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Some results on the finite rings with maximal size of pairwise non-commuting elements is 5

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Abstract. Let R be a finite ring and let X be a non-empty subset of R. If $ab \neq ba$ for any two distinct $a, b \in X$, then X is called a set of pairwise non-commuting elements of R. Moreover, X is said to be a set of pairwise non-commuting elements of R with maximal size if its cardinality is the largest one among all such sets. In this paper, we study the structures for some finite rings with maximal size of pairwise non-commuting elements is 5.

Let R be a ring. We let $\mathbb{Z}_n$ denote the additive group under modulo n. The centralizer of r in R is defined as $C_R(r) = \{s \in R \mid sr = rs\}$ and the center of R is defined as $Z(R) = \{s \in R \mid sr = rs \text{ for any } r \in R\}$. For any subring S of a ring R, we let R/S to represent the additive factor group of $(R, +)$ by $(S, +)$ and let $|R : S|$ to represent the index of $(S, +)$ in $(R, +)$. The isomorphisms considered in this paper are the additive group isomorphisms. Besides that, we denote $\overline{r} = r + Z(R)$ for any $r \in R$ and denote $\overline{S} = S/Z(R)$ for any $S \leqslant R$ containing $Z(R)$.

Let $\mathrm{Cent}(R)$ denote the set of all distinct centralizers in a ring R, and $\mathrm{Cent}(R) = \{C_R(r) \mid r \in R\}$. A ring R is said to be an n-centralizer

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ring if $|\mathrm{Cent}(R)| = n$, where $n \in \mathbb{N}$. The study of n-centralizer ring was first introduced by Dutta et al. [7] in 2015, and it is published in 2022 (see [10]). By the definition of n-centralizer rings, we note that for any ring R, R is a 1-centralizer ring if and only if R is commutative. In [10], Nath et al. proved that there does not exist any 2-centralizer ring and 3-centralizer ring. They also classified all 4-centralizer and 5-centralizer finite rings. Apart from this, Dutta et al. [8] determined the possible values of $|R : Z(R)|$ for any 6-centralizer and 7-centralizer finite rings. Besides that, in [9] Dutta et al. found some characterization of n-centralizer finite rings for $n \leqslant 7$. In [5], Chan et al. obtained a new characterization for all 6-centralizer and 7-centralizer finite rings. In the same paper, Chan et al. also characterized all n-centralizer finite rings for $n = 8, 9$. Motivated by the study of n-centralizer rings, Chan et al. introduced the notion of (m, n)-centralizer rings in [4] and given some characterizations of the (m, n)-centralizer finite rings for $n \leq 7$.

Let R be a finite ring and let X be a non-empty subset of R. If $ab \neq ba$ for any two distinct $a, b \in X$, then X is called a set of pairwise non-commuting elements of R. Moreover, X is said to be a set of pairwise non-commuting elements of R with maximal size if its cardinality is the largest one among all such sets. The definition of pairwise non-commuting elements of rings is introduced by Dutta et al. in [8]. In the same paper, Dutta et al. have obtained some results regarding set of pairwise non-commuting elements having maximal size. Besides that, they completely determined the characterization for all finite rings with maximal size of pairwise non-commuting elements is t, where $t \in \{3, 4\}$.

In [1], Abdollahi et al. have found some interesting relations between centralizers and pairwise non-commuting elements in groups. By motivated by [1] and [3], we are interested to investigate the relationship between centralizers and pairwise non-commuting elements in rings. In this paper, we study the structure for some finite rings with maximal size of pairwise non-commuting elements is 5. To achieve it, we applied the similar techniques which have been used in [2]. We end the paper with our main result as follows.

Theorem 1. *Let R be a finite ring with maximal size of pairwise non-commuting elements is* 5. *If* $|\mathrm{Cent}(R)| > 6$, *then* $|\mathrm{Cent}(R)| = 16$ *and* $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

1. Preliminary results

Here, we establish some results that are helpful for the proof of the main result.

Lemma 1. *Let R be a finite ring. If $r_1, r_2 \in R \setminus Z(R)$ with $C_R(r_1) \cap C_R(r_2) = Z(R)$, then $|\overline{R}| \leqslant |R : C_R(r_1)||R : C_R(r_2)|$.*

Proof. Obviously, $C_R(r_1) + C_R(r_2) \subseteq R$. Thus, $|C_R(r_1) + C_R(r_2)| \leqslant |R|$ and it follows that, $\frac{|C_R(r_1)||C_R(r_2)|}{|C_R(r_1) \cap C_R(r_2)|} \leqslant |R|$. So, we have $|\overline{R}| \leqslant |R : C_R(r_1)|$ $|R : C_R(r_2)|$. □

Lemma 2. *Let R be a finite ring. Let $r_1, r_2 \in R \setminus Z(R)$ with $r_1 r_2 \neq r_2 r_1$. If $C_R(r_1)$ is commutative and $|R : C_R(r_1)| = p$ for some prime p, then $C_R(r_1) \cap C_R(r_2) = Z(R)$.*

Proof. Let $a \in C_R(r_1) \cap C_R(r_2)$. From the given assumption, we know that $C_R(r_1)$ is commutative. Thus, $C_R(r_1) \leqslant C_R(a)$. Since $r_2 \notin C_R(r_1)$ but $r_2 \in C_R(a)$, then $C_R(r_1) < C_R(a)$. Since $|C_R(r_1)| = \frac{|R|}{p}$, then it is clear that there does not exist any $K < R$ such that $C_R(r_1) < K < R$. So, we are forced to conclude that $C_R(a) = R$. This implies that $a \in Z(R)$. Hence, $C_R(r_1) \cap C_R(r_2) \leqslant Z(R)$. Also, it is obvious that $Z(R) \leqslant C_R(r_1) \cap C_R(r_2)$. Consequently, we obtain $C_R(r_1) \cap C_R(r_2) = Z(R)$. □

Lemma 3. *Let R be a finite ring. Let $r \in R \setminus Z(R)$ with $|\overline{C_R(r)}| = pm$ for some prime p and $m \in \mathbb{N}$. If $C_R(r)$ is non-commutative, then the order of $\overline{r}$ is not m.*

Proof. Suppose that the order of $\overline{r}$ is m. Since $C_R(r)$ is non-commutative, then $\overline{C_R(r)}$ satisfies $\{Z(R)\} < \overline{Z(C_R(r))} < \overline{C_R(r)}$. Since $\overline{r} \in \overline{Z(C_R(r))}$, then $|\overline{Z(C_R(r))}|$ is divisible by m. Hence, $|C_R(r) : Z(C_R(r))| = p$. This leads to $C_R(r)/Z(C_R(r))$ is cyclic. It yields that $C_R(r)$ is commutative; a contradiction. □

Lemma 4. *Let R be a finite ring. Let $r_1 \in R \setminus Z(R)$ with $|\overline{C_R(r_1)}| = p_1 p_2 p_3$ for some primes p_1, p_2, p_3. If $C_R(r_1)$ is non-commutative, then $C_R(r_1) \neq C_R(r_2)$ for any $r_2 \in R \setminus Z(R)$ with $\overline{r_1} \notin < \overline{r_2} >$.*

Proof. Assume that $C_R(r_1) = C_R(r_2)$ for some $r_2 \in R \setminus Z(R)$ with $\overline{r_1} \notin < \overline{r_2} >$. Since $C_R(r_1)$ is non-commutative, then $\overline{C_R(r_1)}$ must satisfies $\{Z(R)\} < \overline{Z(C_R(r_1))} < \overline{C_R(r_1)}$. Suppose that $|\overline{Z(C_R(r_1))}| = p_i p_j$ for two distinct $i, j \in \{1, 2, 3\}$, then $|C_R(r_1) : Z(C_R(r_1))| = p_k$ where $k \in$

$\{1,2,3\}\backslash\{i,j\}$. This yields that $C_R(r_1)$ is cyclic. Consequently, $C_R(r_1)$ is commutative, which is a contradiction. Thus, we have $|\overline{Z(C_R(r_1))}| = p_i$ for some $i \in \{1,2,3\}$. This shows that $\overline{Z(C_R(r_1))}$ is cyclic. So, we obtain $\overline{r_1} \in \overline{Z(C_R(r_1))} = \overline{Z(C_R(r_2))} =< \overline{r_2} >$, which leads to a contradiction. □

In the following, we provide some results for finite rings with $|\overline{R}| = 16$.

Lemma 5. *Let R be a finite ring with $|\overline{R}| = 16$. If $r \in R \setminus Z(R)$ with $|R : C_R(r)| \neq 2$, then $C_R(r)$ is commutative.*

Proof. Suppose that there exists some $r \in R \setminus Z(R)$ with $|R : C_R(r)| \neq 2$ such that $C_R(r)$ is non-commutative. Thus, $|R : C_R(r)| = 4$ or 8. Since $C_R(r)$ is non-commutative, then $C_R(r)$ must satisfies $Z(R) < Z(C_R(r)) < C_R(r) < R$. Obviously, $C_R(r)$ is not satisfies $Z(R) < Z(C_R(r)) < C_R(r) < R$ when $|R : C_R(r)| = 8$. If $|R : C_R(r)| = 4$, then $|C_R(r) : Z(C_R(r))| = 2$. This implies that $C_R(r)/Z(C_R(r))$ is cyclic. It follows that $C_R(r)$ is commutative, which is a contradiction. □

Proposition 1. *Let R be a finite ring with $|\mathrm{Cent}(R)| > 6$ and $|\overline{R}| = 16$. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a set of pairwise non-commuting elements of R with maximal size. If $|R : C_R(x_i)| \neq 2$ for any $i \in \{1,2,3,4,5\}$, then $|\mathrm{Cent}(R)| = 16$.*

Proof. From [8, Proposition 2.4(a)], we have $\overline{R} = \bigcup_{i=1}^{5} \overline{C_R(x_i)}$. Given that $|R : C_R(x_i)| \neq 2$ for any $i \in \{1,2,3,4,5\}$ and $|\overline{R}| = 16$. It follows that $|\overline{C_R(x_i)}| = 4$ for any $i \in \{1,2,3,4,5\}$ and $\overline{C_R(x_i)} \cap \overline{C_R(x_j)} = \{Z(R)\}$ for any two distinct $i,j \in \{1,2,3,4,5\}$. By Lemma 5, $C_R(x_i)$ is commutative for any $i \in \{1,2,3,4,5\}$. Thus, it can be checked that for any $r \in R\backslash Z(R)$, $C_R(r)$ is non-commutative if and only if $|\overline{C_R(r)}| = 8$. Since $|\mathrm{Cent}(R)| > 6$, then there exists some $a_1 \in R \setminus Z(R)$, such that $C_R(a_1)$ is non-commutative with $|\overline{C_R(a_1)}| = 8$. Without loss of generality, we assume that $a_1 \in C_R(x_1)$. Therefore, we have

$$\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{a_1}, \overline{x_1 + a_1}\}$$

and

$$\overline{C_R(a_1)} = \{\overline{0}, \overline{x_1}, \overline{a_1}, \overline{x_1 + a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}\}$$

for some $a_2, a_3, a_4, a_5 \in R \setminus Z(R)$. Now, we claim that $|\overline{C_R(x_i)} \cap A| = 1$ for any $i \in \{2,3,4,5\}$ where $A = \{\overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}\}$. Suppose that $|\overline{C_R(x_i)} \cap$

$A| \geqslant 2$ for some $i \in \{2,3,4,5\}$. So, we have $a_{k_1}, a_{k_2} \in C_R(x_i)$ for two distinct $k_1, k_2 \in \{2,3,4,5\}$. If $|C_R(a_{k_1}) \cap C_R(a_{k_2})| = 4|Z(R)|$, then $a_1 \in C_R(x_1) \cap C_R(a_{k_1}) \cap C_R(a_{k_2}) = C_R(x_1) \cap C_R(x_i) = Z(R)$, which is a contradiction. If $|C_R(a_{k_1}) \cap C_R(a_{k_2})| = 8|Z(R)|$, then $C_R(a_{k_1}) = C_R(a_{k_2})$ with $|\overline{C_R(a_{k_1})}| = 8$, which contradicts with Lemma 3 and Lemma 4. So, our claim is true. Without loss of generality, we let $a_i \in C_R(x_i)$ for any $i \in \{2,3,4,5\}$. Hence, we have

$$\overline{C_R(x_i)} = \{\overline{0}, \overline{x_i}, \overline{a_i}, \overline{x_i + a_i}\}$$

for any $i \in \{2,3,4,5\}$. Since $a_1 \in C_R(a_i)$ but $a_1 \notin C_R(x_i)$ for any $i \in \{2,3,4,5\}$, then $C_R(x_i) < C_R(a_i)$ for any $i \in \{2,3,4,5\}$. So, we note that $|\overline{C_R(a_i)}| = 8$ for any $i \in \{2,3,4,5\}$. Next, we claim that $|\overline{C_R(x_j + a_j)}| = 8$ for some $j \in \{1,2,3,4,5\}$. Assume that $|\overline{C_R(x_i + a_i)}| = 4$ for any $i \in \{1,2,3,4,5\}$. Thus, $C_R(x_i + a_i) = C_R(x_i)$ for any $i \in \{1,2,3,4,5\}$. This implies that

$$\overline{C_R(a_2)} = \{\overline{0}, \overline{x_2}, \overline{a_2}, \overline{x_2 + a_2}, \overline{a_1}, \overline{a_3}, \overline{a_4}, \overline{a_5}\}.$$

This shows that $|\overline{C_R(a_1)} \cap \overline{C_R(a_2)}| = 6$, which contradicts the fact that $|\overline{C_R(a_1)} \cap \overline{C_R(a_2)}|$ is divide $|\overline{R}|$. Consequently, we have $|\overline{C_R(x_j + a_j)}| = 8$ for some $j \in \{1,2,3,4,5\}$, as claimed. By Lemma 3 and Lemma 4, we obtain $C_R(a_j) \neq C_R(x_j + a_j)$. It follows that $\overline{C_R(a_j)} \cap \overline{C_R(x_j + a_j)} = \overline{C_R(x_j)}$. Here, we claim that $|\overline{C_R(a_j)} \cap \{\overline{a_i}, \overline{x_i + a_i}\}| = |\overline{C_R(x_j + a_j)} \cap \{\overline{a_i}, \overline{x_i + a_i}\}| = 1$ for any $i \in \{1,2,3,4,5\} \setminus \{j\}$. Assume to the contrary that $|\overline{C_R(a_j)} \cap \{\overline{a_i}, \overline{x_i + a_i}\}| = 2$ or $|\overline{C_R(x_j + a_j)} \cap \{\overline{a_i}, \overline{x_i + a_i}\}| = 2$ for some $i \in \{1,2,3,4,5\} \setminus \{j\}$. Then, we obtain $a_j \in C_R(x_i)$ or $x_j + a_j \in C_R(x_i)$, which is a contradiction. Thus, our claim is true. So, we note that $x_i + a_i \in C_R(a_j)$ or $x_i + a_i \in C_R(x_j + a_j)$ for any $i \in \{1,2,3,4,5\} \setminus \{j\}$. This implies that $|\overline{C_R(x_i + a_i)}| \geqslant 5$ for any $i \in \{1,2,3,4,5\} \setminus \{j\}$. Hence, we have $|\overline{C_R(x_i + a_i)}| = 8$ for any $i \in \{1,2,3,4,5\} \setminus \{j\}$. Consequently, we obtain $|\mathrm{Cent}(R)| = 1 + 5 + 10 = 16$ by Lemma 3 and Lemma 4, as desired. □

Proposition 2. *Let R be a finite ring with $|\mathrm{Cent}(R)| > 6$ and $|\overline{R}| = 16$. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a set of pairwise non-commuting elements of R with maximal size. Let $m = |\{i \mid i \in \{1,2,3,4,5\}, |R : C_R(x_i)| = 2\}|$. If $m \neq 0$, then $m \geqslant 2$.*

Proof. Suppose that $m = 1$. Without loss of generality, we let $|R : C_R(x_1)| = 2$. Thus, $\overline{C_R(x_1)}$ can be written as

$$\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{a}, \overline{b}, \overline{a + b}, \overline{x_1 + a}, \overline{x_1 + b}, \overline{x_1 + a + b}\}$$

for some $a, b \in R \setminus Z(R)$. By [8, Proposition 2.4(a)], we have $\overline{R} = \overset{5}{\underset{i=1}{\cup}} \overline{C_R(x_i)}$. Hence, we obtain $|\overline{C_R(x_i)}| = 4$ for any $i \in \{2,3,4,5\}$ by [6, Theorem 1]. Here, we claim that $ab \neq ba$. Assume that $ab = ba$, then it is clear that $C_R(x_1)$ is commutative. It follows from Lemma 2 and Lemma 1 that $|\overline{R}| \leqslant 2(4) = 8$; a contradiction. So, we conclude that $ab \neq ba$. By using similar arguments as in the proof of Lemma 1, we obtain $|\overline{C_R(x_1)} \cap \overline{C_R(x_i)}| = 2$ for any $i \in \{2,3,4,5\}$. Then, we note that there exist four elements $w_2, w_3, w_4, w_5 \in \{a, b, a+b, x_1 + a, x_1 + b, x_1 + a + b\}$ such that $\overline{w_i} \in \overline{C_R(x_i)}$ for any $i \in \{2,3,4,5\}$. Let $A = \{a, b, a+b, x_1+a, x_1+b, x_1+a+b\} \setminus \{w_2, w_3, w_4, w_5\}$. It is obvious that all the elements in the set A are non-commute with x_2, x_3, x_4, x_5. Now, we claim that $A = \{u_3, x_1+u_3\}$ for some $u_3 \in \{a, b, a+b\}$. Suppose to the contrary that $A \neq \{w, x_1+w\}$ for any $w \in \{a, b, a+b\}$. Hence, we note that there exist two distinct $u, v \in \{a, b, a+b\}$ such that $u, v \in A$, $u, x_1 + v$ or $x_1 + u, x_1 + v \in A$. So, we have $\{\alpha, \beta, x_2, x_3, x_4, x_5\}$ is a set of pairwise non-commuting elements of R where $\alpha \in \{u, x_1 + u\}$ and $\beta \in \{v, x_1 + v\}$. This contradicts with the fact that the maximal size of pairwise non-commuting elements of R is 5. Therefore, $A = \{u_3, x_1+u_3\}$ for some $u_3 \in \{a, b, a+b\}$, as claimed. Without loss of generality, we have

$$\begin{aligned}
\overline{C_R(x_2)} &= \{\overline{0}, \overline{x_2}, \overline{u_1}, \overline{x_2 + u_1}\},\\
\overline{C_R(x_3)} &= \{\overline{0}, \overline{x_3}, \overline{x_1 + u_1}, \overline{x_1 + x_3 + u_1}\},\\
\overline{C_R(x_4)} &= \{\overline{0}, \overline{x_4}, \overline{u_2}, \overline{x_4 + u_2}\},\\
\overline{C_R(x_5)} &= \{\overline{0}, \overline{x_5}, \overline{x_1 + u_2}, \overline{x_1 + x_5 + u_2}\}
\end{aligned}$$

where $u_1, u_2 \in \{a, b, a+b\} \setminus \{u_3\}$ with $u_1 \neq u_2$. Lemma 5 reminds that $C_R(x_i)$ is commutative for any $i \in \{2,3,4,5\}$. Consequently, we obtain

$$\overline{C_R(u_1)} = \{\overline{0}, \overline{x_1}, \overline{u_1}, \overline{x_1 + u_1}, \overline{x_2}, \overline{x_2 + u_1}, \overline{x_4 + u_2}, \overline{x_1 + x_5 + u_2}\}$$

and

$$\begin{aligned}
\overline{C_R(x_1 + u_1)} \quad = \quad & \{\overline{0}, \overline{x_1}, \overline{u_1}, \overline{x_1 + u_1}, \overline{x_3}, \overline{x_1 + x_3 + u_1}, \overline{x_4 + u_2},\\
& \overline{x_1 + x_5 + u_2}\}.
\end{aligned}$$

This shows that $|\overline{C_R(u_1)} \cap \overline{C_R(x_1 + u_1)}| = 6$. We have reached a contradiction as $|\overline{C_R(u_1)} \cap \overline{C_R(x_1 + u_1)}|$ is divide $|\overline{R}|$. $\square$

Proposition 3. *Let R be a finite ring with $|\mathrm{Cent}(R)| > 6$ and $|\overline{R}| = 16$. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a set of pairwise non-commuting elements of R with maximal size. Let $m = |\{i \mid i \in \{1,2,3,4,5\}, |R : C_R(x_i)| = 2\}|$. If $m \geqslant 2$, then $|\mathrm{Cent}(R)| = 16$.*

Proof. Without loss of generality, we let $|R : C_R(x_1)| = |R : C_R(x_2)| = 2$. By using similar arguments as in the proof of Lemma 1, we obtain $|\overline{C_R(x_1)} \cap \overline{C_R(x_2)}| = 4$. Therefore, $\overline{C_R(x_1)} \cap \overline{C_R(x_2)}$ can be written as $\overline{C_R(x_1)} \cap \overline{C_R(x_2)} = \{\overline{0}, \overline{a}, \overline{b}, \overline{a+b}\}$ for some $a, b \in R \setminus Z(R)$. So, we have

$$\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{a}, \overline{b}, \overline{a+b}, \overline{x_1+a}, \overline{x_1+b}, \overline{x_1+a+b}\},$$
$$\overline{C_R(x_2)} = \{\overline{0}, \overline{x_2}, \overline{a}, \overline{b}, \overline{a+b}, \overline{x_2+a}, \overline{x_2+b}, \overline{x_2+a+b}\}.$$

Now, we claim that $ab \neq ba$. Suppose that $ab = ba$, then it is obvious that $C_R(x_1)$ and $C_R(x_2)$ are commutative. By Lemma 2 and Lemma 1, we obtain $|\overline{R}| \leqslant 2(2) = 4$, which is a contradiction. So, our claim is proved. Thus, we have

$$\begin{aligned}
\overline{C_R(a)} &= \{\overline{0}, \overline{a}, \overline{x_1}, \overline{x_2}, \overline{x_1+x_2}, \overline{x_1+a}, \overline{x_2+a}, \overline{x_1+x_2+a}\}, \\
\overline{C_R(b)} &= \{\overline{0}, \overline{b}, \overline{x_1}, \overline{x_2}, \overline{x_1+x_2}, \overline{x_1+b}, \overline{x_2+b}, \overline{x_1+x_2+b}\}, \\
\overline{C_R(a+b)} &= \{\overline{0}, \overline{a+b}, \overline{x_1}, \overline{x_2}, \overline{x_1+x_2}, \overline{x_1+a+b}, \overline{x_2+a+b}, \\
&\quad \overline{x_1+x_2+a+b}\}, \\
\overline{C_R(x_1+x_2)} &= \{\overline{0}, \overline{x_1+x_2}, \overline{a}, \overline{b}, \overline{a+b}, \overline{x_1+x_2+a}, \overline{x_1+x_2+b}, \\
&\quad \overline{x_1+x_2+a+b}\}.
\end{aligned}$$

Apart from this, we have

$$\overline{C_R(u+v)} = \{\overline{0}, \overline{u}, \overline{v}, \overline{u+v}\}$$

for any $u \in \{x_1, x_2, x_1+x_2\}$ and $v \in \{a, b, a+b\}$. Consequently, we have $|\mathrm{Cent}(R)| = 1 + 2 + 4 + 9 = 16$. This completes the proof. □

2. Main Theorem

Lastly, we give the proof for our main result.

Theorem 2. *Let R be a finite ring with maximal size of pairwise non-commuting elements is 5. If $|\mathrm{Cent}(R)| > 6$, then $|\mathrm{Cent}(R)| = 16$ and $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a set of pairwise non-commuting elements of R with maximal size. By [8, Proposition 2.4(b) and (c)] and [3, Theorem 1.2], we have $|R : Z(R)| \leqslant 16$. It follows that $|R : Z(R)| \geqslant 6$. Since $|R : Z(R)| = 6, 7, 10, 11, 13, 14$ or 15, we have $R/Z(R) \cong \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_{10}, \mathbb{Z}_{11}, \mathbb{Z}_{13}, \mathbb{Z}_{14}$ or $\mathbb{Z}_{15}$, which gives $R/Z(R)$ is cyclic. Hence, R is commutative; a contradiction. If $|R : Z(R)| = 9$, then $R/Z(R) \cong \mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. Since $R/Z(R)$ is non-commutative, then $R/Z(R)$ is not cyclic. Hence, $R/Z(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. It follows from [10, Theorem 2.5] that $|\mathrm{Cent}(R)| = 5$; a contradiction.

Now, we claim that if $|R : Z(R)| = 8$ or 12, then $C_R(r)$ is commutative for any $r \in R \setminus Z(R)$. Suppose that $C_R(r)$ is non-commutative for some $r \in R \setminus Z(R)$, then $C_R(r)$ must satisfies $Z(R) < Z(C_R(r)) < C_R(r) < R$. Therefore, $|C_R(r) : Z(C_R(r))| = 2$ or 3. This implies that $C_R(r)/Z(C_R(r))$ is cyclic, which yields $C_R(r)$ is commutative; a contradiction. From [8, Proposition 2.4(a)], we note that for any $r \in R \setminus Z(R)$, $r \in C_R(x_i)$ for some $i \in \{1, 2, 3, 4, 5\}$. Thus, if given $|R : Z(R)| = 8$ or 12, then for any $r \in R \setminus Z(R), C_R(r) = C_R(x_i)$ for some $i \in \{1, 2, 3, 4, 5\}$. It follows that $|\mathrm{Cent}(R)| = 6$, which is a contradiction. Consequently, we obtain $|R : Z(R)| = 16$.

From Propositions 1, 2 and 3, we obtain $|\mathrm{Cent}(R)| = 16$. We let $R/Z(R) \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then, there exists some $a \in R \setminus Z(R)$ such that the order of $\overline{a}$ is 4 or 8. Since $\gcd(3, \text{order of } \overline{a}) = 1$, then it can be shown that $\overline{C_R(a)} = \overline{C_R(3a)}$. It is clear that $\overline{a} \neq \overline{3a}$, therefore, $|\mathrm{Cent}(R)| < |R : Z(R)|$; a contradiction. So, we conclude that $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. This completes the proof. □

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