On nearly $S\Phi$-normal subgroups of finite groups

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Abstract. Let $G$ be a finite group, $H$ a subgroup of $G$ and $H_{sG}$ the subgroup of $H$ generated by all those subgroups of $H$ which are $s$-permutable in $G$. Then we say that $H$ is nearly $S\Phi$-normal in $G$ if $G$ has a normal subgroup $T$ such that $HT \leq G$ and $H \cap T \leq \Phi(H)H_{sG}$. In this paper, we study the structure of group $G$ under the condition that some given subgroups of $G$ are nearly $S\Phi$-normal in $G$. Some known results are generalised.

Introduction

Throughout this paper, all groups are finite. $G$ always denotes a group, $p$ denotes a prime, $\pi$ denotes a set of primes, and $\Phi(G)$ denotes the Frattini subgroup of $G$.

Recall that a subgroup $H$ of a group $G$ is said to be $s$-permutable (or $s$-quasinormal) [17] in $G$ if $HP = PH$ for all Sylow subgroups $P$ of $G$. A subgroup $H$ of $G$ is said to be $c$-normal in $G$ [30] if there exists a normal subgroup $T$ of $G$ such that $HT = G$ and $H \cap T \leq H_G$, where $H_G$ is the largest normal subgroup of $G$ contained in $H$. A subgroup $H$ of $G$ is said to be $\Phi$-$S$-supplemented in $G$ [18, 19] if there exists a subnormal subgroup $T$ of $G$ such that $HT = G$ and $H \cap T \leq \Phi(H)$, where $\Phi(H)$ is Frattini subgroup of $H$. A subgroup $H$ of $G$ is said to be weakly $\Phi$-supplemented in $G$ [20] if there exists a subgroup $T$ of $G$
such that $HT = G$ and $H \cap T \leq \Phi(H)$. A subgroup $H$ of $G$ is said to be nearly $s$-normal \cite{32} in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT \leq G$ and $H \cap T \leq H_{sG}$, where $H_{sG}$ is the subgroup of $H$ generated by all those subgroups of $H$ which are $s$-permutable in $G$. A subgroup $H$ of $G$ is said to be weakly $S\Phi$-supplemented \cite{33} in $G$ if there exists a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq \Phi(H)H_{sG}$. By using these concepts, a large number of results have been obtained (see, for example, \cite{3, 17–22, 30, 32, 33}). As a continuation of the above research, we now introduce the following new notion.

**Definition 1.** Let $H$ be a subgroup of $G$. We say that $H$ is nearly $S\Phi$-normal in $G$ when $G$ has a normal subgroup $T$ which satisfies $HT \leq G$ and $H \cap T \leq \Phi(H)H_{sG}$, where $H_{sG}$ generated by all those subgroups of $H$ which are all $s$-permutable in $G$. Obviously, all normal subgroups, all $s$-permutable subgroups, all $c$-normal subgroups and all nearly $s$-normal subgroups of $G$ are all nearly $S\Phi$-normal in $G$. But the next example implies that the converse does not hold.

**Example 1.** Let $G = S_5$ be the symmetric group of order 120 and $H = \langle (1234) \rangle \leq G$. It is easily see that $H_{sG} = H_G = 1$. As $H$ is not permutable with Sylow 3-subgroup of $G$, $H$ is not $s$-permutable in $G$. Since $HA_5 = G$ and $H \cap A_5 = \Phi(H) = \langle (13)(24) \rangle$, clearly $H$ is nearly $S\Phi$-normal in $G$, but neither nearly $s$-normal in $G$ nor $c$-normal in $G$.

Recall that a class of groups $\mathfrak{F}$ is said to be a formation if it is closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is saturated (respectively solubly saturated) if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$ (respectively $G/\Phi(N) \in \mathfrak{F}$ for a soluble normal subgroup $N$ of $G$) (see for example, \cite[p. 3]{10}). Also, the $\mathfrak{F}$-residual of $G$, denoted by $G^\mathfrak{F}$, is the smallest normal subgroup of $G$ with the quotient in $\mathfrak{F}$.

Following \cite{8, 29} or \cite{12}, for a formation $\mathfrak{F}$, a chief factor $H/K$ of $G$ is said to be $\mathfrak{F}$-central in $G$ if $H/K \cong G/C_G(H/K) \in \mathfrak{F}$. A normal subgroup $N$ of $G$ is $\mathfrak{F}$-central, if either $N = 1$ or $N \neq 1$ and every chief factor of $G$ below $N$ is $\mathfrak{F}$-central in $G$. The $\mathfrak{F}$-hypercentral $Z_\mathfrak{F}(G)$ of $G$ is the product of all normal $\mathfrak{F}$-central subgroups of $G$.

A subgroup $H$ of $G$ has a supersoluble supplement in $G$ if there exists a supersoluble subgroup $L$ of $G$ such that $G = HL$.

We use $U$ and $\mathfrak{N}_p$ to denote the classes of all supersoluble groups and $p$-nilpotent groups respectively. In this paper, we obtain the following results.
Theorem 1. Assume that $\mathcal{F}$ is a solubly saturated formation that contains the class of all supersoluble groups and $E$ a normal soluble subgroup of $G$ satisfies $G/E \in \mathcal{F}$. If every maximal subgroup of each noncyclic Sylow subgroup of $F(E)$ either has a supersoluble supplement in $G$ or is nearly $S\Phi$-normal, then $G \in \mathcal{F}$.

Theorem 2. Assume that $\mathcal{F}$ is a solubly saturated formation which containing the class of all supersoluble groups and $E \trianglelefteq G$ satisfies $G/E \in \mathcal{F}$. Suppose that $X = E$ or $X = F^*(E)$. If for every noncyclic Sylow subgroup $Q$ of $X$ and each cyclic subgroup of $Q$ with order $q$ or order $4$ (when the Sylow 2-subgroup of $X$ is non-abelian) either has a supersoluble supplement or is nearly $S\Phi$-normal in $G$, then $G \in \mathcal{F}$.

We shall prove Theorems 1 and 2 in section 3. In section 4, we give some applications of our results.

The notation and terminology in this paper are standard and the reader is referred to see [5,8,10,12,29].

1. Preliminaries

The following known results will be needed in this paper.

Lemma 1 ([33, Lemma 2.1]). Let $N$ be a normal subgroup of $G$ and $H$ a subgroup of $G$ with $(|H|,|N|) = 1$. Then $\Phi(H)N/N = \Phi(HN/N)$.

Lemma 2. Let $G$ has a subgroup $H$ and $N \trianglelefteq G$.

1. Assume that $H$ is $s$-permutable in $G$. Then $HN/N$ is $s$-permutable in $G/N$ (see [10, Chapter 1, Lemma 5.34]).

2. Assume that $H$ is a $p$-group. So $O^p(G) \leq N_G(H)$ if and only if $H$ is $s$-permutable in $G$ (see [25, Lemma A]).

Lemma 3 ([26, Lemma 2.8]). Let $H$, $K$ be subgroups of $G$ and $H \leq K$, then

1. $H_{sG}$ is a $s$-permutable subgroup of $G$ and $H_G \leq H_{sG}$;
2. $H_{sG} \leq H_{sK}$;
3. If $H \trianglelefteq G$, then $(K/H)_{s(G/H)} = K_{sG}/H$.

Lemma 4. Let $H \trianglelefteq K \leq G$ and $R \trianglelefteq G$, then

1. $H$ is nearly $S\Phi$-normal in $G$ if and only if $N \trianglelefteq G$ satisfies $HN \trianglelefteq G$, $H_G \leq N$, and $H \cap N \leq \Phi(H)H_{sG}$.
2. If $H$ is nearly $S\Phi$-normal in $G$, then $H$ is nearly $S\Phi$-normal in $K$. 

(3) Assume that \( R \leq H \). If \( H \) is nearly \( S\Phi \)-normal in \( G \), then \( H/R \) is nearly \( S\Phi \)-normal in \( G/R \). Moreover, the converse holds when \( R \leq \Phi(H) \).

(4) If \( H \) is nearly \( S\Phi \)-normal in \( G \) with \((|H|, |R|) = 1\), then \( HR/R \) is nearly \( S\Phi \)-normal in \( G/R \).

Proof. Assume that \( H \) is nearly \( S\Phi \)-normal in \( G \). So we there exists \( N \leq G \) satisfies \( HN \leq G \) and \( H \cap N \leq \Phi(H)H_{sG} \).

(1) \( \Rightarrow \) Let \( N_0 = NH_G \). Then clearly, \( N_0 \leq G \) and \( HN_0 = HNH_G = HN \leq G \). It can easily see that \( H \cap N_0 = H \cap NH_G = (H \cap N)H_G \leq \Phi(H)H_{sG} \).

\( \Leftarrow \) Obviously.

(2) Let \( N_0 = N \cap K \). Then \( N_0 \leq K \) and \( HN_0 = H(N \cap K) = HN \leq K \). By Lemma 3(2), we have that \( H \cap N_0 = H \cap (N \cap K) \leq H \cap N \leq \Phi(H)H_{sG} \leq \Phi(H)H_{sK} \), which means that \( H \) is nearly \( S\Phi \)-normal in \( K \).

(3) Now assume that \( H \) is nearly \( S\Phi \)-normal in \( G \). Consider \( \overline{G} = G/R \). Then \( \overline{N} = NR/R \leq G/R = \overline{G} \) and \( \overline{H} \overline{N} = (H/R)(NR/R) = HNR/R \leq G/R = \overline{G} \). Moreover, by [8, A, 9.2(e)] and Lemma 3(3), we have that \( \overline{H} \overline{N} = (H/R) \cap (NR/R) = (H \cap N)R/R \leq \Phi(H)H_{sG}R/R \leq \Phi(H/R)(H/R)_{s(G/R)} = \Phi(\overline{H})(\overline{H})_{sG} \). So \( H/R \) is nearly \( S\Phi \)-normal in \( G/R \).

Now suppose that \( R \leq \Phi(H) \) and \( N/R \leq G/R \) satisfies \( (H/R)(N/R) \leq G/R \) and

\[
(H/R) \cap (N/R) \leq \Phi(H/R)(H/R)_{s(G/R)}.
\]

Then \( HN \leq G \). Since \( R \leq \Phi(H) \), it implies that \( \Phi(H/R) = \Phi(H)/R \) by [8, A, 9.2(e)], and so \( H \cap N \leq \Phi(H)H_{sG} \) by Lemma 3(3). This shows that \( H \) is nearly \( S\Phi \)-normal in \( G \).

(4) Let \( \overline{G} = G/R \), \( \overline{H} = HR/R \) and \( \overline{N} = NR/R \). Then \( \overline{N} \leq \overline{G} \) and \( \overline{H} \overline{N} \leq \overline{G} \).

Since \((|H|, |R|) = 1\), we have

\[
(|HR \cap N : HR \cap N \cap R|, |HR \cap N : HR \cap N \cap H|) = 1.
\]

[12, Lemma 3.8.1] implies that

\[
(HR \cap N) = (HR \cap N \cap R)(HR \cap N \cap H) = (N \cap H)(N \cap R).
\]
It follows that from Lemmas 1 and 2(1) that

\[ H \cap N = (HR/R) \cap (NR/R) \]
\[ = (HR \cap NR)/R \]
\[ = (HR \cap N)R/R \]
\[ = (H \cap N)R/R \]
\[ \leq \Phi(H)H_{sG}R/R \]
\[ \leq \Phi(HR/R)(HR/R)_{s(G/R)}. \]

Hence \( HR/R \) is nearly \( S\Phi \)-normal in \( G/R \). \(\square\)

Let \( P \) be a \( p \)-group. If \( P \) is not a nonabelian 2-group, then we use \( \Omega(P) \) to denote the subgroup \( \Omega_1(P) \). Otherwise, \( \Omega(P) = \Omega_2(P) \).

**Lemma 5** ([13, Lemma 4.3]). Let \( C \) be a Thompson critical subgroup (see [9, p.185]) of a nontrivial \( p \)-group of \( P \).

1. If \( p \) is odd, then the exponent of \( \Omega(C) \) is \( p \).
2. If \( p = 2 \), then the exponent of \( \Omega(C) \) is at most 4. Moreover, If \( P \) is an abelian 2-group, then the exponent of \( \Omega(C) \) is 2.

**Lemma 6** ([28, Theorem B]). Assume that \( \mathfrak{F} \) is a formation and \( N \trianglelefteq G \). If \( F^*(N) \leq Z_{\mathfrak{F}}(G) \), then \( N \leq Z_{\mathfrak{F}}(G) \).

**Lemma 7** ([11, Lemma 3.3]). Let \( \mathfrak{F} \) be a solubly saturated formation which containing \( \mathfrak{U} \) and also \( N \trianglelefteq G \) satisfies \( G/N \in \mathfrak{F} \). If \( N \leq Z_{\mathfrak{U}}(G) \), then \( G \in \mathfrak{F} \). In particular, if \( N \) is cyclic, then \( G \in \mathfrak{F} \).

**Lemma 8** ([6, Lemma 2.12] and [6, Lemma 2.8]). Let \( \mathfrak{F} \) be a solubly saturated formation and \( P \) a normal \( p \)-subgroup of \( G \) and \( C \) is a Thompson critical subgroup of \( P \). If either \( \Omega(C) \leq Z_{\mathfrak{F}}(G) \) or \( P/\Phi(P) \leq Z_{\mathfrak{F}}(G/\Phi(P)) \), then \( P \leq Z_{\mathfrak{F}}(G) \).

**Lemma 9** ([34, Lemma 2.8]). Let \( N \) be a maximal subgroup of \( G \) and \( P \) a normal \( p \)-subgroup of \( G \) such that \( G = NP \), here \( p \) is a prime. Then \( P \cap N \trianglelefteq G \).

**Lemma 10** ([16] or [4, Theorem 12]). Let \( G \) be a minimal non-supersoluble group, then \( G \) is soluble.
2. Proofs of Theorems 1 and 2

We start the proof of Theorem 1 with the following lemma.

**Proposition 1.** Let $G$ has a normal $q$-subgroup $Q$. If every maximal subgroup of $Q$ does not have a supersoluble supplement in $G$ is nearly $S\Phi$-normal, then $Q \leq Z_\mu(G)$.

**Proof.** Suppose that the proposition is false and we can consider a counterexample of $(G, Q)$ for which $|G| + |Q|$ minimal. Clearly, $G \neq Z_\mu(G)$, which means that $G$ is not supersoluble.

(1) There is a unique minimal normal subgroup $L$ of $G$ which contained in $Q$, $Q/L \leq Z_\mu(G/L)$ and $|L| > q$.

Assume that $L$ is a minimal normal subgroup of $G$ which contained in $Q$. Then $(G/L, Q/L)$ satisfies the hypothesis of the proposition by Lemma 4(3). This shows that $Q/L \leq Z_\mu(G/L)$. If $|L| = q$, then we have that $Q \leq Z_\mu(G)$, a contradiction. Hence $|L| > q$. Now consider that if $R$ is another minimal normal subgroup of $G$ in $Q$ satisfies $L \neq R$. With a similar argument, we can get $Q/R \leq Z_\mu(G/R)$. It implies that $LR/R \leq Z_\mu(G/R)$, and thus $|L| = q$, a contradiction. Hence (1) holds.

(2) $\Phi(Q) \neq 1$.

Assume that $\Phi(Q) = 1$, then we get $Q$ is an elementary abelian group. So $L$ has a complement $S$ in $Q$. Let $L_1$ be a maximal subgroup of $L$ satisfies $L_1$ is normal in $G_q$, where $G_q$ is some Sylow $q$-subgroup of $G$. Hence $Q_1 = L_1S$ is a maximal subgroup of $Q$ and $Q_1 \cap L = L_1$. It follows from the hypothesis that either $Q_1$ has a supersoluble supplement in $G$ or is nearly $S\Phi$-normal in $G$. If $Q_1$ is nearly $S\Phi$-normal in $G$, then we get $G$ has a normal subgroup $N$ satisfies $Q_1N \trianglelefteq G$ and $Q_1 \cap N \leq \Phi(Q_1)(Q_1)_sG = (Q_1)_sG$. Thus $Q \cap Q_1N = Q_1(Q \cap N) \trianglelefteq G$. If $Q \cap N = 1$, it follows that $Q_1 \trianglelefteq G$ and we obtain $Q_1 \cap L = L_1 \trianglelefteq G$. Therefore $|L| = q$, which contradicts with claim (1). Hence, $1 \neq Q \cap N \trianglelefteq G$. By claim (1), $L \leq Q \cap N$, and so $Q_1 \cap L \leq Q_1 \cap N \leq (Q_1)_sG$. It follows that $L_1 = Q_1 \cap L = (Q_1)_sG \cap L$ is $s$-permutable in $G$ (see Lemma 3(1) and [17, Proposition 2]). Hence by Lemma 2(2), $O^q(G) \leq N_G(L_1)$ since $L_1$ is a $q$-group. But as $L_1 \leq GL_q$, we obtain $L_1 \leq GL$. This implies that $|L| = q$. So this contradiction means that $Q_1$ has a supersoluble supplement $S$ in $G$. Thus $G = Q_1S = QS$. Since $(G/L)/(Q/L) \cong G/Q \cong S/S \cap Q$ is supersoluble and by (1) $Q/L \leq Z_\mu(G/L)$, we can obtain that $G/L$ is supersoluble. But $G$ is not supersoluble, and so $L \nleq \Phi(G)$, which means that $G$ has a maximal subgroup $M$ satisfies $L \nleq M$ and $G = LM$. Clearly, $Q \cap M \leq QM = G$. If $Q \cap M \neq 1$, then by (1), $L \leq Q \cap M$. 


which is impossible. Hence \( Q \cap M = 1 \). Then \( Q = Q \cap LM = L \) and so \( G = L_1S = LS \). Clearly, \( L \cap S \leq G \). If \( L \cap S \neq 1 \), then \( L \leq S \) and so \( G = S \) is supersoluble. This contradiction shows that \( L \cap S = 1 \). Hence \( L = L \cap L_1S = L_1 \), a contradiction. Hence \( \Phi(Q) \neq 1 \).

(3) Final contradiction.

(1) and (2) show that \( L \leq \Phi(Q) \) and \( Q/L \leq Z_G(G/L) \). It follows that \( Q/\Phi(Q) \leq Z_G(G/\Phi(Q)) \). Then \( Q \leq Z_G(G) \) by Lemma 8. The final contradiction finishes the proof.

**Proof of Theorem 1.** Since \( E \) is soluble, we get \( F(E) \neq 1 \). Let \( Q \) be a Sylow \( q \)-subgroup of \( F(E) \), here \( q \) belongs in \( \pi(F(E)) \). It is easy to see that, \( Q \) is a normal \( q \)-subgroup of \( G \). Suppose that \( Q \) is noncyclic, then for every maximal subgroup of \( Q \) which either has a supersoluble supplement in \( G \) or is nearly \( S \Phi \)-normal in \( G \). Then \( Q \leq Z_G(G) \) by Proposition 1. So we need to consider \( Q \) is cyclic. Suppose that \( K/L \) be any \( G \)-chief factor below \( Q \). Thus \( |K/L| = q \), and therefore \( K/L \) is \( \Phi \)-central in \( G \). So, \( Q \leq Z_G(G) \). This implies that \( F(E) \leq Z_G(G) \). In particular, since \( E \) is soluble. Therefore by [10, Chap. 1, Proposition 5.4], we can get \( F^*(E) = F(E) \). Then we have that \( G \in \mathfrak{S} \) by Lemma 6 and Lemma 7.

The following two propositions are main steps to prove Theorem 2.

**Proposition 2.** Suppose that \( N \trianglelefteq G \) satisfies \( G/N \) is \( q \)-nilpotent, here \( q \) is the smallest prime belongs in \( \pi(G) \). If each cyclic subgroup of \( N \) of order \( q \) or 4 (when the Sylow 2-subgroup of \( N \) is not abelian) is nearly \( S \Phi \)-normal in \( G \), then \( G \) is \( q \)-nilpotent.

*Proof.* Suppose that the proposition is false and we can consider a counterexample of \( G \) for which \( |G| \) minimal. Then

(1) \( G \) is a minimal non-nilpotent group.

Assume that \( M \) is a proper subgroup of \( G \). Since \( M/M \cap N \cong MN/N \leq G/N \), we get \( M/M \cap N \) is \( q \)-nilpotent. Since every cyclic subgroup of \( M \cap N \) of order \( q \) or 4 (when the Sylow 2-subgroup of \( N \) is not abelian) is nearly \( S \Phi \)-normal in \( M \) by the hypothesis and Lemma 4(2). This means that \( M \) satisfies the hypothesis of the proposition. Thus \( M \) is \( q \)-nilpotent by the choice of \( G \). It means that \( G \) is a minimal non-\( q \)-nilpotent group.

By [8, Chap. VII, Theorem 6.18] and [15, Chap. IV, Proposition 5.4], \( G \) is a minimal non-nilpotent group, \( G = Q \times P \), where \( P \) is a Sylow \( p \)-subgroup of \( G \), \( Q = G^{31} \), \( Q/\Phi(Q) \) is a \( G \)-chief factor, \( \exp(Q) \) is \( q \) or 4 (when \( Q \) is a nonabelian 2-group) and \( \Phi(G) = Z_{31}(G) \).
(2) Assume that \( y \) is an element of order \( q \) of \( Q \). Then we have that \( \langle y \rangle \leq \Phi(Q) \).

First we prove that \( Q \leq N \). Since \( G/N \) is \( q \)-nilpotent, \( G = Q \wr P \) and \( G/N = (QN/N) \rtimes (PN/N) \). It shows that \( G/N \) is nilpotent, and so \( Q = G^{\text{nil}} \leq N \). Let \( A = \langle y \rangle \) and \( |\langle y \rangle| = q \). Then \( G \) has a normal subgroup \( T \) such that \( AT \leq G \) and \( A \cap T \leq \Phi(A)A_{sG} = A_{sG} \). It is clear that either \( A_{sG} = A \) or \( A_{sG} = 1 \). If \( A_{sG} = A \), then \( A \) is \( s \)-permutable in \( G \) by Lemma 3(1). It follows form Lemma 2 that \( O^q(G/\Phi(Q)) \leq N_{G/\Phi(Q)}(\Phi(Q)/\Phi(Q)) \). In addition, because \( A\Phi(Q)/\Phi(Q) \leq Q/\Phi(Q) \), we can get \( A\Phi(Q)/\Phi(Q) \leq G/\Phi(Q) \). But \( Q/\Phi(Q) \) is a \( G \)-chief factor, we have that either \( A\Phi(Q) = Q \) or \( A\Phi(Q) = \Phi(Q) \). For the former case, we can get \( A = Q \). Then by Lemma [23, (10.1.9)], \( G \) is \( q \)-nilpotent, a contradiction. So \( A\Phi(Q) = \Phi(Q) \), which means that \( A \leq \Phi(Q) \). If \( A_{sG} = 1 \), then \( AT \leq G \) and \( A \cap T = 1 \). Since \( Q/\Phi(Q) \) is a chief factor of \( G \) and \( (Q \cap AT)\Phi(Q)/\Phi(Q) \leq G/\Phi(Q) \), we have that either \( (Q \cap AT)\Phi(Q) = \Phi(Q) \) or \( (Q \cap AT)\Phi(Q) = Q \). In the former case, it is clear that \( A \leq \Phi(Q) \). Now we consider \( (Q \cap AT)\Phi(Q) = Q \). It implies that \( A(Q \cap T) = Q \). If \( Q \cap T \leq \Phi(Q) \), then \( Q = A \), a contradiction as above. Hence \( Q \cap T \not\leq \Phi(Q) \). But as \( (Q \cap T)\Phi(Q)/\Phi(Q) \leq G/\Phi(Q) \) and \( Q/\Phi(Q) \) is a chief factor of \( G \), we obtain that \( (Q \cap T)\Phi(Q) = Q \), and so \( Q \leq T \). It follows that \( A = A \cap T \leq A_{sG} = 1 \), a contradiction. This contradiction shows that, in any case, we have \( A \leq \Phi(Q) \). Hence (2) holds.

(3) \( Q \) is a nonabelian 2-group.

If either \( Q \) is abelian or \( Q \) is nonabelian and \( q > 2 \), then \( \exp(Q) = q \). (2) means that \( Q \leq \Phi(Q) \), a contradiction. Then \( Q \) is a nonabelian 2-group.

(4) Final contradiction.

Let \( y \) be an arbitrary element of \( Q \) with order 4 and \( H = \langle y \rangle \). Our claim is \( H \leq \Phi(Q) \). If \( H = H_{sG} \), then \( H \leq \Phi(Q) \) due to the same argument as in claim (2). If \( H \not= H_{sG} \), then \( H_{sG} \leq \Phi(H) \). There exists \( T \leq G \) satisfies \( HT \leq G \) and \( H \cap T \leq \Phi(H) \) by the hypothesis. Since \( Q/\Phi(Q) \) is a chief factor of \( G \) and \( (Q \cap HT)\Phi(Q)/\Phi(Q) \leq G/\Phi(Q) \). We have that either \( (Q \cap HT)\Phi(Q) = \Phi(Q) \) or \( (Q \cap HT)\Phi(Q) = Q \). In the former case it is clear that \( H \leq \Phi(Q) \). Now we consider \( (Q \cap HT)\Phi(Q) = Q \). It follows that \( H(Q \cap T) = Q \). If \( Q \cap T \leq \Phi(Q) \), then \( Q = H \). Then by [23, (10.1.9)], \( G \) is \( q \)-nilpotent, a contradiction. Therefore \( Q \cap T \not\leq \Phi(Q) \). But as \( (Q \cap T)\Phi(Q)/\Phi(Q) \leq G/\Phi(Q) \) and \( Q/\Phi(Q) \) is a chief factor of \( G \), we obtain that \( (Q \cap T)\Phi(Q) = Q \), and so \( Q \leq T \). It follows that
\[ H = H \cap T \leq \Phi(H), \text{ which is impossible. From above, we get } H \leq \Phi(Q). \]

But by (3), \( \text{exp}(Q) = 4 \), (2) means that \( Q \leq \Phi(Q) \), a contradiction. This contradiction finishes the proof. \( \square \)

**Proposition 3.** Suppose that \( Q \) is a normal \( q \)-subgroup of \( G \). If each cyclic subgroup of \( Q \) of order \( q \) or order 4 (when \( Q \) is a nonabelian 2-group) either has a supersoluble supplement in \( G \) or is nearly \( S\Phi \)-normal, then \( Q \leq Z_u(G) \).

**Proof.** Let the proposition is false and we can consider a counterexample of \((G, Q)\) for which \( |G| + |Q| \) minimal.

(1) There exists a unique normal subgroup \( N \) of \( G \) satisfies \( Q/N \) is a \( G \)-chief factor, \( N \leq Z_u(G) \), and \( |Q/N| > q \).

Let \( Q/N \) be a \( G \)-chief factor. Then the hypothesis holds for \((G, N)\). So \( N \leq Z_u(G) \). Assume that \( |Q/N| = q \), then \( Q/N \leq Z_u(G/N) \). This contradiction shows that \( |Q/N| > q \). Suppose that \( Q/K \) is a chief factor of \( G \) with \( Q/N \not= Q/K \), then with a similar argument as above, we can get \( K \leq Z_u(G) \). Therefore by Lemma [10, Chap. 1, Theorem 2.6(d)]

\[ Q/N = NK/N \leq NZ_u(G)/N \leq Z_u(G/N). \]

Since \( N \leq Z_u(G) \), we have \( Q \leq Z_u(G) \), a contradiction of supposition.

(2) \( \text{exp}(Q) \) is \( q \) or 4 (when \( Q \) is a nonabelian 2-group).

Assume that \( Q \) has a Thompson critical subgroup \( C \). If \( \Omega(C) < Q \), then \( \Omega(C) \leq N \leq Z_u(G) \) by claim (1). Thus by Lemma 8, \( Q \leq Z_u(G) \), a contradiction. Hence \( Q = \Omega(C) \), and therefore \( \text{exp}(Q) \) is \( q \) or 4 (when \( Q \) is a nonabelian 2-group) by Lemma 5.

(3) Final contradiction.

Since \( (Q/N) \cap Z(G_q/N) > 1 \), here \( G_q \) is some Sylow \( q \)-subgroup of \( G \). Suppose that \( K/N \leq (Q/N) \cap Z(G_q/N) \) and \( |K/N| = q \). Let \( y \in K \setminus N \) and \( H = \langle y \rangle \). Then we get \( K = HN \) and \( |H| = q \) or 4 (when \( Q \) is a nonabelian 2-group) by (2). If \( H = H_{sG} \), then by Lemma 2(1) and Lemma 3(1), we have \( HN/N \) is \( s \)-permutable in \( G/N \). Hence by Lemma 2(2) that \( O^q(G/N) \leq N_{G/N}(HN/N) \). It is easy to see that \( HN/N \leq G_q/N \). So it shows that \( HN/N \leq G/N \). But \( Q/N \) is a \( G \)-chief factor and \( H \not\leq N \), we get \( Q = HN \). So \( |Q/N| = |K/N| = q \), which contradicts with \( |Q/N| > q \). Thus \( H \not= H_{sG} \) and therefore \( H_{sG} \leq \Phi(H) \). By the hypothesis, \( H \) is either nearly \( S\Phi \)-normal in \( G \) or has a supersoluble supplement in \( G \). First assume that \( H \) is nearly \( S\Phi \)-normal in \( G \). So there exists \( T \leq G \) satisfies \( HT \leq G \) and \( H \cap T \leq \Phi(H)H_{sG} = \Phi(H) \).
Since $Q/N$ is a chief factor of $G$ and $(Q \cap HT)N/N \trianglelefteq G/N$, we have that either $(Q \cap HT)N = N$ or $(Q \cap HT)N = Q$. For the former case, we see that $H \leq N$, a contradiction. Hence now we consider $(Q \cap HT)N = Q$. It follows that $H(Q \cap T)N = Q$. If $Q \cap T \leq N$, then $K = HN = Q$, and so $|Q/N| = |K/N| = q$, a contradiction of claim (1). Hence $Q \cap T \nsubseteq N$, and so $N \leq Q \cap T$ by claim (1). But as $(Q \cap T)N/N \trianglelefteq G/N$ and $Q/N$ is a chief factor of $G$, we obtain that $(Q \cap T)N = Q$, and so $Q \leq T$. It follows that $H = H(Q \cap T) \leq \Phi(H)$. This contradiction shows that $H$ has a supersoluble supplement $S$ in $G$. Then $G = HS = QS$. If $Q \leq S$, then $G = S$ is supersoluble, a contradiction. Therefore $G$ has a maximal subgroup $M$ satisfies $S \leq M$ and $G = QM$. Then $Q \cap M$ is normal in $G$ by Lemma 9. By (1), $Q \cap M \leq N$. So

$$Q = Q \cap HS = H(Q \cap S) \leq H(Q \cap M) \leq HN = K,$$

a contradiction. This finishes the proof. □

**Proof of Theorem 2.**

We first prove that the assertion holds for $X = E$. Let the theorem is false and we can consider a counterexample of $(G, E)$ for which $|G| + |E|$ minimal. Then

(1) $E$ is soluble.

If $E$ is not soluble, then Feit-Thompson Theorem implies that $2 \in \pi(E)$. Assume that $E$ has a cyclic Sylow 2-subgroup. By [23, (10.1.9)], we have that $E$ is 2-nilpotent. So $E$ is soluble, a contradiction. Therefore every Sylow 2-subgroup of $E$ is noncyclic. Then every cyclic subgroup of $E$ with order 2 or 4 either has a supersoluble supplement or is nearly $S\Phi$-normal in $G$. If every cyclic subgroup of $E$ of order 2 or 4 is nearly $S\Phi$-normal in $G$, then by Proposition 2, $E$ is 2-nilpotent, so $E$ is soluble, a contradiction. Hence, $E$ has a cyclic subgroup $H$ with order 2 or 4 satisfies $H$ has a supersoluble supplement $S$ in $G$. Thus $G = HS = ES$ and so $G/E = ES/E \cong S/S \cap E \in \mathfrak{U}$. Now we need to show that $G$ is a minimal non-supersoluble group. Assume that $R$ is a proper subgroup of $G$. As $R/R \cap E \cong RE/E \leq G/E$, we obtain that $R/R \cap E \in \mathfrak{U}$. Assume that $\langle y \rangle$ is a cyclic subgroup of a non-cyclic Sylow subgroup of $R \cap E$ with order $q$ or order 4 (when the Sylow 2-subgroup of $R \cap E$ is non-abelian). Then either $\langle y \rangle$ has a supersoluble supplement in $G$ or is nearly $S\Phi$-normal in $G$. If $\langle y \rangle$ is nearly $S\Phi$-normal in $G$, then $\langle y \rangle$ is nearly $S\Phi$-normal in $R$ by Lemma 4(2). If $N$ is a supersoluble supplement of $\langle y \rangle$ in $G$, then $R = \langle y \rangle(N \cap R)$ and $N \cap R \in \mathfrak{U}$. It imply that $(R, R \cap E)$ satisfies the hypothesis. Then
R ∈ 𝔪 by the choice of (G, E). Therefore G is a minimal non-supersoluble group. Thus G is soluble by ([16] or [4, Theorem 12]), and so E is soluble. Hence by this contradiction (1) exists.

(2) G^δ = V is a q-group, V/Φ(V) is a G-chief factor, and exp(V) is q or 4 (when q = 2 and V is nonabelian).

Since G/E ∈ -gun, we have V ⊆ E. Thus by (1), V is soluble. If V ⊆ Φ(G), then V ⊆ S_G for every maximal subgroup S of G, and so G/S_G ∈ -gun. Therefore claim (2) holds by using Semenchuk Theorem (see [27] or [12, Theorem 3.4.2]). Now suppose that V ∉ Φ(G). Let G has a maximal subgroup S such that V ∉ S. Then G = VS = ES and S/S ∩ E ∼ SE/E = G/E ∈ -gun. For every non-cyclic Sylow subgroup P of S ∩ E, we can let ⟨y⟩ be a cyclic subgroup of P of prime order or order 4 (when the Sylow 2-subgroup of S ∩ E is non-abelian). As the same discussion as (1) of the proof, (S, S ∩ E) also satisfies the hypothesis. It means that S ∈ -gun. Then we have (2) by the Semenchuk Theorem ([27] or [12, Theorem 3.4.2]).

(3) Final contradiction.

If V is noncyclic, then by Proposition 3 and (2), we have that V ≤ Z_U(G). Assume that V is cyclic. So, obviously, V ≤ Z_U(G). It follows that V ∈ -gun. This contradiction prove that the theorem is holds for X = E.

Now we prove that the theorem is also true for X = F^*(E).

By Lemma 4(2), (F^*(E), F^*(E)) also satisfies the hypothesis. As above, we know that F^*(E) is supersoluble, and thus F(E) = F^*(E). Assume that H is a Sylow q-subgroup of F(E). Then H is normal in G. Assume that H is non-cyclic, then we have that the hypothesis of Proposition 3 holds for H. Hence H ≤ Z_U(G). Now let H is cyclic, then obviously, H ≤ Z_U(G). This induces that F^*(E) = F(E) ≤ Z_U(G).

Hence from Lemma 6, E ≤ Z_U(G). It implies by Lemma 7 that G ∈ -gun. This finishes the proof.

3. Some applications of our results

From Theorem 1, we obtain the following corollaries.

**Corollary 1** ([1, Theorem 3.2]). Assume that N ⊆ G satisfies G/N ∈ 𝔪 and G is soluble. If each maximal subgroup of every Sylow subgroup of F(N), which are normal in G, then G ∈ 𝔪.

**Corollary 2** ([21, Theorem 2]). Assume that N is a normal soluble
subgroup of $G$ satisfies $G/N \in \mathcal{U}$. Assume that every maximal subgroup of the Sylow subgroups of $F(N)$ are $c$-normal in $G$, then $G \in \mathcal{U}$.

**Corollary 3** ([31, Theorem 1]). Assume that $\mathcal{F}$ is a saturated formation which contains the class of all supersoluble groups and $N$ a normal soluble subgroup of $G$ satisfies $G/N \in \mathcal{F}$. If every maximal subgroup of the Sylow subgroups of $F(N)$ are $c$-normal in $G$, then $G \in \mathcal{F}$.

**Corollary 4** ([1, Theorem 4.2]). Let $G$ be a group and $E$ a soluble normal subgroup of $G$ with supersoluble quotient $G/E$. Suppose that every maximal subgroups of every Sylow subgroup of $F(E)$ is $s$-permutable in $G$. Then $G$ is supersoluble.

Theorem 2 covers a lot of results, in particular:

**Corollary 5** ([32, Theorem 3.8]). Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and $G$ be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup $E$ of $G$ such that $G/E \in \mathcal{F}$ and every cyclic subgroup of every noncyclic Sylow subgroup of $E$ with prime order or order 4 (if the Sylow 2-subgroup is not abelian) not having a supersoluble supplement in $G$ is nearly $s$-normal in $G$.

**Corollary 6** ([30, Theorem 4.2]). If every subgroup of order 4 or all minimal subgroups of $G$ are $c$-normal in $G$, then $G \in \mathcal{U}$.

**Corollary 7** ([22, Theorem 3.4]). Assume that $N \unlhd G$ with supersoluble quotient $G/N$. If every subgroup of order 4 (when the Sylow 2-subgroup of $N$ is non-abelian) or all minimal subgroups of $N$ are $c$-normal in $G$, then $G \in \mathcal{U}$.

**Corollary 8** ([3, Theorem 3.4]). Assume that $\mathcal{F}$ is a saturated formation which contains the class of all supersoluble groups. If every cyclic subgroup with order 4 and all minimal subgroups of $G^\mathcal{F}$ are $c$-normal in $G$, then $G \in \mathcal{F}$.

**Corollary 9** ([2, Theorem 3.1]). If every subgroup of $G$ of prime order and each cyclic subgroup of $G$ with order 4 are $s$-permutable in $G$, then $G \in \mathcal{U}$.

**Corollary 10** ([24, Theorem 3.9]). Assume that $\mathcal{F}$ is a saturated formation which contains the class of all supersoluble groups. Then $N \unlhd G$ satisfies $G/N \in \mathcal{F}$ and if every subgroup of order 4 and every minimal subgroup of $N$ are $c$-normal in $G$ if and only if $G \in \mathcal{F}$. 
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