

On Smith normal forms of q -Varchenko matrices*

N. Boulware, N. Jing, and K. C. Misra

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ABSTRACT. In this paper, we investigate q -Varchenko matrices for some hyperplane arrangements with symmetry in two and three dimensions, and prove that they have a Smith normal form over $\mathbb{Z}[q]$. In particular, we examine the hyperplane arrangement for the regular n -gon in the plane and the dihedral model in the space and Platonic polyhedra. In each case, we prove that the q -Varchenko matrix associated with the hyperplane arrangement has a Smith normal form over $\mathbb{Z}[q]$ and realize their congruent transformation matrices over $\mathbb{Z}[q]$ as well.

Introduction

Hyperplane arrangements are present in various problems in geometry, combinatorics, and algebra [5, 7, 8, 11]. A hyperplane arrangement divides an affine space to disjoint regions. The set of hyperplanes separating two regions R_i and R_j is denoted by $\text{sep}(R_i, R_j)$. In 1993, Varchenko defined a matrix associated with a hyperplane arrangement in [13] which helps to reveal the intrinsic combinatorial and algebraic properties of the hyperplane arrangement. The Varchenko matrix has rows and columns indexed by the regions of the hyperplane arrangement, where each hyperplane is assigned an indeterminate a_H . Then the (i, j) -entry of the Varchenko matrix is obtained by taking the product over all a_H such that $H \in \text{sep}(R_i, R_j)$.

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Gao and Zhang [4] have determined the necessary and sufficient conditions for the Varchenko matrix associated with a hyperplane arrangement to have a diagonal form. In [11], Stanley noted that Gao and Zhang's result also holds for pseudosphere arrangements, which are a generalization of hyperplane arrangements.

q -Varchenko matrices come from these Varchenko matrices by replacing the (i, j) entry by $q^{\#\text{sep}(R_i, R_j)}$. It is known [3] that the number of diagonal entries of the Smith normal form of the q -Varchenko matrix V_q exactly divisible by $(q - 1)^i$ is equal to the i -th Betti number of the complex complement of the arrangement. Applications of the invariant factors in the Smith normal form to combinatorial and discrete mathematics are discussed in [9]. Hyperplane arrangements also provide interesting quantum integrable models [14]. While the Smith normal form of a matrix is guaranteed to exist over a principal ideal domain such as $\mathbb{Q}[q]$, the algorithm is not always computationally practical [12]. Combinatorially, it is important to know whether the congruent transformations can be carried out in $\mathbb{Z}[q]$. In [3], Denham and Hanlon illustrated two problems with implementation of the algorithm: one is that the size of the q -Varchenko matrix becomes extremely large very quickly in proportion to the number of hyperplanes in the arrangement; the second is that the degrees of the polynomial matrix entries "blow up during the intermediate stages in the computation".

Little is known about the existence of a Smith normal form of a matrix over rings that are not principal ideal domains. One of the open problems is whether the hyperplane arrangement based on the root systems of type A has an integral Smith normal form [11].

In this paper we consider an easier problem related to root systems. According to McKay [6], the simply-laced types of root systems are in one-to-one correspondence to the symmetry groups of Platonic polyhedra [10] and their degenerates such as the regular polygons and regular polyhedras. Explicitly, we will study the related hyperplane arrangement models based on Platonic polyhedra and their degenerations. The cyclic model has been studied in a recent paper [2] as an example of peelable hyperplane arrangement. In this paper we will use a different method to approach all hyperplane arrangement models based on regular polyhedra, and show that all of the q -Varchenko matrices have the Smith normal forms over $\mathbb{Z}[q]$ and also the congruent transformations can be realized in $\mathbb{Z}[q]$ as well.

Using the symmetry of these arrangements, we give the q -Varchenko matrices, their Smith normal forms over $\mathbb{Z}[q]$, and the corresponding tran-

sition matrices. Then, the methods used in the study of the q -Varchenko matrix for the octahedron arrangement are modified and applied to show that the q -Varchenko matrix for a hyperplane arrangement corresponding to a pyramid with a square base also has a Smith normal form over $\mathbb{Z}[q]$. Lastly, we consider the hyperplane arrangement corresponding to a pyramid with a regular pentagonal base and show that its q -Varchenko matrix has a Smith normal form over $\mathbb{Z}[q]$. In each case, we give the transition matrices and determine the Smith normal form.

The paper is organized as follows. In Section 1 we discuss some basic notions and the hyperplane arrangements. Next, in Section 2 we revisit the hyperplane arrangement of a regular n -gon in \mathbb{R}^2 which we call the cyclic model; we formulate an algorithm that allows us to use the symmetry of the arrangement to obtain its q -Varchenko matrix and determine the Smith normal form over $\mathbb{Z}[q]$ for an arbitrary n . Then we move into \mathbb{R}^3 and define the dihedral model hyperplane arrangement; we obtain its q -Varchenko matrix using the symmetry of the arrangement, determine the Smith normal form over $\mathbb{Z}[q]$ for an arbitrary n , and give the transition matrices. Finally in Section 3, we consider the hyperplane arrangements in \mathbb{R}^3 corresponding to a tetrahedron, a cube, and an octahedron.

1. Hyperplane arrangements

We first recall some basic notions about hyperplane arrangements following [12]. A real hyperplane is an $n - 1$ dimensional subspace defined by a linear equation in \mathbb{R}^n . Let $\mathcal{A} = \{h_1, h_2, \dots, h_k\}$ be a set of real hyperplanes. If $\mathbb{R}^n \setminus \mathcal{A}$ is a disjoint union of open sets, then we call \mathcal{A} a hyperplane arrangement, and the disjoint open subsets are referred as the regions of \mathcal{A} , and their collection is called the *the set of regions* of \mathcal{A} , denoted by $\mathcal{R}(\mathcal{A})$. The set of hyperplanes separating two regions R, R' is denoted by $\text{sep}(R, R')$, and its cardinality $d(R, R') = \#\text{sep}(R, R')$ forms a metric on $\mathcal{R}(\mathcal{A})$. If we fix a base region R_0 , then the *distance enumerator* of \mathcal{A} with respect to R_0 is

$$D_{\mathcal{A}, R_0}(t) = \sum_{R \in \mathcal{R}(\mathcal{A})} t^{d(R_0, R)}.$$

Fix a base region R_0 . Then the regions of \mathcal{A} form a partial order (weak order) defined by $R \preceq R'$ if $\text{sep}(R_0, R) \subseteq \text{sep}(R_0, R')$. This weak order $\mathcal{R}_{\mathcal{A}}$ is graded by distance. The *intersection poset* $L(\mathcal{A})$ is the set of all nonempty intersections of hyperplanes in \mathcal{A} , including the underlying space itself as the intersection over the empty set. We have $x \leq y$ in the

intersection poset if $y \subseteq x$. Therefore the underlying space is the minimal element, $\hat{0}$.

Let $x \in L(\mathcal{A})$. The subarrangement \mathcal{A}_x is defined as $\{h \in \mathcal{A} : x \subseteq h\}$. The arrangement \mathcal{A}^x is defined by $\mathcal{A}^x = \{x \cap h \neq \emptyset : h \in \mathcal{A} - \mathcal{A}_x\}$. If $\mathcal{A} = \{h_1, h_2, \dots, h_n\}$ is a hyperplane arrangement, then take $x = h_k$ and \mathcal{A}^{h_k} is an arrangement in the affine subspace h_k :

$$\mathcal{A}^{h_k} = \{h_k \cap h_i \neq \emptyset : h_i \in \mathcal{A} - \{h_k\}\}. \quad (1.1)$$

Let h_k be a hyperplane in $\mathcal{A} = \{h_1, h_2, \dots, h_m\}$. $\mathcal{A}' = \mathcal{A} - \{h_k\}$ is called the *deleted arrangement*. The arrangement in h_k defined by $\mathcal{A}'' = \{h_i \cap h_k : h_i \in \mathcal{A}'\}$ is called the *restricted arrangement*. Then the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ can be used to recursively solve the problem of counting the number of regions of \mathcal{A} : the *number of regions* of \mathcal{A} equals the number of regions of \mathcal{A}' plus the number of regions of \mathcal{A}'' [7]. The following result is easy to see.

Proposition 1.1.

$$\text{sep}(R, R') = (\text{sep}(R_0, R) \cup \text{sep}(R_0, R')) - (\text{sep}(R_0, R') \cap \text{sep}(R_0, R)).$$

The hyperplane arrangement of a regular n -gon is the hyperplane arrangement given by taking the edges of a regular n -gon and extending them to lines in \mathbb{R}^2 . Similarly, the hyperplane arrangement of a regular polytope is the hyperplane arrangement given by taking the sides of a regular polytope and extending them to planes in \mathbb{R}^3 .

2. Symmetry models

In this section we develop a simple method to compute the q -Varchenko matrix of a hyperplane arrangement and show that the q -Varchenko matrix has an integral Smith normal form for the simple example of the cyclic model. In particular, we consider the hyperplane arrangement of the affine span of the facets of special polytopes.

2.1. The cyclic model

The Smith normal form for the q -Varchenko matrix associated with the hyperplane arrangement of a regular n -gon, $\mathcal{A} = \{h_1, h_2, \dots, h_n\}$, was presented in [2]. Here the hyperplanes consists of all facets of the regular n -polygons. Our approach to finding the q -Varchenko matrix V_q is different and meant to introduce a method to treat other models. We refer to the

method of arranging hyperplane arrangements of regular n -gons as the cyclic model C_n , using the same symbol for the cyclic group of order n .

Take the base region R_0 to be the center n -gon region, so $\# \text{ sep } (R_0, R_0) = 0$. The distance enumerator for \mathcal{A} (with respect to R_0) is

$$D_{\mathcal{A}, R_0}(t) = 1 + \sum_{k=1}^p nt^k, \text{ where } p = \lfloor \frac{n+1}{2} \rfloor. \tag{2.1}$$

For positive integer r , the n -dimensional vector $\mathbf{q}^r = (q^r, q^r, \dots, q^r)$, is sometimes denoted by \mathbf{q}_n^r to indicate its size if needed. We also use the following notations for the i -dimensional vectors:

$$(q^r)_i = (q^r, q^{r-2}, q^{r-4}, \dots, q^{r-2i+2}), \tag{2.2}$$

$$(q^r)^i = (q^r, q^{r+2}, q^{r+4}, \dots, q^{r+2i-2}). \tag{2.3}$$

For $i < j$, we also define the $(n - j + i + 1)$ -dimensional vector:

$$[q^{i-j}]_{n-j+i+1} = (q^{i-j}, q^{i-j+2}, \dots, q^{i+j-2}, \underbrace{q^{i+j}, \dots, q^{i+j}}_{n+1-i-j}, q^{i+j-2}, \dots, q^{i-j}) \tag{2.4}$$

We will denote the transpose of a matrix M by M^t .

Recall that the circulant matrix $C = C(a_1, a_2, \dots, a_n)$ is defined by:

$$C(a_1, a_2, \dots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}. \tag{2.5}$$

The reverse circulant matrix $RC = RC(a_1, a_2, \dots, a_n)$ is defined by:

$$RC(a_1, a_2, \dots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & a_4 & \cdots & a_1 \\ a_3 & a_4 & a_5 & \cdots & a_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix}. \tag{2.6}$$

So for any integer $1 \leq a \leq n$, the (x', x) -entry of RC_{ij} is equal to the $(\overline{x' + a}, \overline{x - a})$ -entry. The overline refers to modulo n .

Theorem 2.1. For a natural integer n , let p be as in (2.1). The $(1+np) \times (1+np)$ q -Varchenko matrix for the C_n arrangement has the following block form:

$$V_q = \begin{pmatrix} 1 & Q_1 & Q_2 & \dots & Q_p \\ Q_1^t & C_{11} & C_{12} & \dots & C_{1p} \\ Q_2^t & C_{21} & C_{22} & \dots & C_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_p^t & C_{p1} & C_{p2} & \dots & C_{pp} \end{pmatrix} \tag{2.7}$$

where Q_r is the row vector \mathbf{q}_n^r and C_{ij} ($1 \leq i \leq j \leq p$) is the circulant matrix given by

$$C_{ij} = C((q^{j-i})^i, \mathbf{q}_{n+1-i-j}^{i+j}, (q^{j-i+2(i-1)})_{i-1}, \mathbf{q}_{j-i}^{j-i}). \tag{2.8}$$

Proof. By definition the Varchenko matrix encodes the intersection numbers among separable regions in the model: $V_q = (q^{\#\text{sep}(R_i, R_j)})$, where $\#\text{sep}(R_i, R_j)$ is the number of hyperplanes separating region R_i from region R_j .

Now let's carefully count the separating numbers $\#\text{sep}(R_i, R_j)$. For $1 \leq k \leq p$, there are n regions R such that $\#\text{sep}(R_0, R) = k$. For $1 \leq x \leq n$: $R_0 \leq R_x \leq R_{n+x} \leq \dots \leq R_{(p-1)n+x}$ in the weak order of containment. Label the $1+np$ regions of the arrangement iteratively as follows. First of all, the n hyperplanes h_1, \dots, h_n are named so that each h_x forms an edge of the n -gon that shares vertices with the edges formed by $h_{\bar{x} \pm 1}$, and so that h_2 follows h_1 moving along the edges in a counter-clockwise direction; the central region is labeled by R_0 ; label the remaining regions in the following iterative process: for each $k \leq n$ and $m \leq p-1$, choose R' and name it by $R_{(m+1)n+k}$ such that $\text{sep}(R_0, R_{m+((k+1)\text{mod}(n))}) = \text{sep}(R_{mn+k}, R')$. Repeat this process till k reaches n . Here and throughout this section \bar{i} denotes $(i \text{ mod } n)$.

For $1 \leq x \leq n$ and $1 \leq r \leq p$, $\#\text{sep}(R_0, R_{(r-1)n+x}) = r$, so $Q_r = \mathbf{q}_n^r$. Also for any $\bar{a} \equiv \bar{b} + 1$ we have that

$$\#\text{sep}(R_{(i-1)n+1}, R_{(j-1)n+x}) = \#\text{sep}(R_{(i-1)n+\bar{a}}, R_{(j-1)n+\bar{b}})$$

For any integer $1 \leq a \leq n$, the (x', x) -entry of C_{ij} is equal to the $(\overline{x'+a}, \overline{x+a})$ -entry, where the overline refers to modulo n . Therefore, C_{ij} is a circulant matrix.

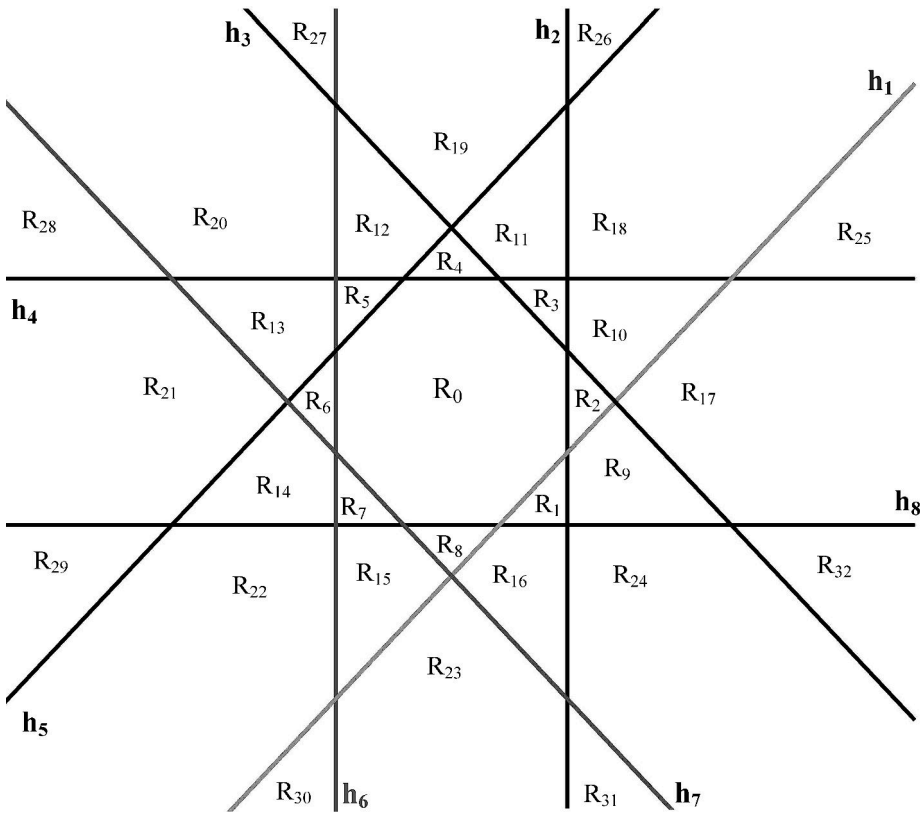


FIGURE 1. The labelled C_8 hyperplane arrangement.

Suppose $n \geq 3$ and $i \leq j$. For $1 \leq x \leq n$, note that C_{1j} is given by $\# \text{sep}(R_1, R_{(j-1)n+x})$, C_{2j} is given by $\# \text{sep}(R_{n+1}, R_{(j-1)n+x})$, and C_{3j} is given by $\# \text{sep}(R_{2n+1}, R_{(j-1)n+x})$ etc.

Now consider C_{ij} where $i \leq j$. Specifically for $i \leq j \leq p$ and $1 \leq x \leq n$,

$$\begin{aligned} & \# \text{sep}(R_{(i-1)n+1}, R_{(j-1)n+x}) \\ &= \begin{cases} j - i, & \text{sep}(R_0, R_{(i-1)n+1}) \subseteq \text{sep}(R_0, R_{(j-1)n+x}), \\ j - i + 2k, & k \text{ elements from } \text{sep}(R_0, R_{(i-1)n+1}) \\ & \text{do not belong to } \text{sep}(R_0, R_{(j-1)n+x}), \\ j + i, & \text{sep}(R_0, R_{(i-1)n+1}) \cap \text{sep}(R_0, R_{(j-1)n+x}) = \emptyset. \end{cases} \end{aligned}$$

By analyzing three cases we have that for $1 \leq k < i$ and $i \leq j$,

$$\# \text{sep}(R_{(i-1)n+1}, R_{(j-1)n+x}) = \tag{2.9}$$

$$\begin{cases} j - i, & x \in \underbrace{\{1, n, n - 1, \dots, n - (j - i - 1)\}}_{j - i + 1 \text{ terms in set}}, \\ j - i + 2k, & x \in \{k + 1, n - (j - i - 1) - k\}, \\ j + i, & x \in \underbrace{\{i + 1, i + 2, \dots, n - (j - 1)\}}_{n + 1 - i - j \text{ terms in set}}. \end{cases}$$

Therefore C_{ij} is indeed as given in (2.8), and the theorem is proved. \square

Theorem 2.2. *For any $n \geq 3$, the Smith normal form of the q -Varchenko matrix for the cyclic model arrangement is*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (1 - q^2)I_n & 0 \\ 0 & 0 & (1 - q^2)^2 I_{n(p-1)} \end{pmatrix}. \tag{2.10}$$

The corresponding left and right transition matrices are (respectively) SP and $(SP)^t$ where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -qI_{n \times 1} & I_n & 0 & 0 & 0 \\ 0 & -qI_n & I_n & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -qI_n & I_n \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & I_n & 0 & \dots & 0 \\ 0 & -qJ & I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -qJ & I_n \end{pmatrix}, \tag{2.11}$$

where J is the $n \times n$ permutation matrix $\begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix}$.

Proof. Note that the first row and column of the q -Varchenko matrix V_q (see (2.7)) for C_n can be transformed into its Smith normal form over $\mathbb{Z}[q]$ in two actions: congruent action by P (i.e. $V \rightarrow PVP^t$) followed by congruent action by S . After left multiplication by P , the Q_k^t blocks in V_q become zeros since for $1 \leq k \leq p$, $Q_k^t - qQ_{k-1}^t = 0$. Also the (i, j) -entry

C_{ij} in V_q becomes

$$C_{ij} - qC_{i-1, j} = \begin{cases} q^{j-i}(1 - q^2) \left(\sum_{k=0}^{j-i} (J^t)^k + \sum_{m=1}^{i-1} q^{2m} J^m \right), & \text{for } 1 \leq i \leq j, \\ q^{j-i}(1 - q^2) J^{j-i} \sum_{k=0}^{i-1} q^{2k} J^k, & \text{for } i > j. \end{cases} \tag{2.12}$$

Then after right multiplication by P^t , the Q_k blocks become zeros since for $1 \leq k \leq p$, $Q_k - qQ_{k-1} = 0$. Furthermore, the (i, j) -entry C_{ij} block matrices in V_q become

$$(C_{ij} - qC_{i-1, j}) - q(C_{i, j-1} - qC_{i-1, j-1}) = (1 - q^2)q^{|i-j|} J^{\overline{i-j}}, \tag{2.13}$$

which can be verified case by case by using (2.12).

Therefore, after congruent action of P the q -Varchenko matrix is transformed to the $(1 + np) \times (1 + np)$ matrix:

$$PV_q P^t = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & (1 - q^2)I_n & q(1 - q^2)J^t & \dots & q^{p-1}(1 - q^2)(J^t)^{p-1} \\ 0 & q(1 - q^2)J & (1 - q^2)I_n & \dots & q^{p-2}(1 - q^2)(J^t)^{p-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & q^{p-1}(1 - q^2)J^{p-1} & q^{p-2}(1 - q^2)J^{p-2} & \dots & (1 - q^2)I_n \end{pmatrix}. \tag{2.14}$$

Then, left multiplication by S to $PV_q P^t$ leaves the first row unchanged. The block matrix entries below the diagonal become zero, the block matrix entries along the diagonal become $(1 - q^2)^2 I$ and the block matrix entries above the diagonal become $q^{j-i}(1 - q^2)^2 (J^t)^{j-i}$.

Multiplication on the right by S^t to the resulting matrix leaves the entries on and below the diagonal unchanged, and the entries above the

diagonal become zeros. Indeed for $1 < i < j$,

$$\begin{aligned}
 & C_{ij} - qC_{i-1,j} - q(C_{i,j-1} - qC_{i-1,j-1}) \\
 & \quad - qJ\left(C_{i-1,j} - qC_{i-2,j} - q(C_{i-1,j-1} - qC_{i-2,j-1})\right) - \\
 & \quad \left(C_{i,j-1} - qC_{i-1,j-1} - q(C_{i,j-2} - qC_{i-1,j-2})\right. \\
 & \quad \left. - qJ\left(C_{i-1,j-1} - qC_{i-2,j-1} - q(C_{i-1,j-2} - qC_{i-2,j-2})\right)\right)(qJ^t) \\
 & = q^{j-i}(1 - q^2)(J^t)^{j-i} - q^{j-1-i}(J^t)^{j-1-i}qJ^t = 0. \tag{2.15}
 \end{aligned}$$

Thus the congruent action by S to the matrix in (2.14) transforms the q -Varchenko matrix V_q to its Smith normal form in (2.10) \square

2.2. The dihedral model D_n

Now we take the hyperplanes from the cyclic model and move them into \mathbb{R}^3 so that the lines forming edges of the regular n -gon become planes forming a regular n -gon shaped cylinder. Then we add one hyperplane h that is perpendicular to h_1, h_2, \dots, h_n . This doubles the number of regions. The distance enumerator for \mathcal{A} (with respect to R_0^+) is

$$D_{\mathcal{A}, R_0^+}(t) = 1 + t + \sum_{k=1}^p n(t^k + t^{k+1}), \text{ where } p = \lfloor \frac{n+1}{2} \rfloor.$$

We will refer to the regions above h as R_x^+ and the regions below h as R_x^- . The R_x^+ regions are labelled according to the cyclic model. Effectively, we label the R_x^- regions by taking a reflection of the cyclic model regions along the axis of symmetry between R_1^+ and R_n^+ .

Theorem 2.3. *The q -Varchenko matrix for the dihedral model arrangement is a $2(np + 1) \times 2(np + 1)$ matrix with the following block form:*

$$V_q = \begin{pmatrix} 1 & \bar{Q}_1 & \bar{Q}_2 & \dots & \bar{Q}_p \\ \bar{Q}_1^t & \bar{C}_{11} & \bar{C}_{12} & \dots & \bar{C}_{1p} \\ \bar{Q}_2^t & \bar{C}_{12} & \bar{C}_{22} & \dots & \bar{C}_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{Q}_p^t & \bar{C}_{1p} & \bar{C}_{2p} & \dots & \bar{C}_{pp} \end{pmatrix}, \tag{2.16}$$

where for $1 \leq i \leq j \leq p$, the blocks are defined by

$$\bar{Q}_i = \begin{pmatrix} Q_i & qQ_i \\ qQ_i & Q_i \end{pmatrix}, 0 \leq i \leq p; \quad \bar{C}_{ij} = \begin{pmatrix} C_{ij} & qJ^{i-1}KC_{ij} \\ qJ^{i-1}KC_{ij} & C_{ij} \end{pmatrix}.$$

Here $Q_0 = 1, Q_r = \mathbf{q}_n^r, C_{ij}$ is defined in (2.8), K is the $n \times n$ skew diagonal matrix or $RC(0, \dots, 1)_{n \times n}$ and $J = C(0, 1, 0, \dots, 0)_{n \times n}$.

Proof. For $1 \leq k \leq p$, there are n regions R^\pm such that $\# \text{sep}(R_0^\pm, R^\pm) = k$. For $1 \leq x \leq n: R_0^\pm \leq R_x^\pm \leq R_{n+x}^\pm \leq \dots \leq R_{(p-1)n+x}^\pm$ in the weak order of containment.

Label the $2 + 2np$ regions of the arrangement in the following way:

- 1) h is the hyperplane perpendicular to the hyperplanes h_1, \dots, h_n ;
- 2) For $1 \leq x \leq n$, the hyperplane h_x forms an edge of the n -gon that shares vertices with the edges formed by $h_{x \pm 1 \pmod n}$ so that h_2 follows h_1 moving along the edges in a counter-clockwise direction;
- 3) the central regions above h is labeled as R_0^+ and below h is labeled as R_0^- ;
- 4) for $1 \leq x \leq n$, label the region R as R_x if $\text{sep}(R_0^+, R) = h_x$ and as R_{n+1-x}^- if $\text{sep}(R_0^-, R) = h_x$;
- 5) For $1 \leq k, m \leq n, m + 1 \leq p - 1$, label the region R as $R_{(m+1)n+k}^+$ if $\text{sep}(R_0^+, R_{m+((k+1) \pmod n)}^+) = \text{sep}(R_{mn+k}^+, R^+)$ and label it $R_{(m+1)n+k}^-$ if $\text{sep}(R_0^-, R_{m+((k+1) \pmod n)}^-) = \text{sep}(R_{mn+k}^-, R^-)$.

Now for $1 \leq x \leq n$ and $1 \leq r \leq p$, it is easy to see that $\# \text{sep}(R_0^+, R_{(r-1)n+x}^+) = r$ and $\# \text{sep}(R_0^+, R_{(r-1)n+x}^-) = r + 1$. Therefore $Q_r = (q^r, \dots, q^r)_{1 \times n} = \mathbf{q}_n^r$.

Note that though the process is identical for the R^+ and R^- regions, the hyperplane involved in each step is not. For example,

$$\begin{aligned} \text{sep}(R_1^+, R_{n+1}^+) &= \{h_2\}, \quad \text{sep}(R_1^-, R_{n+1}^-) = \{h_{n-1}\}; \\ \text{sep}(R_{n+1}^+, R_{2n+1}^+) &= \{h_3\}, \quad \text{sep}(R_{n+1}^-, R_{2n+1}^-) = \{h_{n-2}\}. \end{aligned}$$

However, $\text{sep}(R_0^\pm, R_1^\pm) \subseteq \text{sep}(R_0^\pm, R_{2n+1}^\pm) \subseteq \dots \subseteq \text{sep}(R_0^\pm, R_{(p-1)n+1}^\pm)$.

For $1 \leq i, j \leq p$,

$$\text{sep}(R_0^+, R_{(i-1)n+x}^+) = \{h_x, h_{x+1}, h_{x+2}, \dots, h_{x+(i-1)}\}, \quad (2.17)$$

$$\begin{aligned} \text{sep}(R_0^-, R_{(j-1)n+x}^-) \\ = \{h_{n+1-x}, h_{n+1-(x+1)}, h_{n+1-(x+2)}, \dots, h_{n+1-(x+(j-1))}\}, \quad (2.18) \end{aligned}$$

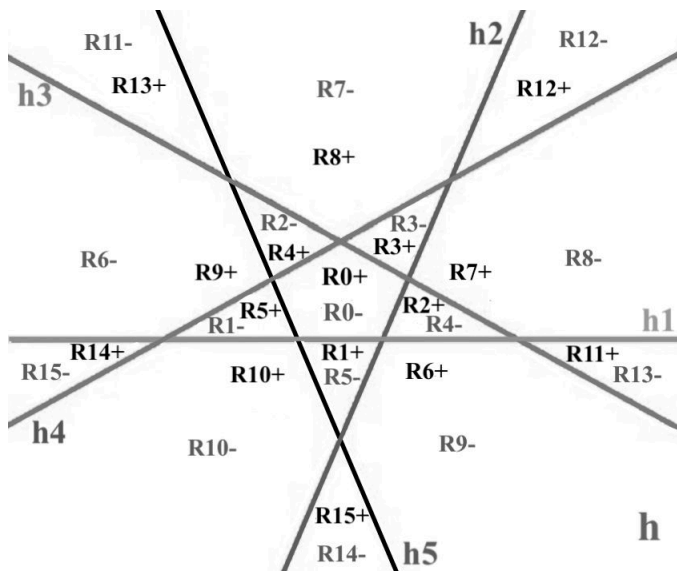


FIGURE 2. The labelled D_5 arrangement, where $k = 5$ and $m = 2$. The hyperplane h is this page. The $+$ regions are on the front side of the page; the $-$ regions are on the back side of the page.

where the index i in h_i is modulo n . Hence the $2(np + 1) \times 2(np + 1)$ q -Varchenko matrix V_q for D_n is:

$$V_q = \begin{pmatrix} 1 & q & Q_1 & Q_2 & \dots & Q_r & Q_{r+1} & \dots & Q_p & Q_{p+1} \\ q & 1 & Q_2 & Q_1 & \dots & Q_{r+1} & Q_r & \dots & Q_{p+1} & Q_p \\ Q_1^t & Q_2^t & C_{1+1+} & C_{1+1-} & \dots & C_{1+r+} & C_{1+r-} & \dots & C_{1+p+} & C_{1+p-} \\ Q_2^t & Q_1^t & C_{1-1+} & C_{1-1-} & \dots & C_{1-r+} & C_{1-r-} & \dots & C_{1-p+} & C_{1-p-} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots \\ Q_r^t & Q_{r+1}^t & C_{r+1+} & C_{r+1-} & \dots & C_{r+r+} & C_{r+r-} & \dots & C_{r+p+} & C_{r+p-} \\ Q_{r+1}^t & Q_r^t & C_{r-1+} & C_{r-1-} & \dots & C_{r-r+} & C_{r-r-} & \dots & C_{r-p+} & C_{r-p-} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_p^t & Q_{p+1}^t & C_{p+1+} & C_{p+1-} & \dots & C_{p+r+} & C_{p+r-} & \dots & C_{p+p+} & C_{p+p-} \\ Q_{p+1}^t & Q_p^t & C_{p-1+} & C_{p-1-} & \dots & C_{p-r+} & C_{p-r-} & \dots & C_{p-p+} & C_{p-p-} \end{pmatrix}, \tag{2.19}$$

where $Q_r = \mathbf{q}_n^r$, $C_{i\pm j\pm} = C_{i\mp j\mp}$, $C_{j\pm i\pm} = C_{i\pm j\pm}^t$, $C_{i\pm j\mp} = C_{i\mp j\pm}$ and $C_{j\mp i\pm} = C_{i\pm j\mp}^t$. As immediate consequences of the indexing method of

the regions, $C_{i\pm j\pm}$ are circulant matrices, and $C_{i\pm j\pm} = C_{ij}$ as stated in Theorem 2.1.

For $1 \leq x \leq n$ and $0 \leq a \leq n$

$$\# \text{sep} \left(R_{(i-1)n+1+a}^{\pm}, R_{(j-1)n+x-a}^{\pm} \right) = \# \text{sep} \left(R_{(i-1)n+1}^{\pm}, R_{(j-1)n+x}^{\pm} \right). \tag{2.20}$$

which is a consequence of the fact that when $x+k = n$ for some $1 \leq k < n$, $R_{(j-1)n+(x+k \bmod n)}^{\pm} = R_{(j-1)n}^{\pm}$.

For $1 \leq x', x \leq n$, the (x', x) -entry of C_{i+j-} in V_q is equal to the number of the hyperplanes separating the regions $R_{(i-1)n+x'}^+$ and $R_{(j-1)n+x}^-$. Now let's compute it as in the proof of Theorem 2.1 starting with

$\# \text{sep}(R_{(i-1)n+x'}, R_{(j-1)n+x})$ with $x' = n - (i - 1)$. This will give us the $n - (i - 1)$ row of C_{i+j-} instead of the first row. However, as a result of the indexing convention, we see that that the $C_{i\pm j\mp}$ blocks are reverse-circulant matrices. Therefore they are determined entirely by the first row, or equivalently, any particular row. For $1 \leq x \leq n$, we have the following observations:

$$\text{sep} \left(R_0^+, R_{(i-1)n+n-(i-1)}^+ \right) = \{h_n, h_{n-1}, h_{n-2}, \dots, h_{n-(i-1)}\}, \tag{2.21}$$

$$\begin{aligned} \text{sep} \left(R_0^+, R_{(j-1)n+x}^- \right) = \\ \{h, h_{n+1-x}, h_{n+1-(x+1)}, h_{n+1-(x+2)}, \dots, h_{n+1-(x+(j-1))}\}. \end{aligned} \tag{2.22}$$

Therefore by Proposition 1.1,

$$\begin{aligned} \text{sep} \left(R_{(i-1)n+n-(i-1)}^+, R_{(j-1)n+x}^- \right) = \\ \{h_n, h_{n-1}, \dots, h_{n-(i-1)}\} \cup \{h, h_{n+1-x}, \dots, h_{n+1-(x+(j-1))}\} - \\ (\{h_n, h_{n-1}, \dots, h_{n-(i-1)}\} \cap \{h, h_{n+1-x}, \dots, h_{n+1-(x+(j-1))}\}). \end{aligned} \tag{2.23}$$

Then we have for $1 \leq k < i \leq j$,

$$\begin{aligned} & \# \text{sep} \left(R_{(i-1)n+n-(i-1)}^+, R_{(j-1)n+x}^- \right) \\ &= \begin{cases} j+1-i, & x \in \underbrace{\{1, n, n-1, \dots, n-(j-i-1)\}}_{j-i+1 \text{ terms in set}}. \\ j+1-i+2k, & x \in \{k+1, n-(j-i-1)-k\}. \\ j+1+i, & x \in \underbrace{\{i+1, i+2, \dots, n-(j-1)\}}_{n+1-i-j \text{ terms in set}}. \end{cases} \end{aligned} \tag{2.24}$$

Comparing it to (2.9), we see that this is precisely $1 + \# \text{sep}(R_{(i-1)n+1}^+, R_{(j-1)n+x}^+)$. Therefore, the $n - (i - 1)$ row of C_{i+j-} equals q times the first row of C_{ij} as defined in Theorem 2.1. Left multiplication by $J^{i-1}K$ acting on the circulant matrix C_{ij} takes the first row to the n th row, and then takes the n th row to the $n - (i - 1)$ row. Since both C_{i+j-} and $qJ^{i-1}KC_{ij}$ are reverse circulant matrices, the equality of one row of each matrix is sufficient to determine the equality of the two matrices. Therefore, $qJ^{i-1}KC_{ij} = C_{i+j-}$. \square

Proposition 2.4. For $i \leq j$, $C_{ij}^t = J^{j-i}C_{ij}$.

Proof. Note that for $1 \leq r, c \leq n$ we have $J_{rc} = \delta_{r+1 \pmod n, c}$, and hence $(J^{j-i})_{rc} = \delta_{r+j-i, c}$. So

$$(J^{j-i}C_{ij})_{rc} = \sum_{k=1}^n (J^{j-i})_{rk}(C_{ij})_{kc} = \sum_{k=1}^n \delta_{r+j-i, k}(C_{ij})_{kc} = (C_{ij})_{r+j-i, c}.$$

Since it is a circulant matrix, $J^{j-i}C_{ij}$ can be written using the $(1 + j - i)$ th row of C_{ij} . Therefore, we have

$$J^{j-i}C_{ij} = C(\underbrace{q^{j-i}, \dots, q^{j-i}}_{j-i+1}, q^{j-i+2}, \dots, q^{j-i+2(i-1)}, \underbrace{q^{j+i}, \dots, q^{j+i}}_{n+1-i-j}, q^{j-i+2(i-1)}, \dots, q^{j-i+2}) = C_{ij}^t. \quad \square$$

Theorem 2.5. For $n \geq 3$, the Smith normal form of the q -Varchenko matrix V_q given in Theorem 2.3 for the dihedral model arrangement is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1 - q^2)I_{n+1} & 0 & 0 \\ 0 & 0 & (1 - q^2)^2 I_{np} & 0 \\ 0 & 0 & 0 & (1 - q^2)^3 I_{n(p-1)} \end{pmatrix}. \quad (2.25)$$

This is obtained by the congruent action of $RTSP$, where

$$P = \left(\begin{array}{cc|cc|c|cc|cc} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -q & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \hline -q1 & 0 & I_n & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -q1 & 0 & I_n & \dots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -qI_n & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -qI_n & \ddots & 0 & 0 & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & \dots & -qI_n & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -qI_n & 0 & I_n \end{array} \right),$$

$$S = \left(\begin{array}{cc|cc|c|cc|cc} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & \dots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -qJ & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -qJ & \ddots & 0 & 0 & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & \dots & -qJ & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -qJ & 0 & I_n \end{array} \right),$$

$$T = \left(\begin{array}{cc|cc|cc|c|cc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & I_n & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -qK & I_n & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_n & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & -qJK & I_n & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \dots & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -qJ^{p-1}K & I_n \end{array} \right),$$

$$R = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & A_2 & & & \dots & 0 \\ 0 & 0 & \ddots & \dots & \dots & 0 \\ 0 & 0 & \dots & A_j & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & & & A_p \end{pmatrix};$$

where

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}, \quad A_j = \begin{pmatrix} I_n & 0 & \dots & 0 \\ 0 & \dots & 0 & I_n \end{pmatrix}_{2n \times np} \quad \text{for } 2 \leq j \leq p.$$

Proof. We will index the first row (resp. column) of the Varchenko matrix V_q with entries Q_k (resp. Q_k^t) as row $i = 0$ (resp. column $j = 0$). The entry C_{11} will be indexed by row $i = 1$ and column $j = 1$. The Varchenko matrix V_q for D_n can be transformed into its Smith normal form over $\mathbb{Z}[q]$ in successive congruent actions of P, S, T and R .

After left multiplication by P , the Q_k^t blocks become zeros since for $1 \leq k \leq p$, $Q_k^t - qQ_{k-1}^t = 0$. Furthermore, the original C_{ij} block matrices have the form given in (2.12) and the original $qJ^{i-1}KC_{ij}$ block matrices have the form

$$\begin{cases} q^j(1 - q^2) \left(\sum_{k=0}^{j-1} J^k \right) K, & \text{for } i = 1, \\ q^{j-i+1}(1 - q^2)J^{j-1} \left(\sum_{k=0}^{j-i} (J^t)^k + \sum_{m=1}^{i-1} q^{2m} J^m \right) K, & \text{for } i > 1. \end{cases} \tag{2.26}$$

After right multiplication by P^t , the Q_k blocks become zeros since for $1 \leq k \leq p$, $Q_k - qQ_{k-1} = 0$ and the first column remains unchanged. The block matrices with the original entry C_{ij} become

$$(1 - q^2)q^{|i-j|} J^{\overline{i-j}}. \tag{2.27}$$

Furthermore, the original $qJ^{i-1}KC_{ij}$ block entries become

$$q^{j-i+1}(1 - q^2)J^{j-1} \left((1 - q^2) \left(\sum_{k=0}^{i-2} q^{2k} J^k \right) + q^{2(i-1)}J^{i-1} \right) K. \quad (2.28)$$

This can be verified by direct computations case by case, depending on the position of the original $qJ^{i-1}KC_{ij}$ blocks.

Then, left multiplication by the matrix S has the effect of beginning with the last two rows and subtracting qJ times the previous two rows; then the same operation is repeated by subtracting qJ times the third-to-last two rows from the second-to-last two rows, and so forth. The qKC_{1j} -entries remain the same: $q^j(1 - q^2)J^{j-1}K$.

For all of the $qJ^{i-1}KC_{ij}$ -entries that are below the diagonal and for which $i \neq j$, left multiplication by S carries out the operation of subtracting qJ times the $qJ^{i-1}KC_{i,j-1}$ -entry from the $qJ^{i-1}KC_{ij}$ -entry. In this case, the result equals 0. For all of the $qJ^{i-1}KC_{ij}$ -entries that are either on or above the diagonal or for which $i = j$, left multiplication by S carries out the operation of subtracting qJ times the $qJ^{i-2}KC_{i-1,j}$ -entry from the $qJ^{i-1}KC_{ij}$ -entry.

Right multiplication by S^t has the effect of beginning with the last two columns and subtracting the previous two columns times qJ^t ; then the same operation is repeated by subtracting the third-to-last two columns times qJ^t from the second-to-last two columns, and so forth. The qKC_{11} -entries remain the same throughout this step, and the $qJ^{i-1}KC_{ii}$ -entries remain $q(1 - q^2)^2J^{i-1}K$, since the corresponding entries to the left of them are zeros. This operation gives us zeros above the block diagonal entries for which $i \neq j$, since for $i > j$, the current $qJ^{i-1}KC_{ij}$ -entry minus the current $qJ^{i-1}KC_{i,j-1}$ -entry times qJ^t is

$$\begin{aligned} & q^{j-i+1}(1 - q^2)^2J^{j-1}K - (q^{(j-1)-i+1}(1 - q^2)^2J^{j-2}K)qJ^t \\ & = q^{j-i+1}(1 - q^2)^2(J^{j-1}K - J^{j-2}KJ^t) = 0. \end{aligned}$$

Left multiplication by T has the effect of beginning with the last two rows, subtracting $qJ^{p-1}K$ times the second to last row from the last row; then the same operation is repeated by subtracting $qJ^{p-2}K$ times the fourth to last row from the third to last row, and so on. The $qJ^{i-1}KC_{ii}$ -entries above the diagonal remain the same, and the ones below the diagonal become zero since each $qJ^{i-1}KC_{ii}$ -entry equals $qJ^{i-1}K$ times the corresponding C_{ii} -entry which is $(1 - q^2)I$ for C_{11} and $(1 - q^2)^2I$ for $i \geq 2$.

Along the diagonal, we get $(1 - q^2)I$ in the first $\overline{C_{11}}$ diagonal position and $(1 - q^2)^2I$ in its second diagonal position:

$$\begin{aligned} (1 - q^2)I - qK (q(1 - q^2)K) &= (1 - q^2)I - q^2(1 - q^2)K^2 \\ &= (1 - q^2)I - q^2(1 - q^2)I = (1 - q^2)^2I. \end{aligned}$$

For $i \geq 2$, we get $(1 - q^2)^2I$ in the first $\overline{C_{ii}}$ diagonal position and $(1 - q^2)^3$ in its second diagonal position:

$$\begin{aligned} (1 - q^2)^2I - qJ^{i-1}K (q(1 - q^2)^2J^{i-1}K) \\ &= (1 - q^2)^2I - q^2(1 - q^2)^2J^{i-1}KJ^{i-1}K \\ &= (1 - q^2)^2I - q^2(1 - q^2)^2I = (1 - q^2)^3I. \end{aligned}$$

At this point, we have calculated $TSPV_q(SP)^t$ which is:

$$\left(\begin{array}{cc|cc|cc|c|c} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & tI_n & qtK & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & t^2I_n & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & t^2I_n & qt^2JK & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t^3I_n & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \dots & t^2I_n & qt^2J^{p-1}K \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & t^3I_n \end{array} \right),$$

where $t = 1 - q^2$.

Right multiplication by T^t has the effect of beginning with the last column and subtracting the previous column times $q(J^{p-1}K)^t = qJ^{p-1}K$; then the same operation is repeated by subtracting the fourth-to-last column times $qJ^{p-1}K$ from the third-to-last column, and so on. The $qJ^{i-1}KC_{ii}$ -entries above the diagonal become zero since each $qJ^{i-1}KC_{ii}$ -entry equals $qJ^{i-1}K$ times the corresponding (i, i) -entry which is $(1 - q^2)I$

for C_{11} and $(1 - q^2)^2I$ for $i \geq 2$. Now the matrix is in the diagonal form:

$$TSPV_q(TSP)^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & tI_n & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & t^2I_n & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & t^2I_n & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t^3I_n & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \dots & t^2I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & t^3I_n \end{pmatrix}$$

Finally the left and right multiplication by the permutation matrices R and R^t (respectively) moves the $(1 - q^2)^3I_n$ diagonal entries to the last $n(p - 1)$ block diagonal places which gives the Smith normal form of the q -Varchenko matrix V_q in this case. □

Example 2.6. The q -Varchenko matrix for D_5 is the 32×32 matrix

$$\begin{pmatrix} 1 & q & Q_1 & qQ_1 & Q_2 & qQ_2 & Q_3 & qQ_3 \\ q & 1 & qQ_1 & Q_1 & qQ_2 & Q_2 & qQ_3 & Q_3 \\ \hline Q_1^t & qQ_1^t & C_{11} & qKC_{11} & C_{12} & qKC_{12} & C_{13} & qKC_{13} \\ qQ_1^t & Q_1^t & qKC_{11} & C_{11} & qKC_{12} & C_{12} & qKC_{13} & C_{13} \\ \hline Q_2^t & qQ_2^t & C_{12}^t & qKC_{12} & C_{22} & qJKC_{22} & C_{23} & qJKC_{23} \\ qQ_2^t & Q_2^t & qKC_{12} & C_{12}^t & qJKC_{22} & C_{22} & qJKC_{23} & C_{23} \\ \hline Q_3^t & qQ_3^t & C_{13}^t & qKC_{13} & C_{23}^t & qJKC_{23} & C_{33} & qJ^2KC_{33} \\ qQ_3^t & Q_3^t & qKC_{13} & C_{13}^t & qJKC_{23} & C_{23}^t & qJ^2KC_{33} & C_{33} \end{pmatrix}$$

The circulant matrices C_{ij} , $1 \leq j \leq 3$ and the 1×5 row vectors Q_k , $k = 1, 2, 3$, are as defined in Theorem 2.1. Here, RC denotes reverse circulant matrices.

$$\begin{aligned}
 qKC_{11} &= RC(q^3, q^3, q^3, q^3, q), & qKC_{12} &= RC(q^4, q^4, q^4, q^2, q^2), \\
 qKC_{13} &= RC(q^5, q^5, q^3, q^3, q^3), & qJKC_{22} &= RC(q^5, q^5, q^3, q, q^3), \\
 qJKC_{23} &= RC(q^6, q^4, q^2, q^2, q^4), & qJ^2KC_{33} &= RC(q^5, q^3, q, q^3, q^5).
 \end{aligned}$$

$$P = \left(\begin{array}{cc|cc|cc|cc}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -q & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \hline
 -q1 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 \\
 0 & -q1 & 0 & I_5 & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & -qI_5 & 0 & I_5 & 0 & 0 & 0 \\
 0 & 0 & 0 & -qI_5 & 0 & I_5 & 0 & 0 \\
 \hline
 0 & 0 & 0 & 0 & -qI_5 & 0 & I_5 & 0 \\
 0 & 0 & 0 & 0 & 0 & -qI_5 & 0 & I_5
 \end{array} \right),$$

$$S = \left(\begin{array}{cc|cc|cc|cc}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & I_5 & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & -qJ & 0 & I_5 & 0 & 0 & 0 \\
 0 & 0 & 0 & -qJ & 0 & I_5 & 0 & 0 \\
 \hline
 0 & 0 & 0 & 0 & -qJ & 0 & I_5 & 0 \\
 0 & 0 & 0 & 0 & 0 & -qJ & 0 & I_5
 \end{array} \right),$$

$$T = \left(\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -qK & I_5 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -qJK & I_5 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & I_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -qJ^2K & I_5 \end{array} \right),$$

$$R = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_5 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_5 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_5 \end{array} \right).$$

$$SPV_q P^t = \left(\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & tI & qtK & qtJ^4 & q^2tJK & q^2tJ^3 & q^3tJ^2K \\ 0 & 0 & qtK & tI & q^2tJK & qtJ^4 & q^3tJ^2K & q^2tJ^3 \\ \hline 0 & 0 & 0 & 0 & t^2I & qt^2JK & qt^2J^4 & qt^2J^2K \\ 0 & 0 & 0 & 0 & qt^2JK & t^2I & q^2t^2J^2K & qt^2J^4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & t^2I & qt^2J^2K \\ 0 & 0 & 0 & 0 & 0 & 0 & qt^2J^2K & t^2I \end{array} \right).$$

where $t = 1 - q^2$. The Smith normal form is

$$RTSPV_q(RTSP)^t = \text{diag}(1, tI_6, t^2I_{15}, t^3I_{10}).$$

3. Polyhedral models

There are five regular Platonic regular polyhedra whose symmetry groups are the exceptional Kleinian subgroups [10]. This section treats the exceptional ones. In each case we give the q -Varchenko matrix V_q , its Smith normal form over $\mathbb{Z}[q]$ and its transition matrices. It can be checked by direct calculations that multiplying V_q on the left and right by the given transition matrices we obtain the corresponding Smith normal form. Further details on these calculations can be seen in [1].

Denote the region R_i as i ; similarly, let ij (respectively, ijk) denote the region R separated from R_0 by the hyperplanes h_i and h_j (respectively, h_i, h_j and h_k).

3.1. Tetrahedron

The distance enumerator with respect to R_0 is $D_{\mathcal{A},R_0}(t) = 1 + 4t + 6t^2 + 4t^3$. Let R_0 denote the central region of the hyperplane arrangement. Label the four regions R_1, R_2, R_3, R_4 so that h_i separates R_0 from R_i as shown in Fig. 3.

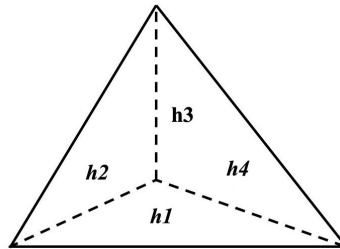


FIGURE 3. Tetrahedron. Italicized hyperplanes are not visible from this view.

The 15 regions are indexed in the following order, and denoted by the hyperplanes in $\text{sep}(R, R_0)$:

$$0 \left\| 1 \ 2 \ 3 \ 4 \right\| 12 \ 23 \ 34 \ 14 \left| 13 \ 24 \right\| 123 \ 234 \ 341 \ 412 \quad (3.1)$$

$$V_q = \left(\begin{array}{c|cccc|cccc|cc|cccc} 1 & & & & & q^2 & q^2 & q^2 & q^2 & q^2 & q^2 & & q^3 & q^3 & q^3 & q^3 \\ \hline q & 1 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q & q & q^3 & & q^2 & q^4 & q^2 & q^2 \\ q & q^2 & 1 & q^2 & q^2 & q & q & q^3 & q^3 & q^3 & q & & q^2 & q^2 & q^4 & q^2 \\ q & q^2 & q^2 & 1 & q^2 & q^3 & q & q & q^3 & q & q^3 & & q^2 & q^2 & q^2 & q^4 \\ q & q^2 & q^2 & q^2 & 1 & q^3 & q^3 & q & q & q^3 & q & & q^4 & q^2 & q^2 & q^2 \\ \hline q^2 & q & q & q^3 & q^3 & 1 & q^2 & q^4 & q^2 & q^2 & q^2 & & q & q^3 & q^3 & q \\ q^2 & q^3 & q & q & q^3 & q^2 & 1 & q^2 & q^4 & q^2 & q^2 & & q & q & q^3 & q^3 \\ q^2 & q^3 & q^3 & q & q & q^4 & q^2 & 1 & q^2 & q^2 & q^2 & & q^3 & q & q & q^3 \\ q^2 & q & q^3 & q^3 & q & q^2 & q^4 & q^2 & 1 & q^2 & q^2 & & q^3 & q^3 & q & q \\ \hline q^2 & q & q^3 & q & q^3 & q^2 & q^2 & q^2 & q^2 & 1 & q^4 & & q & q^3 & q & q^3 \\ q^2 & q^3 & q & q^3 & q & q^2 & q^2 & q^2 & q^2 & q^4 & 1 & & q^3 & q & q^3 & q \\ \hline q^3 & q^2 & q^2 & q^2 & q^4 & q & q & q^3 & q^3 & q & q^3 & & 1 & q^2 & q^2 & q^2 \\ q^3 & q^4 & q^2 & q^2 & q^2 & q^3 & q & q & q^3 & q^3 & q & & q^2 & 1 & q^2 & q^2 \\ q^3 & q^2 & q^4 & q^2 & q^2 & q^3 & q^3 & q & q & q & q^3 & & q^2 & q^2 & 1 & q^2 \\ q^3 & q^2 & q^2 & q^4 & q^2 & q & q^3 & q^3 & q & q^3 & q & & q^2 & q^2 & q^2 & 1 \end{array} \right).$$

The Smith normal form is

$$\text{diag} (1, (1 - q^2)I_4, (1 - q^2)^2I_6, (1 - q^2)^3I_4).$$

The left and right transformation matrices are (respectively) P and P^t , where

$$P = \left(\begin{array}{c|cccc|c|c|c|c} 1 & & & & & 0 & & & & 0 \\ \hline -q1 & & & & & I_4 & & & & 0 \\ \hline q^21 & & & & & -q(I + J) & & & & I_4 & & & & & 0 \\ q^21 & & & & & -q(I_2 | I_2) & & & & 0 & & & & & I_2 & & & 0 \\ \hline -q^31 & & & & & q^2(I + J + J^2) & & & & -q(I + J) & & & & & -q(I_2 | I_2)^t & & & I_4 \end{array} \right), \quad (3.2)$$

and

$$(I_2 | I_2) = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array} \right), \quad J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

3.2. Cube

Let R_0 denote the central region of the hyperplane arrangement $\mathcal{A} = \{h_1, h_2, h_3, h_4, h_5, h_6\}$. The distance enumerator with respect to R_0 is $D_{\mathcal{A}, R_0}(t) = 1 + 6t + 12t^2 + 8t^3$. Define each of the 6 regions R_i such that $\text{sep}(R_i, R_0) = h_i$ for $1 \leq i \leq 6$ where the hyperplanes h_i are as shown in Fig. 4.

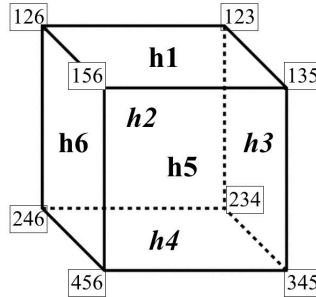


FIGURE 4. Cube with labelled hyperplanes and separating set hyperplanes for regions corresponding to vertices. Italicized hyperplanes are not visible from this view.

The 27 regions are indexed in the following order, and denoted by the hyperplanes in $\text{sep}(R, R_0)$:

$$0 \left\| 1 \ 2 \ 3 \ 4 \ 5 \ 6 \right\| 12 \ 23 \ 34 \ 45 \ 56 \ 61 \left| 13 \ 24 \ 35 \ 46 \ 15 \ 26 \right\| \quad (3.3)$$

$$123 \ 234 \ 345 \ 456 \ 561 \ 612 \left| 135 \ 246 \right.$$

$$V_q = \begin{pmatrix} 1 & Q & Q_2 & Q_2 & Q_3 & Q'_3 \\ Q^t & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \\ Q_2^t & v_{21} & v_{22} & v_{23} & v_{24} & v_{25} \\ Q_2^t & v_{31} & v_{32} & v_{33} & v_{34} & v_{35} \\ Q_3^t & v_{41} & v_{42} & v_{43} & v_{44} & v_{45} \\ Q_3^t & v_{51} & v_{52} & v_{53} & v_{54} & v_{55} \end{pmatrix}$$

where $v_{ji} = v_{ij}^t$ is a row-circulant matrix, $Q_k = (q^k, q^k, q^k, q^k, q^k, q^k) = \mathbf{q}_6^k$, and $Q'_k = (q^k, q^k) = \mathbf{q}_2^k$.

$$v_{11} = C(1, q^2, q^2, q^2, q^2, q^2)_{6 \times 6}, \quad v_{12} = C(q, q^3, q^3, q^3, q^3, q)_{6 \times 6},$$

$$\begin{aligned}
 v_{13} &= C(q, q^3, q^3, q^3, q, q^3)_{6 \times 6}, & v_{14} &= C(q^2, q^4, q^4, q^4, q^2, q^2)_{6 \times 6}, \\
 v_{15} &= C(q^2, q^4)_{6 \times 2}, & v_{22} &= C(1, q^2, q^4, q^4, q^4, q^2)_{6 \times 6}, \\
 v_{23} &= C(q^2, q^2, q^4, q^4, q^2, q^2)_{6 \times 6}, & v_{24} &= C(q, q^3, q^5, q^5, q^3, q)_{6 \times 6}, \\
 v_{25} &= C(q^3, q^3)_{6 \times 2}, & v_{33} &= C(1, q^4, q^2, q^4, q^2, q^4)_{6 \times 6}, \\
 v_{34} &= C(q, q^3, q^3, q^5, q^3, q^3)_{6 \times 6}, & v_{35} &= C(q, q^5)_{6 \times 2}, \\
 v_{44} &= C(1, q^2, q^4, q^6, q^4, q^2)_{6 \times 6}, & v_{45} &= C(q^2, q^4)_{6 \times 2}, \\
 v_{55} &= C(1, q^6)_{2 \times 2},
 \end{aligned}$$

where $C_{6 \times 2}$ means the 6×2 block of the circulant matrix.

The Smith normal form is

$$\text{diag} (1, (1 - q^2)I_6, (1 - q^2)^2I_{12}, (1 - q^2)^3I_8).$$

The left transition matrix is U and the right transition matrix is U^t , where

$$U = \left(\begin{array}{c|c|c|c|c|c}
 1 & 0 & 0 & 0 & 0 & 0 \\
 \hline
 -q1 & I_6 & 0 & 0 & 0 & 0 \\
 \hline
 q^21 & -q(I + J) & I_6 & 0 & 0 & 0 \\
 \hline
 q^21 & -q(I + J^2) & 0 & I_6 & 0 & 0 \\
 \hline
 -q^31 & q^2(I + J + J^2) & -q(I + J) & -qI & I_6 & 0 \\
 \hline
 -q^31 & q^2(I_2 | I_2 | I_2) & 0 & -q(I_2 | I_2 | I_2) & 0 & I_2
 \end{array} \right);$$

$$(I_2 | I_2 | I_2) = \left(\begin{array}{c|c|c}
 1 & 0 & 1 & 0 & 1 & 0 \\
 \hline
 0 & 1 & 0 & 1 & 0 & 1
 \end{array} \right), \text{ and } J = \left(\begin{array}{c}
 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 \\
 1 & 0 & 0 & 0 & 0 & 0
 \end{array} \right).$$

3.3. Octahedron

Let R_0 denote the central region of the hyperplane arrangement

$$\mathcal{A} = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8\}.$$

The distance enumerator with respect to R_0 is

$$D_{\mathcal{A}, R_0}(t) = 1 + 8t + 12t^2 + 24t^3 + 14t^4.$$

The hyperplanes in Fig. 5 are labelled according to the order of the regions R in the cube arrangement such that $\#\text{sep}(R_0, R) = 3$ as shown in Fig. 4 and indexed in Eq. 3.3. The last six regions in the octahedron arrangement are indexed by the order of the hyperplanes in the cube arrangement in Fig. 4.

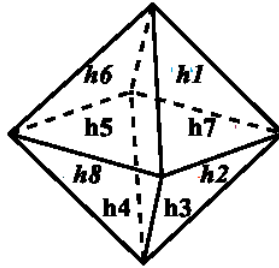


FIGURE 5. Octahedron with labelled hyperplanes. Hyperplanes labelled in italics are not visible from this view.

The 59 regions are indexed in the following order, and denoted by the hyperplanes in $\text{sep}(R, R_0)$:

$$\begin{aligned}
 &0 \left\| 1 \ 2 \ 3 \ 4 \ 5 \ 6 \right| 7 \ 8 \left\| 16 \ 12 \ 23 \ 34 \ 45 \ 56 \right| 17 \ 28 \ 37 \ 48 \ 57 \ 68 \left\| \right. \\
 &167 \ 128 \ 237 \ 348 \ 457 \ 568 \left| 157 \ 268 \ 137 \ 248 \ 357 \ 468 \right\| \\
 &567 \ 168 \ 127 \ 238 \ 347 \ 458 \left| 156 \ 126 \ 123 \ 234 \ 345 \ 456 \right\| \\
 &1267 \ 1238 \ 2347 \ 3458 \ 4567 \ 1568 \left| 1357 \ 2468 \right\| \\
 &1567 \ 1268 \ 1237 \ 2348 \ 3457 \ 4568
 \end{aligned}$$

$$V_q = \begin{pmatrix} 1 & Q_1 & Q'_1 & Q_2 & Q_2 & Q_3 & Q_3 & Q_3 & Q_3 & Q_4 & Q'_4 & Q_4 \\ Q_1^t & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & v_{16} & v_{17} & v_{18} & v_{19} & v_{1,10} & v_{1,11} \\ Q_1^t & v_{21} & v_{22} & v_{23} & v_{24} & v_{25} & v_{26} & v_{27} & v_{28} & v_{29} & v_{2,10} & v_{2,11} \\ Q_2^t & v_{31} & v_{32} & v_{33} & v_{34} & v_{35} & v_{36} & v_{37} & v_{38} & v_{39} & v_{3,10} & v_{3,11} \\ Q_2^t & v_{41} & v_{42} & v_{43} & v_{44} & v_{45} & v_{46} & v_{47} & v_{48} & v_{49} & v_{4,10} & v_{4,11} \\ Q_3^t & v_{51} & v_{52} & v_{53} & v_{54} & v_{55} & v_{56} & v_{57} & v_{58} & v_{59} & v_{5,10} & v_{5,11} \\ Q_3^t & v_{61} & v_{62} & v_{63} & v_{64} & v_{65} & v_{66} & v_{67} & v_{68} & v_{69} & v_{6,10} & v_{6,11} \\ Q_3^t & v_{71} & v_{72} & v_{73} & v_{74} & v_{75} & v_{76} & v_{77} & v_{78} & v_{79} & v_{7,10} & v_{7,11} \\ Q_3^t & v_{81} & v_{82} & v_{83} & v_{84} & v_{85} & v_{86} & v_{87} & v_{88} & v_{89} & v_{8,10} & v_{8,11} \\ Q_4^t & v_{91} & v_{92} & v_{93} & v_{94} & v_{95} & v_{96} & v_{97} & v_{98} & v_{99} & v_{9,10} & v_{9,11} \\ Q_4^t & v_{10,1} & v_{10,2} & v_{10,3} & v_{10,4} & v_{10,5} & v_{10,6} & v_{10,7} & v_{10,8} & v_{10,9} & v_{10,10} & v_{10,11} \\ Q_4^t & v_{11,1} & v_{11,2} & v_{11,3} & v_{11,4} & v_{11,5} & v_{11,6} & v_{11,7} & v_{11,8} & v_{11,9} & v_{11,10} & v_{11,11} \end{pmatrix},$$

where $v_{ji} = v_{ij}^t$ is a row-circulant matrix, $Q_k = (q^k, q^k, q^k, q^k, q^k, q^k) = \mathbf{q}_6^k$ and $Q'_k = (q^k, q^k) = \mathbf{q}_2^k$.

$$\begin{aligned} v_{11} &= C(1, q^2, q^2, q^2, q^2, q^2)_{6 \times 6}, & v_{12} &= C(q^2, q^2)_{6 \times 2}, \\ v_{13} &= C(q, q, q^3, q^3, q^3, q^3)_{6 \times 6}, & v_{14} &= C(q, q^3, q^3, q^3, q^3, q^3)_{6 \times 6}, \\ v_{15} &= C(q^2, q^2, q^4, q^4, q^4, q^4)_{6 \times 6}, & v_{16} &= C(q^2, q^4, q^2, q^4, q^4, q^4)_{6 \times 6}, \\ v_{17} &= C(q^4, q^2, q^2, q^4, q^4, q^4)_{6 \times 6}, & v_{18} &= C(q^2, q^2, q^2, q^4, q^4, q^4)_{6 \times 6}, \\ v_{19} &= C(q^3, q^3, q^5, q^5, q^5, q^3)_{6 \times 6}, & v_{1,10} &= C(q^3, q^5)_{6 \times 2}, \\ v_{1,11} &= C(q^3, q^3, q^3, q^5, q^5, q^5)_{6 \times 6}, & v_{22} &= C(1, q^2)_{2 \times 2}, \\ v_{23} &= C(q^3, q^3, q^3, q^3, q^3, q^3)_{2 \times 6}, & v_{24} &= C(q, q^3, q, q^3, q, q^3)_{2 \times 6}, \\ v_{25} &= C(q^2, q^4, q^2, q^4, q^2, q^4)_{2 \times 6}, & v_{26} &= C(q^2, q^4, q^2, q^4, q^2, q^4)_{2 \times 6}, \\ v_{27} &= C(q^2, q^4, q^2, q^4, q^2, q^4)_{2 \times 6}, & v_{28} &= C(q^4, q^4, q^4, q^4, q^4, q^4)_{2 \times 6}, \\ v_{29} &= C(q^3, q^5, q^3, q^5, q^3, q^5)_{2 \times 6}, & v_{2,10} &= C(q^3, q^5)_{2 \times 2}, \\ v_{2,11} &= C(q^3, q^5, q^3, q^5, q^3, q^5)_{2 \times 6}, & v_{33} &= C(1, q^2, q^4, q^4, q^4, q^2)_{6 \times 6}, \\ v_{34} &= C(q^2, q^4, q^4, q^4, q^4, q^2)_{6 \times 6}, & v_{35} &= C(q, q^3, q^5, q^5, q^5, q^3)_{6 \times 6}, \\ v_{36} &= C(q^3, q^3, q^3, q^5, q^5, q^3)_{6 \times 6}, & v_{37} &= C(q^3, q, q^3, q^5, q^5, q^5)_{6 \times 6}, \\ v_{38} &= C(q, q, q^3, q^5, q^5, q^3)_{6 \times 6}, & v_{39} &= C(q^2, q^4, q^6, q^6, q^4, q^2)_{6 \times 6}, \\ v_{3,10} &= C(q^4, q^4)_{6 \times 2}, & v_{3,11} &= C(q^2, q^2, q^4, q^6, q^6, q^4)_{6 \times 6}, \\ v_{44} &= C(1, q^4, q^2, q^4, q^2, q^4)_{6 \times 6}, & v_{45} &= C(q, q^3, q^3, q^5, q^3, q^5)_{6 \times 6}, \\ v_{46} &= C(q, q^5, q, q^5, q^3, q^5)_{6 \times 6}, & v_{47} &= C(q^3, q^3, q, q^5, q^3, q^5)_{6 \times 6}, \end{aligned}$$

$$\begin{aligned}
 v_{48} &= C(q^3, q^3, q^3, q^5, q^5, q^5)_{6 \times 6}, & v_{49} &= C(q^2, q^4, q^4, q^6, q^4, q^4)_{6 \times 6}, \\
 v_{4,10} &= C(q^2, q^6)_{6 \times 2}, & v_{4,11} &= C(q^2, q^4, q^2, q^6, q^4, q^6)_{6 \times 6}, \\
 v_{55} &= C(1, q^4, q^4, q^6, q^4, q^4)_{6 \times 6}, & v_{56} &= C(q^2, q^4, q^2, q^6, q^4, q^4)_{6 \times 6}, \\
 v_{57} &= C(q^2, q^2, q^2, q^6, q^4, q^6)_{6 \times 6}, & v_{58} &= C(q^2, q^2, q^4, q^6, q^6, q^4)_{6 \times 6}, \\
 v_{59} &= C(q, q^5, q^5, q^7, q^3, q^3)_{6 \times 6}, & v_{5,10} &= C(q^3, q^5)_{6 \times 2}, \\
 v_{5,11} &= C(q, q^3, q^3, q^7, q^5, q^5)_{6 \times 6}, & v_{66} &= C(1, q^6, q^2, q^6, q^2, q^6)_{6 \times 6}, \\
 v_{67} &= C(q^2, q^4, q^2, q^6, q^4, q^4)_{6 \times 6}, & v_{68} &= C(q^2, q^4, q^4, q^6, q^4, q^4)_{6 \times 6}, \\
 v_{69} &= C(q^3, q^5, q^5, q^5, q^3, q^3)_{6 \times 6}, & v_{6,10} &= C(q, q^7)_{6 \times 2}, \\
 v_{6,11} &= C(q, q^5, q^3, q^7, q^3, q^5)_{6 \times 6}, & v_{77} &= C(1, q^4, q^4, q^6, q^4, q^4)_{6 \times 6}, \\
 v_{78} &= C(q^2, q^4, q^6, q^6, q^4, q^2)_{6 \times 6}, & v_{79} &= C(q^3, q^7, q^5, q^5, q, q^3)_{6 \times 6}, \\
 v_{7,10} &= C(q^3, q^5)_{6 \times 2}, & v_{7,11} &= C(q, q^5, q^5, q^7, q^3, q^3)_{6 \times 6}, \\
 v_{88} &= C(1, q^2, q^4, q^6, q^4, q^2)_{6 \times 6}, & v_{89} &= C(q^3, q^5, q^7, q^5, q^3, q)_{6 \times 6}, \\
 v_{8,10} &= C(q^3, q^5)_{6 \times 2}, & v_{8,11} &= C(q, q^3, q^5, q^7, q^5, q^3)_{6 \times 6}, \\
 v_{99} &= C(1, q^4, q^4, q^8, q^4, q^4)_{6 \times 6}, & v_{9,10} &= C(q^4, q^4)_{6 \times 2}, \\
 v_{9,11} &= C(q^2, q^2, q^2, q^6, q^6, q^6)_{6 \times 6}, & v_{10,10} &= C(1, q^2)_{2 \times 2}, \\
 v_{10,11} &= C(q^2, q^6, q^2, q^6, q^2, q^6)_{2 \times 6}, & v_{11,11} &= C(1, q^4, q^4, q^8, q^4, q^4)_{6 \times 6},
 \end{aligned}$$

where $C_{6 \times 2}$ means the 6×2 block of the circulant matrix.

The Smith normal form is

$$\text{diag} (1, (1 - q^2)I_8, (1 - q^2)^2 I_{24}, (1 - q^2)^3 I_{20}, (1 - q^2)^2 (I - q^8)I_6).$$

The left transition matrix is LU and the right transition matrix is $U^t R$.

$$U = \begin{pmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -qI & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -qI & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 q^2 I & -q(I + J^5) & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 q^2 I & -qI & -qI_2^t & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & q^2 I & 0 & -qI & -qI & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & q^2 I_2^t & 0 & -q(I + J^4) & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\
 0 & -q^4 I + q^2 J^4 & 0 & q^3 I - qJ^5 & q^3 I - qJ^4 & -q^2 I & 0 & I_6 & 0 & 0 & 0 & 0 \\
 0 & q^2 J^5 & -q^4 I_2^t & -q(I + J^5) & q^3(I + J^4) & 0 & -q^2 I & 0 & I_6 & 0 & 0 & 0 \\
 0 & -q^3 I & 0 & q^2(I + J) & q^2 I & -qI & 0 & -qJ^2 & -qJ & I_6 & 0 & 0 \\
 0 & 0 & -q^3 I_2 & 0 & q^2 I_2 & 0 & -qI_2 & 0 & 0 & 0 & I_2 & 0 \\
 -q^4 I & q^3(J^4 + J^5) & 0 & -q^2 J^5 & q^2 I & -qI & -qI & 0 & 0 & 0 & 0 & I_6
 \end{pmatrix} \tag{3.4}$$

where

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{6 \times 6} \quad \text{and} \quad \mathbf{I}_2 = \left(\begin{array}{c|c|c|c} 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline \end{array} \right)_{2 \times 6}.$$

$$L = \left(\begin{array}{c|ccc|ccc} I_{33} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_6 & q^2 I_6 & 0 & 0 & -q I_6 \\ 0 & 0 & I_6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_2 & 0 \\ 0 & -q I_6 & q^5 I_6 & 0 & 0 & [2] I_6 \end{array} \right), \quad R = \left(\begin{array}{c|ccc|ccc} I_{33} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_6 & -q^4 I_6 & 0 & 0 & q^5 I_6 \\ 0 & 0 & I_6 & 0 & 0 & -q I_6 \\ \hline 0 & 0 & 0 & I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & -q I_6 & 0 & 0 & [2] I_6 \end{array} \right).$$

where $[2] = 1 + q^2$.

3.4. Pyramids

Square base ($n=4$). Let $\mathcal{A} = \{h_1, h_2, h_3, h_4, h_5\}$ be a hyperplane arrangement in \mathbb{R}^3 of a pyramid with a square base, where h_5 is the hyperplane that forms the base and the remaining hyperplanes are labelled according to Fig. 6.

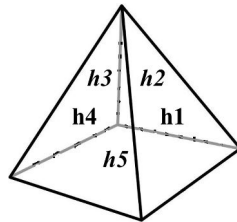


FIGURE 6. Square base pyramid with labelled hyperplanes. Hyperplanes labelled in italics are not visible from this view.

Here the regions are indexed in the following order by the hyperplanes in their separating sets from R_0 :

5	15	25	35	45	125	235	345	145	1235	2345	1345	1245	12345
0	1	2	3	4	12	23	34	41	123	234	341	412	1234

The Smith normal form of the q -Varchenko matrix for this arrangement is

$$\text{diag} (1, (1 - q^2)I_5, (1 - q^2)^2I_{10}, (1 - q^2)^3I_6, (1 - q^2)^2(1 - q^8)),$$

with the left transition matrix $WLTP$ and the right transition matrix $(TP)^tRW^t$.

$$P = \begin{pmatrix} I_9 & -qI_9 & 0 \\ 0 & I_9 & 0 \\ 0 & 0 & I_5 \end{pmatrix}.$$

$$P V_q(\mathcal{A}) P^t = \begin{pmatrix} (1 - q^2)V_q(\mathcal{A}^{h_5}) & 0 \\ 0 & V_q(\mathcal{A} - \{h_5\}) \end{pmatrix}$$

where $V_q(\mathcal{A}^{h_5}) = V_q$ for the C_4 arrangement.

$$V_q(\mathcal{A} - \{h_5\}) =$$

1	q	q	q	q	q^2	q^2	q^2	q^2	q^3	q^3	q^3	q^3	q^4
q	1	q^2	q^2	q^2	q	q^3	q^3	q	q^2	q^4	q^2	q^2	q^3
q	q^2	1	q^2	q^2	q	q	q^3	q^3	q^2	q^2	q^4	q^2	q^3
q	q^2	q^2	1	q^2	q^3	q	q	q^3	q^2	q^2	q^2	q^4	q^3
q	q^2	q^2	q^2	1	q^3	q^3	q	q	q^4	q^2	q^2	q^2	q^3
q^2	q	q	q^3	q^3	1	q^2	q^4	q^2	q	q^3	q^3	q	q^2
q^2	q^3	q	q	q^3	q^2	1	q^2	q^4	q	q	q^3	q^3	q^2
q^2	q^3	q^3	q	q	q^4	q^2	1	q^2	q^3	q	q	q^3	q^2
q^2	q	q^3	q^3	q	q^2	q^4	q^2	1	q^3	q^3	q	q	q^2
q^3	q^2	q^2	q^2	q^4	q	q	q^3	q^3	1	q^2	q^2	q^2	q
q^3	q^4	q^2	q^2	q^2	q^3	q	q	q^3	q^2	1	q^2	q^2	q
q^3	q^2	q^4	q^2	q^2	q^3	q^3	q	q	q^2	q^2	1	q^2	q
q^3	q^2	q^2	q^4	q^2	q	q^3	q^3	q	q^2	q^2	q^2	1	q
q^4	q^3	q^3	q^3	q^3	q^2	q^2	q^2	q^2	q	q	q	q	1

Here the regions are indexed in the following order by the hyperplanes in their separating sets from R_0 :

$$0 \mid 1 \ 2 \ 3 \ 4 \mid 12 \ 23 \ 34 \ 41 \mid 123 \ 234 \ 341 \ 412 \mid 1234$$

$$T = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline -q1 & I_4 & 0 & 0 \\ \hline q^2 1 & -q(I+J) & I_4 & 0 \\ \hline 0 & 0 & 0 & U_2 U_1 \end{array} \right),$$

$$\text{where } U_1 = \left(\begin{array}{c|cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ \hline -q1 & I_4 & 0 & 0 & 0 \\ \hline q^2 1 & -q(I+J) & I_4 & 0 & 0 \\ \hline 0 & q^2 J & -q(I+J) & I_4 & 0 \\ \hline -q^4 & u_{41} & u_{42} & u_{43} & 1 \end{array} \right),$$

$$u_{41} = \begin{pmatrix} q^3 & 0 & 0 & q^3 \end{pmatrix}, \quad u_{42} = \begin{pmatrix} 0 & q^2 & 0 & -q^2 \end{pmatrix}, \quad u_{43} = \begin{pmatrix} -q & -q & 0 & 0 \end{pmatrix};$$

$$U_2 = \left(\begin{array}{c|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_2 & 0 & 0 \\ \hline 0 & 0 & 0 & -q^2 I_2 & I_2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

$$L = \left(\begin{array}{c|ccc} I_{20} & 0 & 0 & 0 \\ \hline 0 & 1 & q^2 & -q \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & -q & q^5 & 1+q^2 \end{array} \right), \quad R = \left(\begin{array}{c|ccc} I_{20} & 0 & 0 & 0 \\ \hline 0 & 1 & -q^4 & q^5 \\ \hline 0 & 0 & 1 & -q \\ \hline 0 & 0 & -q & 1+q^2 \end{array} \right).$$

$$W = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_4 & 0 & 0 & 0 \\ 0 & I_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\ 0 & 0 & I_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Pentagonal base (n=5). $\mathcal{A} = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ is an arrangement in \mathbb{R}^3 of a pyramid with a regular pentagonal base where h_6 forms the base and the remaining hyperplanes are labelled as in Fig. 7.

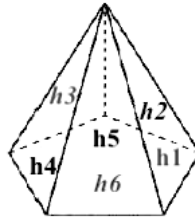


FIGURE 7. Pentagonal base pyramid with labelled hyperplanes. Hyperplanes labelled in italics are not visible from this view.

The Smith normal form is

$$\text{diag} (1, (1 - q^2)I_6, (1 - q^2)^2I_{15}, (1 - q^2)^3I_8, (1 - q^2)^2(1 - q^{10})I_3).$$

with the left transition matrix $WLTP$ and the right transition matrix $(TP)^tRW^t$.

Here the regions are indexed in the following order by the hyperplanes in their separating sets from R_0 :

6	16	26	36	46	56	126	236	346	456	156
0	1	2	3	4	5	12	23	34	45	15
1236	2346	3456	1456	1256	12346	23456	13456	12456	12356	123456
123	234	345	145	125	1234	2345	1345	1245	1235	12345

$$P = \begin{pmatrix} I_{16} & -qI_{16} & 0 \\ 0 & I_{16} & 0 \\ 0 & 0 & I_6 \end{pmatrix}.$$

$$P V_q(\mathcal{A}) P^t = \begin{pmatrix} (1 - q^2)V_q(\mathcal{A}^{h_6}) & 0 \\ 0 & V_q(\mathcal{A} - \{h_6\}) \end{pmatrix}$$

where $V_q(\mathcal{A}^{h_6}) = V_q$ for the C_5 arrangement.

$$V_q(\mathcal{A} - \{h_6\}) = \begin{pmatrix} 1 & Q & Q_2 & Q_3 & Q_4 & q^5 \\ Q^t & v_{11} & v_{12} & v_{13} & v_{14} & Q_4^t \\ Q_2^t & v_{21} & v_{22} & v_{23} & v_{24} & Q_3^t \\ Q_3^t & v_{31} & v_{32} & v_{33} & v_{34} & Q_2^t \\ Q_4^t & v_{41} & v_{42} & v_{43} & v_{44} & Q^t \\ q^5 & Q_4 & Q_3 & Q_2 & Q & 1 \end{pmatrix}$$

where $v_{ji} = v_{ij}^t$ is a row-circulant matrix and $Q_k = (q^k, q^k, q^k, q^k, q^k) = \mathbf{q}_5^k$.

$$\begin{aligned} v_{11} = v_{44} &= C(1, q^2, q^2, q^2, q^2)_{5 \times 5}, & v_{12} = v_{34} &= C(q, q^3, q^3, q^3, q)_{5 \times 5}, \\ v_{13} = v_{24} &= C(q^2, q^4, q^4, q^2, q^2)_{5 \times 5}, & v_{14} &= C(q^3, q^5, q^3, q^3, q^3)_{5 \times 5}, \\ v_{22} = v_{33} &= C(1, q^2, q^4, q^4, q^2)_{5 \times 5}, & v_{23} &= C(q, q^3, q^5, q^3, q)_{5 \times 5}. \end{aligned}$$

Here the regions are indexed in the following order by the hyperplanes in their separating sets from R_0 :

$$0 \mid 1 \ 2 \ 3 \ 4 \ 5 \mid 12 \ 23 \ 34 \ 45 \ 15 \mid 123 \ 234 \ 345 \ 145 \ 125 \mid \\ 1234 \ 2345 \ 1345 \ 1245 \ 1235 \mid 12345$$

$$T = \left(\begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline -q & I_5 & 0 & 0 & 0 \\ \hline q^2 & -q(I + J) & I_5 & 0 & 0 \\ \hline 0 & q^2 J^2 & -q(I + J) & I_5 & 0 \\ \hline 0 & 0 & 0 & 0 & U_2 U_1 \end{array} \right),$$

$$U_1 = \left(\begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline -q & I_5 & 0 & 0 & 0 & 0 \\ \hline q^2 & -q(I + J) & I_5 & 0 & 0 & 0 \\ \hline 0 & q^2 J & -q(I + J) & I_5 & 0 & 0 \\ \hline 0 & -q^5 J^t & q^2 J + q^4 (J^3 + J^t) & -q(I + J) - q^3 J^3 & I_5 & 0 \\ \hline -q^5 & u_{51} & u_{52} & u_{53} & u_{54} & 1 \end{array} \right);$$

$$u_{51} = (q^4 \ 0 \ 0 \ 0 \ q^4), u_{52} = (0 \ 0 \ 0 \ 0 \ -q^3), u_{53} = (0 \ q^2 \ 0 \ 0 \ 0),$$

$$\text{and } u_{54} = (-q \ -q \ 0 \ 0 \ 0);$$

$$\text{and } U_2 = \left(\begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_5 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_5 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_5 & 0 \\ \hline 0 & 0 & 0 & u & 0 & 1 \end{array} \right); u = (0 \ 0 \ 0 \ -q^2(1 - q^2) \ -q^2(1 - q^2)).$$

$$L = \begin{pmatrix} I_{32} & 0 \\ 0 & L_0 \end{pmatrix}, \text{ where}$$

$$L_0 = \begin{pmatrix} 1 & 0 & -1 & -q^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -(1+q^2) & 0 \\ 0 & q^2 & 1 & -(1+q^4 - q^8) & -q^2(1+q^2) & -q(1-q^4) \\ 0 & -q^3 & -q & -q(1-q^2 - q^4 + q^8) & q^3(1+q^2) & 1+q^2 - q^6 \\ 0 & -q^2(1+q^2) & -q^2 & q^2(1-q^6 - q^8) & 1+2q^2+2q^4+q^6 & -q^5(1+q^2) \\ -q^2 & -2q^2 & 0 & 1+q^2 - q^8 & 1+q^2+q^4 & -q(1+q^4) \end{pmatrix}.$$

$$R = \begin{pmatrix} I_{32} & 0 \\ 0 & R_0 \end{pmatrix},$$

where

$$R_0 = (r_1 \vec{r}_2 \vec{r}_3 \vec{r}_4 \vec{r}_5 \vec{r}_6);$$

$$r_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} q^4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 1 + q^4 + q^6 + q^8 \\ q^2(1 + q^2) \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\vec{r}_4 = \begin{bmatrix} q(2 + 2q^4 + 3q^6 + q^8 + 2q^{10} + 2q^{12} + q^{14}) \\ q^3(2 + q^2 + q^4 + 2q^6 + q^8) \\ q(1 + q^6) \\ 0 \\ -q(1 - q^2) \\ 1 \end{bmatrix},$$

$$\vec{r}_5 = \begin{bmatrix} -(1 - 2q^2 - 2q^8 - q^{10}) \\ 1 + q^2 + q^4 + q^6 \\ -(1 - q^4) \\ -1 \\ 1 - q^2 \\ q \end{bmatrix}, \quad \vec{r}_6 = \begin{bmatrix} 1 - q^2 + q^6 \\ -q^2(1 + q^2) \\ 1 + q^2 \\ 1 + q^2 \\ q^2 \\ -q(2 + q^2) \end{bmatrix}.$$

$$W = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_5 & 0 & 0 & 0 \\ 0 & I_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{10} & 0 & 0 \\ 0 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_3 \end{pmatrix}.$$

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CONTACT INFORMATION

Naomi Boulware

Department of Mathematics,
Earlham College,
801 W. National Rd.
Richmond, IN 47374, USA
E-Mail(s): boulwna@earlham.edu

Naihuan Jing

Department of Mathematics,
North Carolina State University,
Raleigh, NC 27695, USA
E-Mail(s): jing@ncsu.edu
Web-page(s): <https://math.sciences.ncsu.edu/people/jing/>

Kailash C. Misra

Department of Mathematics,
North Carolina State University,
Raleigh, NC 27695, USA
E-Mail(s): misra@ncsu.edu
Web-page(s): <https://math.sciences.ncsu.edu/people/misra/>

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