $m=i+s$ and $n=j+s$.

Therefore we obtain $\mathfrak{b}(i,j,[s))=(i+s,j+s,[k-s))$ for any $i,j\in\mathbb{Z}$ and an arbitrary $s=0,1,\ldots,k$, which implies that $\mathfrak{b}=\widetilde{\mathfrak{a}}$. Then

$$\mathfrak{a}=\mathfrak{a}\circ\mathfrak{h}_{-p}\circ\mathfrak{h}_p=\mathfrak{b}\circ\mathfrak{h}_p=\widetilde{\mathfrak{a}}\circ\mathfrak{h}_p,$$

which completes the proof of the proposition. □

The following lemma describes the relation between automorphisms $\widetilde{\mathfrak{a}}$ and $\mathfrak{h}_1$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ in the case when $\mathscr{F}=\{[0),\ldots,[k)\}$.

Lemma 4. *Let k be any positive integer and $\mathscr{F}=\{[0),\ldots,[k)\}$. Then*

$$\widetilde{\mathfrak{a}}\circ\widetilde{\mathfrak{a}}=\underbrace{\mathfrak{h}_1\circ\cdots\circ\mathfrak{h}_1}_{k\text{-times}}=\mathfrak{h}_k\quad\textit{and}\quad\widetilde{\mathfrak{a}}^{-1}=\underbrace{\mathfrak{h}_1^{-1}\circ\cdots\circ\mathfrak{h}_1^{-1}}_{k\text{-times}}\circ\widetilde{\mathfrak{a}}=\mathfrak{h}_{-k}\circ\widetilde{\mathfrak{a}}.$$

Proof. For any $i,j\in\mathbb{Z}$ and an arbitrary $s=0,1,\ldots,k$ we have that

$$\begin{aligned}(\widetilde{\mathfrak{a}}\circ\widetilde{\mathfrak{a}})(i,j,[s))&=\widetilde{\mathfrak{a}}(i+s,j+s,[k-s))=\\&=\widetilde{\mathfrak{a}}(i+s+k-s,j+s+k-s,[k-(k-s)))\\&=(i+k,j+k,[s))=\mathfrak{h}_k(i,j,[s)),\end{aligned}$$

Also, by the equality $\widetilde{\mathfrak{a}} \circ \widetilde{\mathfrak{a}} = \mathfrak{h}_k$ we get that $\widetilde{\mathfrak{a}} = \mathfrak{h}_1^k \circ \widetilde{\mathfrak{a}}^{-1}$, and hence

$$\widetilde{\mathfrak{a}}^{-1} = \left(\mathfrak{h}_1^k\right)^{-1} \circ \widetilde{\mathfrak{a}} = \underbrace{\mathfrak{h}_1^{-1} \circ \cdots \circ \mathfrak{h}_1^{-1}}_{k\text{-times}} \circ \widetilde{\mathfrak{a}} = \mathfrak{h}_{-k} \circ \widetilde{\mathfrak{a}},$$

which completes the proof. □

For any positive integer k we denote the following group $G_k = \langle x, y \mid xy = yx,\ y^2 = x^k \rangle$.

Lemma 5. *For any positive integer k the group $G_k = \langle x, y \mid xy = yx,\ y^2 = x^k \rangle$. is isomorphic to the additive groups of integers $\mathbb{Z}(+)$.*

Proof. In the case when $k = 2p$ for some positive integer p we have that $y^2 = x^{2p}$, and hence x is a generator of G_k such that $y = x^p$.

In the case when $k = 2p + 1$ for some $p \in \omega$ we have that $z = y \cdot x^{-k}$ is a generator of G_k such that $x = z^2$ and $y = z^{2p+1}$. □

Theorem 3. *Let k be any positive integer and $\mathscr{F} = \{[0), \ldots, [k)\}$. Then the group $\mathbf{Aut}(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ of automorphisms of the semigroup $\boldsymbol{B}_{Z}^{\mathscr{F}}$ isomorphic to the group G_k, and hence to the additive groups of integers $\mathbb{Z}(+)$.*

Proof. By Proposition 8 for any automorphism $\mathfrak{a}$ of $\boldsymbol{B}_{Z}^{\mathscr{F}}$ we have that either $\mathfrak{a}(0, 0, [0)) \in \boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$ or $\mathfrak{a}(0, 0, [0)) \in \boldsymbol{B}_{\mathbb{Z}}^{\{[k)\}}$.

Suppose that $\mathfrak{a}(0, 0, [0)) \in \boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$. Then $\mathfrak{a}(0, 0, [0))$ is an idempotent and hence by Lemma 1(2) of [9], $\mathfrak{a}(0, 0, [0)) = (-p, -p, [0))$ for some integer p. Similar arguments as in the proof of Theorem 2 imply that $\mathfrak{a} = \mathfrak{h}_p = \underbrace{\mathfrak{h}_1 \circ \cdots \circ \mathfrak{h}_1}_{p\text{-times}}$.

Suppose that $\mathfrak{a}(0, 0, [0)) \in \boldsymbol{B}_{\mathbb{Z}}^{\{[k)\}}$. Then by Proposition 9 there exists an integer p such that $\mathfrak{a} = \mathfrak{h}_p \circ \widetilde{\mathfrak{a}} = \widetilde{\mathfrak{a}} \circ \mathfrak{h}_p$.

Since $\widetilde{\mathfrak{a}}$ and $\mathfrak{h}_p$ commute, the above arguments imply that any automorphism $\mathfrak{a}$ of $\boldsymbol{B}_{Z}^{\mathscr{F}}$ is a one of the following forms:

- $\mathfrak{a} = \mathfrak{h}_p = (\mathfrak{h}_1)^p$ for some integer p; or
- $\mathfrak{a} = \mathfrak{h}_p \circ \widetilde{\mathfrak{a}} = \widetilde{\mathfrak{a}} \circ \mathfrak{h}_p = \widetilde{\mathfrak{a}} \circ (\mathfrak{h}_1)^p$ for some integer p.

This implies that the map $\mathfrak{A}\colon \mathbf{Aut}(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}) \to G_k$ defined by the formulae $\mathfrak{A}((\mathfrak{h}_1)^p) = x^p$ and $\mathfrak{A}(\widetilde{\mathfrak{a}} \circ (\mathfrak{h}_1)^p) = yx^p$, $p \in \mathbb{Z}$, is a group isomorphism. Next we apply Lemma 4. □

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