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RESEARCH ARTICLE

The comb-like representations of cellular ordinal balleans

Igor Protasov and Ksenia Protasova

Abstract. Given two ordinal λ and γ, let $f : [0, \lambda) \to [0, \gamma)$ be a function such that, for each $\alpha < \gamma$, $\sup\{f(t) : t \in [0, \alpha]\} < \gamma$. We define a mapping $d_f : [0, \lambda) \times [0, \lambda) \longrightarrow [0, \gamma)$ by the rule: if $x < y$ then $d_f(x, y) = d_f(y, x) = \sup\{f(t) : t \in (x, y]\}$, $d(x, x) = 0$. The pair $([0, \lambda), d_f)$ is called a $\gamma-$comb defined by f. We show that each cellular ordinal ballean can be represented as a $\gamma-$comb. In *General Asymptology*, cellular ordinal balleans play a part of ultrametric spaces.

Introduction

In [3], a function $f : [0, 1] \to [0, \infty)$ is called a *comb* if, for every $\varepsilon > 0$, the set $\{t \in [0, 1] : f(t) \geqslant \varepsilon\}$ is finite. Each comb f defines a pseudo-metric d_f on the set $I_f = \{t \in [0, 1] : f(t) = 0\}$ by the rule: if $x < y$ then

$$d_f(x, y) = \max\{f(t) : t \in (x, y)\},$$

$$d_f(y, x) = d_f(x, y), \quad d(x, x) = 0.$$

After some reduced completion of (I_f, d_f), the authors get a compact ultrametric space and show that each compact ultrametric space with no isolated points can be obtained in this way.

In this note, we modify the basic construction from [3] to get the comb-like representations of cellular ordinal balleans which, in *General Asymptology* [7], play a part of ultrametric spaces.

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1. Balleans

Following [5], [7], we say that a *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets, and for all $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radii*.

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we set

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\},$$

$$B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha) \quad \text{and} \quad B^*(A, \alpha) = \bigcup_{a \in A} B^*(a, \alpha).$$

A ball structure $\mathcal{B} = (X, P, B)$ is called a *ballean* if

- for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha') \quad \text{and} \quad B^*(x, \beta) \subseteq B(x, \beta');$$

- for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma);$$

- for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$.

We note that a ballean can be considered as an asymptotic counterpart of a uniform space, and could be defined [8] in terms of the entourages of the diagonal $\Delta_X = \{(x, x) : x \in X\}$ in $X \times X$. In this case a ballean is called a *coarse structure*.

For categorical look at the balleans and coarse structures as "two faces of the same coin" see [2].

Let $\mathcal{B} = (X, P, B)$, $\mathcal{B}' = (X', P', B')$ be balleans. A mapping $f\colon X \to X'$ is called a $\prec$-*mapping* if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that, for every $x \in X$, $f(B(x, \alpha)) \subseteq B'(f(x), \alpha')$.

A bijection $f\colon X \to X'$ is called an *asymorphism* between $\mathcal{B}$ and $\mathcal{B}'$ if f and f^{-1} are $\prec$-mappings. In this case $\mathcal{B}$ and $\mathcal{B}'$ are called *asymorphic*.

Given a ballean $\mathcal{B} = (X, P, B)$, we define a preodering $<$ on P by the rule: $\alpha < \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for each $x \in X$. A subset P' of P is called *cofinal* if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that $\alpha < \alpha'$. A ballean $\mathcal{B}$ is called *ordinal* if there exists a cofinal well-ordered (by $<$) subset P' of P.

For a ballean $\mathcal{B} = (X, P, B)$, $x, y \in X$ and $\alpha \in P$, we say that x and y are α-*path connected* if there exists a finite sequence $x_0, \dots, x_n$, $x_0 = x$,

$x_n = y$ such that $x_{i+1} \in B(x_i, \alpha)$ for each $i \in \{0, \dots, n-1\}$. For any $x \in X$ and $\alpha \in P$, we set

$$B^{\diamond}(x, \alpha) = \{y \in X : x, y \text{ are } \alpha\text{-path connected}\},$$

and say that the ballean $\mathcal{B}^{\diamond} = (X, P, B^{\diamond})$ is a *cellularization* of $\mathcal{B}$. A ballean $\mathcal{B}$ is called *cellular* if the identity $id\colon X \to X$ is an asymorphism between $\mathcal{B}$ and $\mathcal{B}^{\diamond}$.

Each metric space (X, d) defines a metric ballean

$$\mathcal{B}(X, d) = (X, \mathbb{R}^+, B_\alpha),$$

where $B_d(x, r) = \{y \in X : d(x, y) \leqslant r\}$. Clearly, $\mathcal{B}(X, d)$ is ordinal and, if d is an ultrametric then $\mathcal{B}(X, d)$ is cellular.

For examples, decompositions and classification of cellular ordinal balleans see [1], [2], [4], [6].

2. Representations

For ordinals α, β, $\alpha < \beta$, we use the standard notations

$$[\alpha, \beta] = \{t : \alpha \leqslant t \leqslant \beta\}, \qquad [\alpha, \beta) = \{t : \alpha \leqslant t < \beta\},$$
$$(\alpha, \beta] = \{t : \alpha < t \leqslant \beta\}.$$

Let X be a set and γ be an ordinal. We say that a mapping $d\colon X \times X \to [0, \gamma)$ is a γ-*ultrametric* if $d(x, x) = 0$, $d(x, y) = d(y, x)$ and

$$d(x, y) \leqslant \max\{d(x, z), d(z, y)\}.$$

Clearly, each ultrametric space with integer valued metric is an ω-ultrametric space. By [7, Theorem 3.1.1], every cellular metrizable ballean is asymorphic to some ω-ultrametric space.

Given two γ-ultrametric spaces (X, d), (X', d'), a bijection $h\colon X \to X'$ is called an *isometry* if, for any $x, y \in X$, we have

$$d(x, y) = d'(h(x), h(y)).$$

Now let λ, γ be ordinal and $f\colon [0, \lambda) \to [0, \gamma)$ be a function such that, for each $\alpha < \lambda$, $\sup\{f(t) : t \in [0, \alpha]\} < \gamma$. We define a mapping $d_f\colon [0, \lambda) \times [0, \lambda) \to [0, \gamma)$ by the rule: if $x < y$ then

$$d_f(x, y) = d_f(y, x) = \sup\{f(t) : t \in (x, y]\}, d(x, x) = 0,$$

and say that $([0, \lambda), d_f)$ is a γ-*comb* determined by f. Evidently, each γ-comb is a γ-ultrametric space.

Theorem. *Every γ-ultrametric space (X, d) is isometric to some γ-comb $([0, \lambda), d_f)$.*

Proof. We proceed on induction by γ. For $\gamma = 1$, we just enumerate X as $[0, \lambda)$ and take $f \equiv 0$. Assume that we have proved the statement for all ordinals less than γ and consider two cases.

Case 1. Let γ is not a limit ordinal, so $\gamma = \gamma' + 1$. We partition $X = \bigcup\{X_\delta : \delta \in [0, \nu)\}$ into classes of the equivalence $\sim$ defined by $x \sim y$ if and only if $d(x, y) < \gamma'$. If $\delta < \delta' < \nu$ and $x \in X_\delta$, $y \in X_{\delta'}$ then $d(x, y) = \gamma'$.

By the inductive hypothesis, each X_δ is isometric to some γ'-comb $([0, \lambda_\delta), d_{f_\delta})$. We replace inductively each $\delta \in [0, \nu)$ with consecutive intervals $\{[l_\delta, l_\delta + \lambda_\delta) : \delta \in [0, \nu)\}$, $l_0 = 0$ and define a function $f\colon [0, \lambda) \to [0, \gamma)$, $[0, \lambda) = \bigcup\{[l_\delta,\ l_\delta + \lambda_\delta) : \delta \in [0, \nu)\}$ as follows. We put $f = f_0$ on $[0, \lambda_0)$. If $\delta > 0$ then we put $f(l_\delta) = \gamma'$ and $f(l_\delta + x) = f_\delta(x)$ for $x \in (0, \lambda_\delta)$.

After $|\nu|$ steps, we get the desired γ-comb $([0, \lambda), d_f)$.

Case 2. γ is a limit ordinal. We fix some $x_0 \in X$ and, for each $\delta < \gamma$, denote $X_\delta = \{x \in X : d(x_0, x) < \delta\}$. By the inductive hypothesis, there is an isometry $h_\delta\colon X_\delta \to ([0, \lambda_\delta), d_{f_\delta})$. Moreover, in view of Case 1, $f_{\delta+1}$ and $h_{\delta+1}$ can be chosen as the extensions of f_δ and h_δ. Hence, we can use induction by δ to get the desired γ-comb and isometry. □

Every γ-ultrametric space (X, d) can be considered as the ballean $(X, [0, \gamma), B_d)$, where $B_d(x, \alpha) = \{y \in X : d(x, y) \leqslant \alpha\}$.

On the other hand, let (X, P, B) be a cellular ordinal ballean. We may suppose that $P = [0, \gamma)$ and $B(x, \alpha) = B^\diamond(x, \alpha)$ for all $x \in X$, $\alpha \in [0, \gamma)$. We define a γ-ultrametric d on X by $d(x, y) = \min\{\alpha \in [0, \gamma) : y \in B(x, \alpha)\}$. Then (X, P, B) is asymorphic to (X, d).

Corollary. *Every cellular ordinal ballean is asymorphic to some γ-comb.*

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Contact information

I. V. Protasov,
K. D. Protasova

Taras Shevchenko National University of Kyiv, Department of Cybernetics, Volodymyrska 64, 01033, Kyiv Ukraine

E-Mail(s): i.v.protasov@gmail.com, k.d.ushakova@gmail.com

Web-page(s): do.unicyb.kiev.ua/index.php/uk/2014-08-31-19-03-19/38, is.unicyb.kiev.ua/uk/staff.protasova.html

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