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On nearly $S\Phi$-normal subgroups of finite groups

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Abstract. Let G be a finite group, H a subgroup of G and H_{sG} the subgroup of H generated by all those subgroups of H which are s-permutable in G. Then we say that H is *nearly $S\Phi$-normal* in G if G has a normal subgroup T such that $HT \trianglelefteq G$ and $H \cap T \leq \Phi(H)H_{sG}$. In this paper, we study the structure of group G under the condition that some given subgroups of G are nearly $S\Phi$-normal in G. Some known results are generalised.

Introduction

Throughout this paper, all groups are finite. G always denotes a group, p denotes a prime, π denotes a set of primes, and $\Phi(G)$ denotes the Frattini subgroup of G.

Recall that a subgroup H of a group G is said to be *s-permutable* (or *s-quasinormal*) [17] in G if $HP = PH$ for all Sylow subgroups P of G. A subgroup H of G is said to be *c-normal* in G [30] if there exists a normal subgroup T of G such that $HT = G$ and $H \cap T \leq H_G$, where H_G is the largest normal subgroup of G contained in H. A subgroup H of G is said to be *Φ-S-supplemented* in G [18, 19] if there exists a subnormal subgroup T of G such that $HT = G$ and $H \cap T \leq \Phi(H)$, where $\Phi(H)$ is Frattini subgroup of H. A subgroup H of G is said to be *weakly Φ-supplemented* in G [20] if there exists a subgroup T of G

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such that $HT = G$ and $H \cap T \leq \Phi(H)$. A subgroup H of G is said to be *nearly s-normal* [32] in G if there exists a normal subgroup T of G such that $HT \trianglelefteq G$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. A subgroup H of G is said to be *weakly $S\Phi$-supplemented* [33] in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H)H_{sG}$. By using these concepts, a large number of results have been obtained (see, for example, [3, 17–22, 30, 32, 33]). As a continuation of the above research, we now introduce the following new notion.

Definition 1. *Let H be a subgroup of G. We say that H is nearly $S\Phi$-normal in G when G has a normal subgroup T which satisfies $HT \trianglelefteq G$ and $H \cap T \leq \Phi(H)H_{sG}$, here H_{sG} generated by all those subgroups of H which are all s-permutable in G.*

Obviously, all normal subgroups, all s-permutable subgroups, all c-normal subgroups and all nearly s-normal subgroups of G are all nearly $S\Phi$-normal in G. But the next example implies that the converse does not hold.

Example 1. Let $G = S_5$ be the symmetric group of order 120 and $H = \langle(1234)\rangle \leq G$. It is easily see that $H_{sG} = H_G = 1$. As H is not permutable with Sylow 3-subgroup of G, H is not s-permutable in G. Since $HA_5 = G$ and $H \cap A_5 = \Phi(H) = \langle(13)(24)\rangle$, clearly H is nearly $S\Phi$-normal in G, but neither nearly s-normal in G nor c-normal in G.

Recall that a class of groups $\mathfrak{F}$ is said to be a *formation* if it is closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is *saturated* (respectively *solubly saturated*) if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$ (respectively $G/\Phi(N) \in \mathfrak{F}$ for a soluble normal subgroup N of G) (see for example, [10, p. 3]). Also, the $\mathfrak{F}$-*residual* of G, denoted by $G^{\mathfrak{F}}$, is the smallest normal subgroup of G with the quotient in $\mathfrak{F}$.

Following [8, 29] or [12], for a formation $\mathfrak{F}$, a chief factor H/K of G is said to be $\mathfrak{F}$-*central* in G if $H/K \rtimes G/C_G(H/K) \in \mathfrak{F}$. A normal subgroup N of G is $\mathfrak{F}$-central, if either $N = 1$ or $N \neq 1$ and every chief factor of G below N is $\mathfrak{F}$-central in G. The $\mathfrak{F}$-*hypercentral* $Z_{\mathfrak{F}}(G)$ of G is the product of all normal $\mathfrak{F}$-central subgroups of G.

A subgroup H of G has a *supersoluble supplement* in G if there exists a supersoluble subgroup L of G such that $G = HL$.

We use $\mathfrak{U}$ and $\mathfrak{N}_{\mathfrak{p}}$ to denote the classes of all supersoluble groups and p-nilpotent groups respectively. In this paper, we obtain the following results.

Theorem 1. *Assume that $\mathfrak{F}$ is a solubly saturated formation that contains the class of all supersoluble groups and E a normal soluble subgroup of G satisfies $G/E \in \mathfrak{F}$. If every maximal subgroup of each noncyclic Sylow subgroup of $F(E)$ either has a supersoluble supplement in G or is nearly $S\Phi$-normal, then $G \in \mathfrak{F}$.*

Theorem 2. *Assume that $\mathfrak{F}$ is a solubly saturated formation which containing the class of all supersoluble groups and $E \trianglelefteq G$ satisfies $G/E \in \mathfrak{F}$. Suppose that $X = E$ or $X = F^*(E)$. If for every noncyclic Sylow subgroup Q of X and each cyclic subgroup of Q with order q or order 4 (when the Sylow 2-subgroup of X is non-abelian) either has a supersoluble supplement or is nearly $S\Phi$-normal in G, then $G \in \mathfrak{F}$.*

We shall prove Theorems 1 and 2 in section 3. In section 4, we give some applications of our results.

The notation and terminology in this paper are standard and the reader is referred to see [5, 8, 10, 12, 29].

1. Preliminaries

The following known results will be needed in this paper.

Lemma 1 ([33, Lemma 2.1])**.** *Let N be a normal subgroup of G and H a subgroup of G with $(|H|, |N|) = 1$. Then $\Phi(H)N/N = \Phi(HN/N)$.*

Lemma 2. *Let G has a subgroup H and $N \trianglelefteq G$.*

(1) *Assume that H is s-permutable in G. Then HN/N is s-permutable in G/N (see* [10, Chapter 1, Lemma 5.34]*).*

(2) *Assume that H is a p-group. So $O^p(G) \leq N_G(H)$ if and only if H is s-permutable in G (see* [25, Lemma A]*).*

Lemma 3 ([26, Lemma 2.8])**.** *Let H, K be subgroups of G and $H \leq K$, then*

(1) *H_{sG} is a s-permutable subgroup of G and $H_G \leq H_{sG}$;*

(2) *$H_{sG} \leq H_{sK}$;*

(3) *If $H \trianglelefteq G$, then $(K/H)_{s(G/H)} = K_{sG}/H$.*

Lemma 4. *Let $H \leq K \leq G$ and $R \trianglelefteq G$, then*

(1) *H is nearly $S\Phi$-normal in G if and only if $N \trianglelefteq G$ satisfies $HN \trianglelefteq G$, $H_G \leq N$, and $H \cap N \leq \Phi(H)H_{sG}$.*

(2) *If H is nearly $S\Phi$-normal in G, then H is nearly $S\Phi$-normal in K.*

(3) *Assume that* $R \leq H$. *If* H *is nearly* $S\Phi$*-normal in* G, *then* H/R *is nearly* $S\Phi$*-normal in* G/R. *Moreover, the converse holds when* $R \leq \Phi(H)$.

(4) *If* H *is nearly* $S\Phi$*-normal in* G *with* $(|H|, |R|) = 1$, *then* HR/R *is nearly* $S\Phi$*-normal in* G/R.

Proof. Assume that H is nearly $S\Phi$-normal in G. So we there exists $N \trianglelefteq G$ satisfies $HN \trianglelefteq G$ and $H \cap N \leq \Phi(H)H_{sG}$.

(1) $\Longrightarrow$ Let $N_0 = NH_G$. Then clearly, $N_0 \trianglelefteq G$ and $HN_0 = HNH_G = HN \trianglelefteq G$. It can easily see that $H \cap N_0 = H \cap NH_G = (H \cap N)H_G \leq \Phi(H)H_{sG}$.

$\Longleftarrow$ Obviously.

(2) Let $N_0 = N \cap K$. Then $N_0 \trianglelefteq K$ and $HN_0 = H(N \cap K) = HN \cap K \trianglelefteq K$. By Lemma 3(2), we have that $H \cap N_0 = H \cap (N \cap K) \leq H \cap N \leq \Phi(H)H_{sG} \leq \Phi(H)H_{sK}$, which means that H is nearly $S\Phi$-normal in K.

(3) Now assume that H is nearly $S\Phi$-normal in G. Consider $\overline{G} = G/R$. Then $\overline{N} = NR/R \trianglelefteq G/R = \overline{G}$ and $\overline{H}\ \overline{N} = (H/R)(NR/R) = HNR/R \trianglelefteq G/R = \overline{G}$. Moreover, by [8, A, 9.2(e)] and Lemma 3(3), we have that $\overline{H} \cap \overline{N} = (H/R) \cap (NR/R) = (H \cap N)R/R \leq \Phi(H)H_{sG}R/R \leq \Phi(H/R)(H/R)_{s(G/R)} = \Phi(\overline{H})\overline{H}_{s\overline{G}}$. So H/R is nearly $S\Phi$-normal in G/R.

Now suppose that $R \leq \Phi(H)$ and $N/R \trianglelefteq G/R$ satisfies $(H/R)(N/R) \trianglelefteq G/R$ and

$$(H/R) \cap (N/R) \leq \Phi(H/R)(H/R)_{s(G/R)}.$$

Then $HN \trianglelefteq G$. Since $R \leq \Phi(H)$, it implies that $\Phi(H/R) = \Phi(H)/R$ by[8, A, 9.2(e)], and so $H \cap N \leq \Phi(H)H_{sG}$ by Lemma 3(3). This shows that H is nearly $S\Phi$-normal in G.

(4) Let $\overline{G} = G/R$, $\overline{H} = HR/R$ and $\overline{N} = NR/R$. Then $\overline{N} \trianglelefteq \overline{G}$ and $\overline{H}\ \overline{N} \trianglelefteq \overline{G}$.

Since $(|H|, |R|) = 1$, we have

$$(|HR \cap N : HR \cap N \cap R|, |HR \cap N : HR \cap N \cap H|) = 1.$$

[12, Lemma 3.8.1] implies that

$$(HR \cap N) = (HR \cap N \cap R)(HR \cap N \cap H) = (N \cap H)(N \cap R).$$

It follows that from Lemmas 1 and 2(1) that

$$\begin{aligned}\overline{H}\cap\overline{N} &= (HR/R)\cap(NR/R)\\ &= (HR\cap NR)/R\\ &= (HR\cap N)R/R\\ &= (H\cap N)R/R\\ &\leq \Phi(H)H_{sG}R/R\\ &\leq \Phi(HR/R)(HR/R)_{s(G/R)}.\end{aligned}$$

Hence HR/R is nearly $S\Phi$-normal in G/R. □

Let P be a p-group. If P is not a nonabelian 2-group, then we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

Lemma 5 ([13, Lemma 4.3]). *Let C be a Thompson critical subgroup* (see [9, p.185]) *of a nontrivial p-group of P.*

(1) *If p is odd, then the exponent of $\Omega(C)$ is p.*

(2) *If $p = 2$, then the exponent of $\Omega(C)$ is at most 4. Moreover, If P is an abelian 2-group, then the exponent of $\Omega(C)$ is 2.*

Lemma 6 ([28, Theorem B]). *Assume that $\mathfrak{F}$ is a formation and $N \trianglelefteq G$. If $F^*(N) \leq Z_{\mathfrak{F}}(G)$, then $N \leq Z_{\mathfrak{F}}(G)$.*

Lemma 7 ([11, Lemma 3.3]). *Let $\mathfrak{F}$ be a solubly saturated formation which containing $\mathfrak{U}$ and also $N \trianglelefteq G$ satisfies $G/N \in \mathfrak{F}$. If $N \leq Z_{\mathfrak{U}}(G)$, then $G \in \mathfrak{F}$. In particular, if N is cyclic, then $G \in \mathfrak{F}$.*

Lemma 8 ([6, Lemma 2.12] and [6, Lemma 2.8]). *Let $\mathfrak{F}$ be a solubly saturated formation and P a normal p-subgroup of G and C is a Thompson critical subgroup of P. If either $\Omega(C) \leq Z_{\mathfrak{F}}(G)$ or $P/\Phi(P) \leq Z_{\mathfrak{F}}(G/\Phi(P))$, then $P \leq Z_{\mathfrak{F}}(G)$.*

Lemma 9 ([34, Lemma 2.8]). *Let N be a maximal subgroup of G and P a normal p-subgroup of G such that $G = NP$, here p is a prime. Then $P \cap N \trianglelefteq G$.*

Lemma 10 ([16] or [4, Theorem 12]). *Let G be a minimal non-supersoluble group, then G is soluble.*

2. Proofs of Theorems 1 and 2

We start the proof of Theorem 1 with the following lemma.

Proposition 1. *Let G has a normal q-subgroup Q. If every maximal subgroup of Q does not have a supersoluble supplement in G is nearly $S\Phi$-normal, then $Q \leq Z_{\mathfrak{U}}(G)$.*

Proof. Suppose that the proposition is false and we can consider a counterexample of (G, Q) for which $|G| + |Q|$ minimal. Clearly, $G \neq Z_{\mathfrak{U}}(G)$, which means that G is not supersoluble.

(1) There is a unique minimal normal subgroup L of G which contained in Q, $Q/L \leq Z_{\mathfrak{U}}(G/L)$ and $|L| > q$.

Assume that L is a minimal normal subgroup of G which contained in Q. Then $(G/L, Q/L)$ satisfies the hypothesis of the proposition by Lemma 4(3). This shows that $Q/L \leq Z_{\mathfrak{U}}(G/L)$. If $|L| = q$, then we have that $Q \leq Z_{\mathfrak{U}}(G)$, a contradiction. Hence $|L| > q$. Now consider that if R is another minimal normal subgroup of G in Q satisfies $L \neq R$. With a similar argument, we can get $Q/R \leq Z_{\mathfrak{U}}(G/R)$. It implies that $LR/R \leq Z_{\mathfrak{U}}(G/R)$, and thus $|L| = q$, a contradiction. Hence (1) holds.

(2) $\Phi(Q) \neq 1$.

Assume that $\Phi(Q) = 1$, then we get Q is an elementary abelian group. So L has a complement S in Q. Let L_1 be a maximal subgroup of L satisfies L_1 is normal in G_q, where G_q is some Sylow q-subgroup of G. Hence $Q_1 = L_1S$ is a maximal subgroup of Q and $Q_1 \cap L = L_1$. It follows from the hypothesis that either Q_1 has a supersoluble supplement in G or is nearly $S\Phi$-normal in G. If Q_1 is nearly $S\Phi$-normal in G, then we get G has a normal subgroup N satisfies $Q_1N \trianglelefteq G$ and $Q_1 \cap N \leq \Phi(Q_1)(Q_1)_{sG} = (Q_1)_{sG}$. Thus $Q \cap Q_1N = Q_1(Q \cap N) \trianglelefteq G$. If $Q \cap N = 1$, it follows that $Q_1 \trianglelefteq G$ and we obtain $Q_1 \cap L = L_1 \trianglelefteq G$. Therefore $|L| = q$, which contradicts with claim (1). Hence, $1 \neq Q \cap N \trianglelefteq G$. By claim (1), $L \leq Q \cap N$, and so $Q_1 \cap L \leq Q_1 \cap N \leq (Q_1)_{sG}$. It follows that $L_1 = Q_1 \cap L = (Q_1)_{sG} \cap L$ is s-permutable in G (see Lemma 3(1) and [17, Proposition 2]). Hence by Lemma 2(2), $O^q(G) \leq N_G(L_1)$ since L_1 is a q-group. But as $L_1 \trianglelefteq G_q$, we obtain $L_1 \trianglelefteq G$. This implies that $|L| = q$. So this contradiction means that Q_1 has a supersoluble supplement S in G. Thus $G = Q_1S = QS$. Since $(G/L)/(Q/L) \cong G/Q \cong S/S \cap Q$ is supersoluble and by (1) $Q/L \leq Z_{\mathfrak{U}}(G/L)$, we can obtain that G/L is supersoluble. But G is not supersoluble, and so $L \nleq \Phi(G)$, which means that G has a maximal subgroup M satisfies $L \nleq M$ and $G = LM$. Clearly, $Q \cap M \trianglelefteq QM = G$. If $Q \cap M \neq 1$, then by (1), $L \leq Q \cap M$, which is impossible. Hence $Q \cap M = 1$. Then $Q = Q \cap LM = L$ and so $G = L_1 S = LS$. Clearly, $L \cap S \trianglelefteq G$. If $L \cap S \neq 1$, then $L \leq S$ and so $G = S$ is supersoluble. This contradiction shows that $L \cap S = 1$. Hence $L = L \cap L_1 S = L_1$, a contradiction. Hence $\Phi(Q) \neq 1$.

(3) Final contradiction.

(1) and (2) show that $L \leq \Phi(Q)$ and $Q/L \leq Z_{\mathfrak{U}}(G/L)$. It follows that $Q/\Phi(Q) \leq Z_{\mathfrak{U}}(G/\Phi(Q))$. Then $Q \leq Z_{\mathfrak{U}}(G)$ by Lemma 8. The final contradiction finishes the proof. □

Proof of Theorem 1. Since E is soluble, we get $F(E) \neq 1$. Let Q be a Sylow q-subgroup of $F(E)$, here q belongs in $\pi(F(E))$. It is easy to see that, Q is a normal q-subgroup of G. Suppose that Q is noncyclic, then for every maximal subgroup of Q which either has a supersoluble supplement in G or is nearly $S\Phi$-normal in G. Then $Q \leq Z_{\mathfrak{U}}(G)$ by Proposition 1. So we need to consider Q is cyclic. Suppose that K/L be any G-chief factor below Q. Thus $|K/L| = q$, and therefore K/L is $\mathfrak{U}$-central in G. So, $Q \leq Z_{\mathfrak{U}}(G)$. This implies that $F(E) \leq Z_{\mathfrak{U}}(G)$. In particular, since E is soluble. Therefore by [10, Chap. 1, Proposition 5.4], we can get $F^*(E) = F(E)$. Then we have that $G \in \mathfrak{F}$ by Lemma 6 and Lemma 7.

The following two propositions are main steps to prove Theorem 2.

Proposition 2. *Suppose that $N \trianglelefteq G$ satisfies G/N is q-nilpotent, here q is the smallest prime belongs in $\pi(G)$. If each cyclic subgroup of N of order q or 4 (when the Sylow 2-subgroup of N is non-abelian) is nearly $S\Phi$-normal in G, then G is q-nilpotent.*

Proof. Suppose that the proposition is false and we can consider a counterexample of G for which $|G|$ minimal. Then

(1) G is a minimal non-nilpotent group.

Assume that M is a proper subgroup of G. Since $M/M \cap N \cong MN/N \leq G/N$, we get $M/M \cap N$ is q-nilpotent. Since every cyclic subgroup of $M \cap N$ of order q or 4 (when the Sylow 2-subgroup of N is not abelian) is nearly $S\Phi$-normal in M by the hypothesis and Lemma 4(2). This means that M satisfies the hypothesis of the proposition. Thus M is q-nilpotent by the choice of G. It means that G is a minimal non-q-nilpotent group.

By [8, Chap. VII, Theorem 6.18] and [15, Chap. IV, Proposition 5.4], G is a minimal non-nilpotent group, $G = Q \rtimes P$, where P is a Sylow p-subgroup of G, $Q = G^{\mathfrak{N}}$, $Q/\Phi(Q)$ is a G-chief factor, $exp(Q)$ is q or 4 (when Q is a nonabelian 2-group) and $\Phi(G) = Z_{\mathfrak{N}}(G)$.

(2) Assume that y is an element of order q of Q. Then we have that $\langle y\rangle \leq \Phi(Q)$.

First we prove that $Q \leq N$. Since G/N is q-nilpotent, $G = Q \rtimes P$ and $G/N = (QN/N) \rtimes (PN/N)$. It shows that G/N is nilpotent, and so $Q = G^{\mathfrak{N}} \leq N$. Let $A = \langle y\rangle$ and $|\langle y\rangle| = q$. Then G has a normal subgroup T such that $AT \trianglelefteq G$ and $A \cap T \leq \Phi(A)A_{sG} = A_{sG}$. It is clear that either $A_{sG} = A$ or $A_{sG} = 1$. If $A_{sG} = A$, then A is s-permutable in G by Lemma 3(1). It follows form Lemma 2 that $O^q(G/\Phi(Q)) \leq N_{G/\Phi(Q)}(A\Phi(Q)/\Phi(Q))$. In addition, because $A\Phi(Q)/\Phi(Q) \trianglelefteq Q/\Phi(Q)$, we can get $A\Phi(Q)/\Phi(Q) \trianglelefteq G/\Phi(Q)$. But $Q/\Phi(Q)$ is a G-chief factor, we have that either $A\Phi(Q) = Q$ or $A\Phi(Q) = \Phi(Q)$. For the former case, we can get $A = Q$. Then by Lemma [23, (10.1.9)], G is q-nilpotent, a contradiction. So $A\Phi(Q) = \Phi(Q)$, which means that $A \leq \Phi(Q)$. If $A_{sG} = 1$, then $AT \trianglelefteq G$ and $A \cap T = 1$. Since $Q/\Phi(Q)$ is a chief factor of G and $(Q \cap AT)\Phi(Q)/\Phi(Q) \trianglelefteq G/\Phi(Q)$. We have that either $(Q \cap AT)\Phi(Q) = \Phi(Q)$ or $(Q \cap AT)\Phi(Q) = Q$. In the former case, it is clear that $A \leq \Phi(Q)$. Now we consider $(Q \cap AT)\Phi(Q) = Q$. It implies that $A(Q \cap T) = Q$. If $Q \cap T \leq \Phi(Q)$, then $Q = A$, a contradiction as above. Hence $Q \cap T \nleq \Phi(Q)$. But as $(Q \cap T)\Phi(Q)/\Phi(Q) \trianglelefteq G/\Phi(Q)$ and $Q/\Phi(Q)$ is a chief factor of G, we obtain that $(Q \cap T)\Phi(Q) = Q$, and so $Q \leq T$. It follows that $A = A \cap T \leq A_{sG} = 1$, a contradiction. This contradiction shows that, in any case, we have $A \leq \Phi(Q)$. Hence (2) holds.

(3) Q is a nonabelian 2-group.

If either Q is abelian or Q is nonabelian and $q > 2$, then $exp(Q)$ is q. (2) means that $Q \leq \Phi(Q)$, a contradiction. Then Q is a nonabelian 2-group.

(4) Final contradiction.

Let y be an arbitrary element of Q with order 4 and $H = \langle y\rangle$. Our claim is $H \leq \Phi(Q)$. If $H = H_{sG}$, then $H \leq \Phi(Q)$ due to the same argument as in claim (2). If $H \neq H_{sG}$, then $H_{sG} \leq \Phi(H)$. There exists $T \trianglelefteq G$ satisfies $HT \trianglelefteq G$ and $H \cap T \leq \Phi(H)$ by the hypothesis. Since $Q/\Phi(Q)$ is a chief factor of G and $(Q \cap HT)\Phi(Q)/\Phi(Q) \trianglelefteq G/\Phi(Q)$. We have that either $(Q \cap HT)\Phi(Q) = \Phi(Q)$ or $(Q \cap HT)\Phi(Q) = Q$. In the former case it is clear that $H \leq \Phi(Q)$. Now we consider $(Q \cap HT)\Phi(Q) = Q$. It follows that $H(Q \cap T) = Q$. If $Q \cap T \leq \Phi(Q)$, then $Q = H$. Then by [23, (10.1.9)], G is q-nilpotent, a contradiction. Therefore $Q \cap T \nleq \Phi(Q)$. But as $(Q \cap T)\Phi(Q)/\Phi(Q) \trianglelefteq G/\Phi(Q)$ and $Q/\Phi(Q)$ is a chief factor of G, we obtain that $(Q \cap T)\Phi(Q) = Q$, and so $Q \leq T$. It follows that

$H = H \cap T \leq \Phi(H)$, which is impossible. From above, we get $H \leq \Phi(Q)$. But by (3), $exp(Q)$ is 4, (2) means that $Q \leq \Phi(Q)$, a contradiction. This contradiction finishes the proof. $\square$

Proposition 3. *Suppose that Q is a normal q-subgroup of G. If each cyclic subgroup of Q of order q or order 4 (when Q is a nonabelian 2-group) either has a supersoluble supplement in G or is nearly $S\Phi$-normal, then $Q \leq Z_{\mathfrak{U}}(G)$.*

Proof. Let the proposition is false and we can consider a counterexample of (G, Q) for which $|G| + |Q|$ minimal.

(1) There exists a unique normal subgroup N of G satisfies Q/N is a G-chief factor, $N \leq Z_{\mathfrak{U}}(G)$, and $|Q/N| > q$.

Let Q/N be a G-chief factor. Then the hypothesis holds for (G, N). So $N \leq Z_{\mathfrak{U}}(G)$. Assume that $|Q/N| = q$, then $Q/N \leq Z_{\mathfrak{U}}(G/N)$, and so $Q \leq Z_{\mathfrak{U}}(G)$. This contradiction shows that $|Q/N| > q$. Suppose that Q/K is a chief factor of G with $Q/N \neq Q/K$, then with a similar argument as above, we can get $K \leq Z_{\mathfrak{U}}(G)$. Therefore by Lemma [10, Chap. 1, Theorem 2.6(d)]

$$Q/N = NK/N \leq NZ_{\mathfrak{U}}(G)/N \leq Z_{\mathfrak{U}}(G/N).$$

Since $N \leq Z_{\mathfrak{U}}(G)$, we have $Q \leq Z_{\mathfrak{U}}(G)$, a contradiction of supposition.

(2) $exp(Q)$ is q or 4 (when Q is a nonabelian 2-group).

Assume that Q has a Thompson critical subgroup C. If $\Omega(C) < Q$, then $\Omega(C) \leq N \leq Z_{\mathfrak{U}}(G)$ by claim (1). Thus by Lemma 8, $Q \leq Z_{\mathfrak{U}}(G)$, a contradiction. Hence $Q = \Omega(C)$, and therefore $exp(Q)$ is q or 4 (when Q is a nonabelian 2-group) by Lemma 5.

(3) Final contradiction.

Since $(Q/N) \cap Z(G_q/N) > 1$, here G_q is some Sylow q-subgroup of G. Suppose that $K/N \leq (Q/N) \cap Z(G_q/N)$ and $|K/N| = q$. Let $y \in K \backslash N$ and $H = \langle y \rangle$. Then we get $K = HN$ and $|H| = q$ or 4 (when Q is a nonabelian 2-group) by (2). If $H = H_{sG}$, then by Lemma 2(1) and Lemma 3(1), we have HN/N is s-permutable in G/N . Hence by Lemma 2(2) that $O^q(G/N) \leq N_{G/N}(HN/N)$. It is easy to see that $HN/N \trianglelefteq G_q/N$. So it shows that $HN/N \trianglelefteq G/N$. But Q/N is a G-chief factor and $H \nleq N$, we get $Q = HN$. So $|Q/N| = |K/N| = q$, which contradicts with $|Q/N| > q$. Thus $H \neq H_{sG}$ and therefore $H_{sG} \leq \Phi(H)$. By the hypothesis, H is either nearly $S\Phi$-normal in G or has a supersoluble supplement in G. First assume that H is nearly $S\Phi$-normal in G. So there exists $T \trianglelefteq G$ satisfies $HT \trianglelefteq G$ and $H \cap T \leq \Phi(H)H_{sG} = \Phi(H)$.

Since Q/N is a chief factor of G and $(Q\cap HT)N/N\trianglelefteq G/N$, we have that either $(Q\cap HT)N=N$ or $(Q\cap HT)N=Q$. For the former case, we see that $H\leq N$, a contradiction. Hence now we consider $(Q\cap HT)N=Q$. It follows that $H(Q\cap T)N=Q$. If $Q\cap T\leq N$, then $K=HN=Q$, and so $|Q/N|=|K/N|=q$, a contradiction of claim (1). Hence $Q\cap T\nleq N$, and so $N\leq Q\cap T$ by claim (1). But as $(Q\cap T)N/N\trianglelefteq G/N$ and Q/N is a chief factor of G, we obtain that $(Q\cap T)N=Q$, and so $Q\leq T$. It follows that $H=H\cap T\leq \Phi(H)$. This contradiction shows that H has a supersoluble supplement S in G. Then $G=HS=QS$. If $Q\leq S$, then $G=S$ is supersoluble, a contradiction. Therefore G has a maximal subgroup M satisfies $S\leq M$ and $G=QM$. Then $Q\cap M$ is normal in G by Lemma 9. By (1), $Q\cap M\leq N$. So

$$Q=Q\cap HS=H(Q\cap S)\leq H(Q\cap M)\leq HN=K,$$

a contradiction. This finishes the proof. $\square$

Proof of Theorem 2.

We first prove that the assertion holds for $X=E$. Let the theorem is false and we can consider a counterexample of (G,E) for which $|G|+|E|$ minimal. Then

(1) E is soluble.

If E is not soluble, then Feit-Thompson Theorem implies that $2\in\pi(E)$. Assume that E has a cyclic Sylow 2-subgroup. By [23, (10.1.9)], we have that E is 2-nilpotent. So E is soluble, a contradiction. Therefore every Sylow 2-subgroup of E is noncyclic. Then every cyclic subgroup of E with order 2 or 4 either has a supersoluble supplement or is nearly $S\Phi$-normal in G. If every cyclic subgroup of E of order 2 or 4 is nearly $S\Phi$-normal in G, then by Proposition 2, E is 2-nilpotent, so E is soluble, a contradiction. Hence, E has a cyclic subgroup H with order 2 or 4 satisfies H has a supersoluble supplement S in G. Thus $G=HS=ES$ and so $G/E=ES/E\cong S/S\cap E\in\mathfrak{U}$. Now we need to show that G is a minimal non-supersoluble group. Assume that R is a proper subgroup of G. As $R/R\cap E\cong RE/E\leq G/E$, we obtain that $R/R\cap E\in\mathfrak{U}$. Assume that $\langle y\rangle$ is a cyclic subgroup of a non-cyclic Sylow subgroup of $R\cap E$ with order q or order 4 (when the Sylow 2-subgroup of $R\cap E$ is non-abelian). Then either $\langle y\rangle$ has a supersoluble supplement in G or is nearly $S\Phi$-normal in G. If $\langle y\rangle$ is nearly $S\Phi$-normal in G, then $\langle y\rangle$ is nearly $S\Phi$-normal in R by Lemma 4(2). If N is a supersoluble supplement of $\langle y\rangle$ in G, then $R=\langle y\rangle(N\cap R)$ and $N\cap R\in\mathfrak{U}$. It imply that $(R,R\cap E)$ satisfies the hypothesis. Then

$R \in \mathfrak{U}$ by the choice of (G, E). Therefore G is a minimal non-supersoluble group. Thus G is soluble by ([16] or [4, Theorem 12]), and so E is soluble. Hence by this contradiction (1) exists.

(2) $G^{\mathfrak{F}} = V$ is a q-group, $V/\Phi(V)$ is a G-chief factor, and $exp(V)$ is q or 4 (when $q = 2$ and V is nonabelian).

Since $G/E \in \mathfrak{F}$, we have $V \subseteq E$. Thus by (1), V is soluble. If $V \subseteq \Phi(G)$, then $V \subseteq S_G$ for every maximal subgroup S of G, and so $G/S_G \in \mathfrak{F}$. Therefore claim (2) holds by using Semenchuk Theorem (see [27] or [12, Theorem 3.4.2]). Now suppose that $V \nsubseteq \Phi(G)$. Let G has a maximal subgroup S such that $V \nsubseteq S$. Then $G = VS = ES$ and $S/S \cap E \cong SE/E = G/E \in \mathfrak{F}$. For every non-cyclic Sylow subgroup P of $S \cap E$, we can let $\langle y \rangle$ be a cyclic subgroup of P of prime order or order 4 (when the Sylow 2-subgroup of $S \cap E$ is non-abelian). As the same discussion as (1) of the proof, $(S, S \cap E)$ also satisfies the hypothesis. It means that $S \in \mathfrak{F}$. Then we have (2) by the Semenchuk Theorem ([27] or [12, Theorem 3.4.2]).

(3) Final contradiction.

If V is noncyclic, then by Proposition 3 and (2), we have that $V \leq Z_{\mathfrak{U}}(G)$. Assume that V is cyclic. So, obviously, $V \leq Z_{\mathfrak{U}}(G)$. It follows that $V \in \mathfrak{F}$. This contradiction prove that the theorem is holds for $X = E$.

Now we prove that the theorem is also true for $X = F^*(E)$.

By Lemma 4(2), $(F^*(E), F^*(E))$ also satisfies the hypothesis. As above, we know that $F^*(E)$ is supersoluble, and thus $F(E) = F^*(E)$. Assume that H is a Sylow q-subgroup of $F(E)$. Then H is normal in G. Assume that H is non-cyclic, then we have that the hypothesis of Proposition 3 holds for H. Hence $H \leq Z_{\mathfrak{U}}(G)$. Now let H is cyclic, then obviously, $H \leq Z_{\mathfrak{U}}(G)$. This induces that $F^*(E) = F(E) \leq Z_{\mathfrak{U}}(G)$. Hence from Lemma 6, $E \leq Z_{\mathfrak{U}}(G)$. It implies by Lemma 7 that $G \in \mathfrak{F}$. This finishes the proof.

3. Some applications of our results

From Theorem 1, we obtain the following corollaries.

Corollary 1 ([1, Theorem 3.2]). *Assume that $N \trianglelefteq G$ satisfies $G/N \in \mathfrak{U}$ and G is soluble. If each maximal subgroup of every Sylow subgroup of $F(N)$, which are normal in G, then $G \in \mathfrak{U}$.*

Corollary 2 ([21, Theorem 2]). *Assume that N is a normal soluble* *subgroup of G satisfies $G/N \in \mathfrak{U}$. Assume that every maximal subgroup of the Sylow subgroups of $F(N)$ are c-normal in G, then $G \in \mathfrak{U}$.*

Corollary 3 ([31, Theorem 1])**.** *Assume that $\mathfrak{F}$ is a saturated formation which contains the class of all supersoluble groups and N a normal soluble subgroup of G satisfies $G/N \in \mathfrak{F}$. If every maximal subgroup of the Sylow subgroups of $F(N)$ are c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 4 ([1, Theorem 4.2])**.** *Let G be a group and E a soluble normal subgroup of G with supersoluble quotient G/E. Suppose that every maximal subgroups of every Sylow subgroup of $F(E)$ is s-permutable in G. Then G is supersoluble.*

Theorem 2 covers a lot of results, in particular:

Corollary 5 ([32, Theorem 3.8])**.** *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G be a group. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup E of G such that $G/E \in \mathfrak{F}$ and every cyclic subgroup of every noncyclic Sylow subgroup of E with prime order or order 4 (if the Sylow 2-subgroup is not abelian) not having a supersoluble supplement in G is nearly s-normal in G.*

Corollary 6 ([30, Theorem 4.2])**.** *If every subgroup of order 4 or all minimal subgroups of G are c-normal in G, then $G \in \mathfrak{U}$.*

Corollary 7 ([22, Theorem 3.4])**.** *Assume that $N \trianglelefteq G$ with supersoluble quotient G/N. If every subgroup of order 4 (when the Sylow 2-subgroup of N is non-abelian) or all minimal subgroups of N are c-normal in G, then $G \in \mathfrak{U}$.*

Corollary 8 ([3, Theorem 3.4])**.** *Assume that $\mathfrak{F}$ is a saturated formation which contains the class of all supersoluble groups. If every cyclic subgroup with order 4 and all minimal subgroups of $G^{\mathfrak{F}}$ are c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 9 ([2, Theorem 3.1])**.** *If every subgroup of G of prime order and each cyclic subgroup of G with order 4 are s-permutable in G, then $G \in \mathfrak{U}$.*

Corollary 10 ([24, Theorem 3.9])**.** *Assume that $\mathfrak{F}$ is a saturated formation which contains the class of all supersoluble groups. Then $N \trianglelefteq G$ satisfies $G/N \in \mathfrak{F}$ and if every subgroup of order 4 and every minimal subgroup of N are c-normal in G if and only if $G \in \mathfrak{F}$.*

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