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Finite intersection of valuation overrings of polynomial rings in at most three variables

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Abstract. The group of divisibility of an integral domain is the multiplicative group of nonzero principal fractional ideals of the domain and is a partially ordered group under reverse inclusion. We study the group of divisibility of a finite intersection of valuation overrings of polynomial rings in at most three variables and we classify all semilocal lattice-ordered groups which are realizable over $k[x_1, x_2, ..., x_n]$ for $n \leq 3$.

Introduction

Let R be an integral domain with quotient field K, and let $K^* = K - \{0\}$. The *group of divisibility* $G(R)$ of R is defined as the multiplicative group of nonzero principal fractional ideals of R. The ring R is the identity element of $G(R)$. We define a partial order on $G(R)$ by setting $xR \leq yR$ if and only if $yR \subseteq xR$. For background on the group of divisibility see [10]. If R is a Bézout domain (meaning every finitely generated ideal is principal), then the group of divisibility $G(R)$ is an lattice-ordered group (ℓ-group).

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Let D be an integral domain with quotient field K. Any ring R such that $D \subseteq R \subseteq K$, is called an *overring* of D. If D is integrally closed, then D is the intersection of all valuation overrings of D [4, Theorem 3.1.3]. A domain D is called a *semilocal domain* if the domain D has a finitely many maximal ideals. If D is a finite intersection of valuation domains over the same field K, then D is a semilocal Bézout domain [5, Theorem 1.7 Chapter II and Theorem 5.1 Chapter II]. In the case of semilocal Bézout domain, the group of divisibility is a semilocal ℓ-group (having a finite number of maximal filters) [3, Theorem 7]. In this article, we study the group of divisibility of a finite intersection of valuation overrings of a polynomial ring in at most three variables.

We recall that a *valuation* on K is a mapping ν of K onto a totally ordered group $G \cup \{\infty\}$, where ∞ is a symbol such that $g+\infty = \infty+g = \infty+\infty = \infty$ and $g < \infty$ for all $g \in G$, for which the following conditions are satisfied.

(i) $\nu(K - \{0\}) = G$, $\nu(0) = \infty$.

(ii) $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in K$.

(ii) $\nu(x + y) \geq \inf\{\nu(x), \nu(y)\}$ for all x, $y \in K$.

Then $R_\nu = \{x \in K^* : \nu(x) \geq 0\} \cup \{0\}$ is a subring of K. Moreover, $G(R_\nu) \cong G$ [2, p. 103]. The domain R_ν is called a *valuation domain.* If $G \cong \mathbb{Z} \times_\ell \mathbb{Z} \times_\ell \cdots \times_\ell \mathbb{Z}$, then G is called *discrete value group*, where the product $\times_\ell$ is a *lexicographic product*. We define the *lexicographic order* on $\mathbb{Z} \times_l \mathbb{Z} \times_l ... \times_l \mathbb{Z}$ as follows: $(\alpha_1, \alpha_2, ..., \alpha_n) \geq (\beta_1, \beta_2, ..., \beta_n)$ if $\alpha_1 > \beta_1$ or if for some $k > 1, \alpha_i = \beta_i$ for $i = 1, 2, ..., k-1$ and $\alpha_k > \beta_k$. If $G(R_\nu) \cong \mathbb{Z} \times_\ell \mathbb{Z} \times_\ell \cdots \times_\ell \mathbb{Z}$, then R_ν is called a *discrete valuation ring.* If $G \cong \mathbb{Z}$, then R_ν is called a rank one discrete valuation ring DVR. Two valuation rings V_1 and V_2 of K are said to be *independent* if K is the only common overring of both V_1 and V_2. Otherwise, V_1 and V_2 are *dependent.* Any valuation ring of Krull dimension one is independent with other incomparable valuation rings.

Let $\{G_i : i \in I\}$ denote a collection of lattice-ordered groups. The group $\prod_{i \in I} G_i$ with pointwise ordering is an ℓ-group called the *cardinal product* of the G_i. The group $\bigoplus_{i \in I} G_i$ with pointwise ordering is an ℓ-group called the *cardinal sum* of the G_i. For an ℓ-group G the *rational rank* of G is the dimension of $\mathbb{Q} \otimes_{\mathbb{Z}} G$ as a vector space over $\mathbb{Q}$ and is denoted by rat.rank(G). The *rank* of totally ordered group G is the order type of the set of proper convex subgroups of G. For a valuation ν we denote by *rank*(G_ν) the rank of its totally ordered value group G_ν. The Krull dimension of a valuation domain is equal to rank of its value group [14, Corollary, p. 5]. We have $\mathrm{rank}(G_\nu) \leq \mathrm{rat.rank}(G_\nu)$ [14, p. 8].

A short exact sequence of partially ordered groups

$$0 \longrightarrow A \stackrel{\alpha}{\longrightarrow} B \stackrel{\beta}{\longrightarrow} C \longrightarrow 0$$

is called *lexicographically exact* if

$$B_+ = \{b \in B : b \geq 0\} = \{b \in B : \beta(b) > 0\} \cup \{\alpha(a) : a \in A, a \geq 0\}.$$

The group B is called a *lexicographic extension (or lex-extension)* of A by C [3, p. 714].

An ℓ-group G is called *realizable* over a domain D if there exists a Bézout overring R of D such that $G(R) \cong G$ as ℓ-groups. By the Krull-Kaplansky-Jaffard-Ohm Theorem every abelian ℓ-group can be realized as the group of divisibility of a Bézout domain. Doering and Lequain in [3, Theorem 12] proved that every finitely generated ℓ-group can be realized as the group of divisibility of a semilocal Bézout overring of a polynomial ring over a field k in infinitely many variables, where each of the valuation rings appearing in the finite intersection has residue field k. In [13, Theorem 4.2], we show that every finitely generated ℓ-group can be realized over a polynomial ring in finitely many variables, where the number of variables depends on the rational rank of ℓ-group. An ℓ-group G is called *weakly realizable over* $k[x_1, x_2, ..., x_n]$ if there exists a Bézout overring R of $k[x_1, x_2, ..., x_n]$ such that G and $G(R)$ admit a lexico-cardinal decomposition (see [3, p. 723]) of the same form. If G is order isomorphic to a group of the form lex-extension of A by B, then to be weakly realizable means, the group $G(R)$ is order isomorphic to a group of the form lex-extension of A by B.

In this work, we characterize the ℓ-groups which appear as the group of divisibility of a finite intersection of valuation overrings of a polynomial ring in at most three variables. Also, we discuss the ℓ-groups which can be realized as the group of divisibility of a finite intersection of valuation overrings of a polynomial ring in at most three variables. In proposition 2, we show ℓ-group which appears as the group of divisibility of a finite intersection of valuation overrings of $k[x]$, and conversely, we show the ℓ-group is realizable over $k[x]$. In theorem 2, we show ℓ-group which appears as the group of divisibility of a finite intersection of valuation overrings of $k[x_1, x_2]$, and conversely, we show the ℓ-group is realizable over $k[x_1, x_2]$. In theorem 3, we show ℓ-group which appears as the group of divisibility of a finite intersection of valuation overrings of $k[x_1, x_2, x_3]$, and conversely, we show the ℓ-group is weakly realizable over $k[x_1, x_2, x_3]$. For two and three variables case, we use weak approximation theorem for dependent valuation rings [3, Theorem 4] to describe group of divisibility and composite of valuations [9, p. 486] to construct valuation overrings. Moreover, we construct each of the valuation rings that appear in a finite intersection to have the same residue field k using the composite of valuations.

1. Preliminaries

Let K be a field. If we have two valuation rings of K with one contained in the other, then the next result shows existence of a lexicographically exact sequence.

Lemma 1 ([3, Lemma 1]). *Let K be a field and let V and W be two valuation rings of K. Let $V \subsetneq W$. Then the sequence*

$$0 \longrightarrow U(W)/U(V) \xrightarrow{\alpha} G(V) \xrightarrow{\beta} G(W) \longrightarrow 0 \tag{1}$$

is a lexicographically exact sequence, where $U(V)$ and $U(W)$ denote the units of the rings V and W respectively. Moreover, if the sequence (1) splits, then it splits lexicographically.

Proposition 1 ([6, Exercise 6, p. 285]). *Let $\{\nu_i\}_{i=1}^n$ be a finite collection of valuations on a field K. For each i, let V_i be the valuation ring of ν_i and let G_i be the value group of ν_i. Assume that the valuation rings V_i are pairwise independent. Then $G(\bigcap_{i=1}^n V_i) \cong \bigoplus_{i=1}^n G_i$.*

In the above proposition, if the valuation rings are dependent, then the map $\phi : G(R) \to \bigoplus_{i=1}^n G(V_i)$ defined by $\phi(xR) = (xV_1, xV_2, ..., xV_n)$ is not surjective [3, p. 711], and hence finding the group of divisibility is more complicated. Doering and Lequain in 1999 introduced a weak approximation theorem for dependent valuation rings [3, Theorem 4]. They showed that if each of the valuation domains in the intersection has a finitely generated value group then the group of divisibility of the intersection can be calculated explicitly.

Let K be a field, and let $\mathscr{F}$ be a set of finite family of valuation rings of K. Let $\mathscr{N}(\mathscr{F}) = \{(V, V^{'}) : V, V^{'} \in \mathscr{F}\} \cup \{K\}$, where $(V, V^{'})$ is the smallest valuation ring that contains both V and $V^{'}$. Let $\sigma, \tau \in \mathscr{N}(\mathscr{F})$. Then σ is called a *predecessor* of τ if $\sigma \supseteq \tau$ and is called *immediate predecessor* if there is no other valuation ring in between σ and τ. Let $H_{\sigma,\tau} = \ker(G(\tau) \to G(\sigma)) = U(\sigma)/U(\tau)$, where the map is the canonical homomorphism with the order induced from the order of $G(\tau)$. The *weighted dependency tree* of $\mathscr{F}$ is defined by $\mathscr{T}(\mathscr{F}; K) := (\mathscr{N}(\mathscr{F}), \{([\sigma, \tau], H_{\sigma,\tau}) : \sigma, \tau \in \mathscr{N}(\mathscr{F}), \tau$ immediate successor of $\sigma\})$. The elements of $\mathscr{N}(\mathscr{F})$ are the nodes of the tree, K is the root, and the elements of $\mathscr{F}$ are the end nodes. The elements $([\sigma, \tau], H_{\sigma,\tau})$ are the weighted edges of the tree. The *dependency dimension* of $\mathscr{F}$ is defined by

$$d = \text{ dependency dimension } (\mathscr{F}; K) = \max\{l_V - 1 : V \in \mathscr{F}\},$$

where

$$l_V = \textit{ cardinality of } \{[\sigma, \tau] : \tau \supseteq V, \sigma \text{ an immediate predecessor of } \tau\}$$

is the length of the line of predecessors of V. More details on the weighted dependency tree can be found in [3].

The following theorem, known as *Weak Approximation Theorem*, shows that the group of divisibility of the intersection of a finite family of valuation rings having the same quotient field with finite dependency dimension can be expressed in terms of cardinal products and lexicographic extensions, where the factor groups in the lexicographic extensions are totally ordered.

Theorem 1 ([3, Theorem 4]). *Let K be a field, $\mathscr{F}$ be a finite family of valuation rings of K, and G be the divisibility group of $\bigcap_{V \in \mathscr{F}} V$. Let $\mathscr{T}(\mathscr{F}; K)$ be the weighted dependency tree of $\mathscr{F}$ and d be the dependency dimension of $\mathscr{F}$. For every node σ in $\mathscr{N}(\mathscr{F})$, let $\mathscr{S}(\sigma) := \{\tau \in \mathscr{N}(\mathscr{F});$ τ is an immediate successor of $\sigma\}$. Then G is order isomorphic to a group of the form*

$$\prod_{\sigma_1 \in \mathscr{S}(K)} \Bigg(\textit{lex-extension of }_1 \Big[\prod_{\sigma_2 \in \mathscr{S}(\sigma_1)} \Bigg(\textit{ lex-extension of }_2 \Big[\prod_{\sigma_3 \in \mathscr{S}(\sigma_2)}$$

$$\Big(\cdots \prod_{\sigma_d \in \mathscr{S}(\sigma_{d-1})} \Big(\textit{lex-extension of } {}_d\Big[\prod_{\sigma_{d+1}\in\mathscr{S}(\sigma_d)} H_{\sigma_d,\sigma_{d-1}}\Big]_d \textit{ by } H_{\sigma_{d-1},\sigma_d}\Big)\cdots\Big)\Big]_2 \textit{ by } H_{\sigma_1,\sigma_2}\Big)\Big]_1 \textit{ by } H_{K,\sigma_1}\Big).$$

2. Finite intersections of valuation overrings of $\mathbf{k[x_1, x_2, ..., x_n]}$

We shall discuss those ℓ-groups which arise as the group of divisibility of a semilocal Bézout overring of $k[x_1, x_2, ..., x_n]$ for $n = 1, 2$ and 3. We describe explicitly these ℓ-groups for $n = 1$ and $n = 2$. In this chapter, a field k will be assumed to be an infinite field.

2.1. Valuation overrings of $\mathbf{k[x]}$

Let $D = k[x]$. Then by [4, Theorem 2.1.4], all the valuation overrings of $k[x]$ are obtained by localizing $k[x]$ at some prime ideal P. Let $V = k[x]_P$. Since $k[x]$ is a Noetherian domain, V is a Noetherian valuation domain of Krull dimension ≤ 1, and hence $V = k(x)$ or V is a DVR and $G(V) = \mathbb{Z}$ [7, Proposition 6.4.4]. Let $R = \bigcap_{i=1}^{n} V_i$, where each V_i is a distinct nontrivial valuation overring of $k[x]$. Since $k[x]$ is a PID, each nonzero prime ideal P is generated by a prime element. If P is generated by a linear irreducible polynomial, then each $k[x]_P$ has residue field k. Since each V_i has Krull dimension one, the V_i are independent. The group of divisibility of R is $G(R) = \mathbb{Z} \times_c \mathbb{Z} \times_c \cdots \times_c \mathbb{Z}$ by using the proposition 1. Moreover, we can construct valuation rings appearing in a finite intersection in such a way that each of them has residue field k because k is an infinite field. Thus we have the following proposition.

Proposition 2. *Let k be a field and x be an indeterminate of k. A nonzero semilocal ℓ-group G can be realized over $k[x]$ if and only if $G \cong \mathbb{Z} \times_c \mathbb{Z} \times_c \cdots \times_c \mathbb{Z}$ for some finite number of copies of $\mathbb{Z}$.*

2.2. Valuation overrings of $\mathbf{k[x_1, x_2]}$

Let $D = k[x_1, x_2]$. Then D is a two-dimensional Noetherian domain. The following proposition shows that there exist three types of valuation overrings of D.

Proposition 3 ([1, Theorem 1]). *Each valuation overring of $k[x_1, x_2]$ belongs to one of the following three sets.*

a) Valuation rings with rational value group; i.e., the value group is isomorphic to a subgroup of $\mathbb{Q}$.

b) Valuation rings with finitely generated value group of rank 1 and rational rank 2.

c) Valuation rings with discrete value group of rank two.

The following lemma describes a semilocal ℓ-group realizable over $k[x_1, x_2]$.

Lemma 2. *Let G be a semilocal ℓ-group realizable over $k[x_1, x_2]$. Let $\{V_1, V_2, ..., V_n\}$ be a finite collection of valuation overrings of $k[x_1, x_2]$ such that $G(\bigcap_{i=1}^{n} V_i) = G$. Then the following statement hold.*

(a) If each V_i has rank two, then G is isomorphic to a finite cardinal product of groups of the form $\mathbb{Z} \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z})$.

(b) If each V_i has value group a subgroup of $\mathbb{Q}$, then G is isomorphic to a finite cardinal product of subgroups of $\mathbb{Q}$.

(c) If each V_i has a finitely generated value group which is a subgroup of $\mathbb{R}$ having rational rank two, then G is isomorphic to a finite cardinal product of finitely generated subgroups of $\mathbb{R}$ having rational rank two.

Proof. Denote the set $\{V_1, V_2, ..., V_n\}$ by $\mathscr{F}$. Suppose each V_i has rank two. Let $\mathscr{N}(\mathscr{F}) = \left\{(V, V'); V, V' \in \mathscr{F}\right\} \cup \{k(x_1, x_2)\}$, where (V, V') is the smallest valuation ring that contains both V and V'. Let $\mathscr{F}_1, \mathscr{F}_2, ..., \mathscr{F}_m, m \leq n$, be the dependency classes of $\mathscr{F}$. For each $j = 1, 2, ..., m$, let W_j be a nontrivial valuation ring in $\mathscr{N}(\mathscr{F})$ that contains all the valuation rings in $\mathscr{F}_j$. Since W_j is an overring of V for some $V \in \mathscr{F}$, $G(W_j) \cong G(V)/H$, where H is a nonzero convex subgroup of $G(V)$ [14, Proposition 1.11]. Since $G(V) = \mathbb{Z} \times_\ell \mathbb{Z}$, we have $G(W_j) = \mathbb{Z}$. Moreover, for each $V \in \mathscr{F}_j$, $H_{W_j,V} = \ker(G(V) \to G(W_j))$ and $H_{W_j,V} = \mathbb{Z}$ since $\ker(G(V) \to G(W_j))$ is a nonzero convex subgroup of $G(V)$. Also, $H_{k(x_1,x_2),W_j} = \ker(G(W_j) \to G(k(x_1, x_2))) = G(W_j) = \mathbb{Z}$ since

$G(k(x_1,x_2)) = 0$. Let $S_j = \bigcap_{V\in\mathscr{F}_j} V$. Let $n_j = |\mathscr{F}_j|$. Then by Theorem 1,

$$G(S_j) \text{ is a lex-extension of } \prod_{V\in\mathscr{F}_j}{}_c H_{W_j,V} \text{ by } H_{k(x_1,x_2),W_j},$$

which is order isomorphic to a group of the form

$$\text{lex-extension of } (\mathbb{Z} \times_c \mathbb{Z} \times_c \dots \times_c \mathbb{Z}) \text{ by } \mathbb{Z}.$$

Since $\mathbb{Z}$ is projective, the lex-exact sequence splits, so $G(S_j) = \mathbb{Z} \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c \dots \times_c \mathbb{Z})$, where the cardinal product has n_j copies of $\mathbb{Z}$.

Now, by Theorem 1,

$$\begin{aligned} G(R) &= \prod_{j=1}^{m}{}_c G(S_j) \\ &= \prod_{j=1}^{m}{}_c \mathbb{Z} \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c \dots \times_c \mathbb{Z}), \end{aligned}$$

where each cardinal product has n_j copies of $\mathbb{Z}$. This proves (a).

Suppose each V_i has value group a subgroup of $\mathbb{Q}$. Since the V_i are distinct and have dimension one, the V_i are independent. So by Proposition 1,

$$G(\bigcap_{i=1}^{n} V_i) = G(V_1) \times_c G(V_2) \times_c \dots \times_c G(V_n).$$

Thus $G(\bigcap_{i=1}^{n} V_i)$ is a finite cardinal product of subgroups of $\mathbb{Q}$. This proves (b).

Suppose each V_i has a finitely generated value group which is a subgroup of $\mathbb{R}$ having rational rank two. Since the V_i are distinct and rank one, the V_i are independent. Then by Proposition 1,

$$G(\bigcap_{i=1}^{n} V_i) = G(V_1) \times_c G(V_2) \times_c \dots \times_c G(V_n).$$

So $G(\bigcap_{i=1}^{n} V_i)$ is a finite cardinal product of subgroups of $\mathbb{R}$ having rational rank two. This proves (c). □

The next theorem describes the group of divisibility of a finite intersection of valuation overrings of $k[x_1, x_2]$.

Theorem 2. *Let k be an infinite field. A semilocal ℓ-group G can be realized over $k[x_1, x_2]$ if and only if $G = G_1 \times_c G_2 \times_c G_3$, where each G_i, if nonzero, is a semilocal ℓ-group such that*

- *G_1 is isomorphic to a finite cardinal product of subgroups of $\mathbb{Q}$,*
- *G_2 is isomorphic to a finite cardinal product of finitely generated subgroups of $\mathbb{R}$ having rational rank two, and*
- *G_3 is isomorphic to a finite cardinal product of ℓ-groups of the form $\mathbb{Z} \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z})$.*

Moreover, each of valuation rings appearing in the finite intersection that realizes G can be chosen with residue field k.

Proof. Let G be an ℓ-group and suppose $G = G_1 \times_c G_2 \times_c G_3$, where G_1, G_2 and G_3 are as in theorem, and possibly any of the G_i are 0.

If $G_1 = 0$, let $V = k(x_1, x_2)$ so that $G(V) = 0 = G_1$.

Assume that $G_1 \neq 0$ and write $G_1 = H_1 \times_c H_2 \times_c ... \times_c H_n$, where $H_1, H_2, ..., H_n$ are subgroups of $\mathbb{Q}$. First, we want to realize H_i.

Let $a_1, a_2, ..., a_n$ be distinct elements of k. If $H_i \cong \mathbb{Z}$, then we can construct a DVR overring of $k[x_1, x_2]$ whose maximal ideal is generated by (x_1+a_i) as follows. There exists $y \in (x_1+a_i)k[[x_1+a_i]]$ transcendental over $k(x_1 + a_i)$ [3, Lemma 13]. Let $K = k(x_1 + a_i, y)$. Then $S_i^{'} = k[[x_1 + a_i]] \cap K$ is a DVR of K. Moreover,

$$
\begin{aligned}
k &\hookrightarrow S_i^{'}/\mu_{S_i^{'}} \\
&= S_i^{'}/\left(S_i^{'} \cap (x_1 + a_i)k[[x_1 + a_i]]\right) \\
&\cong \left(S_i^{'} + (x_1 + a_i)k[[x_1 + a_i]]\right)/(x_1 + a_i)k[[x_1 + a_i]] \\
&\subseteq k[[x_1 + a_i]]/(x_1 + a_i)k[[x_1 + a_i]] \\
&= k.
\end{aligned}
$$

This shows $S_i^{'}$ has residue field k. Let $\psi : K \to k(x_1, x_2)$ be the field isomorphism defined by $\psi(x_1 + a_i) = x_1 + a_i$ and $\psi(y) = x_2$. Then $S_i = \psi(S_i^{'})$ is a rank one discrete valuation overring of $k[x_1, x_2]$ having k as residue field.

If H_i is not isomorphic to $\mathbb{Z}$, that is, H_i is not finitely generated, then we realize H_i as follows. Let $S = k[x_1 + a_i]_{(x_1+a_i)}$. Then S is a DVR

[15, Corollary 2, page 42] and $G(S)=\mathbb{Z}$. Let ν be the corresponding valuation. Since $H_i \subseteq \mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ is an infinite torsion group, then $H_i/\mathbb{Z}$ is an infinite torsion group, since otherwise H_i will be finitely generated and hence isomorphic to $\mathbb{Z}$. Then by [8, Proposition 3.17], there exists an extension ω of ν to $k(x_1+a_i, x_2)$ such that ω has value group H_i, residue field k and $\omega(x_2)>0$. Denote by S_i the corresponding valuation ring. Let μ_{S_i} denote the maximal ideal of S_i. Since $x_1+a_i, x_2 \in \mu_{S_i}$, then $\mu_{S_i} \cap k[x_1, x_2] = (x_1+a_i, x_2)$.

Let $R = \bigcap\limits_{i=1}^{n} S_i$. Observe that for $j \neq i \in \{1,2,...,n\}, (x_1+a_j, x_2) = \mu_{S_j} \cap k[x_1,x_2] \neq \mu_{S_i} \cap k[x_1,x_2] = (x_1+a_i, x_2)$. Thus the S_i are distinct. Since each S_i has Krull dimension one, the S_i are independent, so by Proposition 1, the group of divisibility of R_1 is

$$\begin{aligned} G(R_1) &= G(S_1) \times_c G(S_2) \times_c ... \times_c G(S_n) \\ &= H_1 \times_c H_2 \times_c ... \times_c H_n \\ &= G_1. \end{aligned} \tag{1}$$

Next, we realize G_2, where $G_2 \neq 0$. Suppose $G_2 = A_1 \times_c A_2 \times_c ... \times_c A_m$, where for each $j = 1,2,...,m$, A_j is a finitely generated subgroup of $\mathbb{R}$ having rational rank two.

Let $c_1, c_2, ..., c_m$ be distinct elements of $k - \{a_1, a_2, ..., a_n\}$. Since A_j is a finitely generated subgroup of $\mathbb{R}$ having rational rank two, we can write $A_j = \mathbb{Z} + r_j\mathbb{Z}$, where r_j is an irrational number. Then we can realize A_j over $k[x_1, x_2]$ by a valuation ring W_j centered on (x_1+c_j, x_2) and having residue field k [9, p. 512].

Let $R_2 = \bigcap\limits_{j=1}^{m} W_j$. Observe that for $i \neq j \in \{1,2,...,m\}, (x_1+c_i, x_2) = \mu_{W_i} \cap k[x_1,x_2] \neq \mu_{W_j} \cap k[x_1,x_2] = (x_1+c_j, x_2)$. Thus the W_j are distinct. Since each W_j has Krull dimension one, the W_j are independent, so by Proposition 1, the group of divisibility of R_2 is

$$\begin{aligned} G(R_2) &= G(W_1) \times_c G(W_2) \times_c ... \times_c G(W_m) \\ &= A_1 \times_c A_2 \times_c ... \times_c A_m \\ &= G_2. \end{aligned} \tag{2}$$

Finally, we realize G_3 when $G_3 \neq 0$. Suppose $G_3 = B_1 \times_c B_2 \times_c ... \times_c B_q$, where $B_t = \mathbb{Z} \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z})$ for each $t = 1,2,...,q$. Assume that B_t has n_t copies of $\mathbb{Z}$ in the cardinal product.

Let $e_1, e_2, ..., e_q$ be nonzero elements in $k-\{a_1, a_2, ..., a_n, c_1, c_2, ..., c_m\}$. Let $p = (x_1 + e_t)$. Then $V_t = D_p = k[x_1, x_2]_{(x_1+e_t)}$ is a DVR [15, Corollary 2, p. 42]. Let $\phi : V_t \to V_t/\mu_{V_t}$ be the canonical homomorphism, where μ_{V_t} denotes the maximal ideal of V_t. Now, we define a valuation on the field V_t/μ_{V_t}. Let $\bar{x}_2 = x_2 + \mu_{V_t}$. Clearly, $k[\bar{x}_2] \subseteq V_t/\mu_{V_t}$. Let $V'_{ti} = k[\bar{x}_2]_{(\bar{x}_2+\alpha_i)}$, where $\alpha_i \in k$ are distinct for $i = 1, 2, ..., n_t$. Then each V'_{ti} has residue field k. Since the V'_{ti} are independent, $V'_{ti}V'_{tj} = k(\bar{x}_2)$ for $i \neq j$.

Let $V_{ti} = \phi^{-1}(V'_{ti})$. Then $V_{ti} \subset V_t$ and by [14, p. 9], the group of divisibility of V_{ti} is $G(V_{ti}) = \mathbb{Z} \times_\ell \mathbb{Z}$ and the residue field of V_{ti} is k. By construction, each V_{ti} contains $k[x_1, x_2]$ and the V_{ti} are dependent but they are centered on different maximal ideals of $k[x_1, x_2]$.

Let $T_t = V_{t1} \cap V_{t2} \cap \cdots \cap V_{tn_t}$. Let $H_{V_t,V_{ti}} = \ker(G(V_{ti}) \to G(V_t))$. Then by [14, Proposition 1.11], $H_{V_t,V_{ti}}$ is a nonzero convex subgroup of $\mathbb{Z} \times_\ell \mathbb{Z}$ because V_t is an overring of V_{ti}. Thus $H_{V_t,V_{ti}} = 0 \times_\ell \mathbb{Z}$ for all $i = 1, 2, ..., n_t$. Since $G(k(x_1, x_2)) = 0$, $H_{k(x_1,x_2),V_t} = \ker(G(V_t) \to G(k(x_1, x_2))) = G(V_t) = \mathbb{Z}$. Then by Theorem 1, where the dependency dimension is $d = 2 - 1 = 1$, the group of divisibility of T_t is

$$G(T_t) \quad = \quad G(V_{t1} \cap V_{t2} \cap \cdots \cap V_{tr})$$

and $G(T_t)$ is order isomorphic to a group of the form

$$\text{lex-extension of } (H_{V_t,V_{t1}} \times_c H_{V_t,V_{t2}} \times_c ... \times_c H_{V_t,V_{tr}}) \text{ by } H_{k(x_1,x_2),V_t}.$$

The group $G(T_t)$ is then order isomorphic to a group of the form

$$\text{lex-extension of } (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z}) \text{ by } \mathbb{Z}.$$

Since $\mathbb{Z}$ is projective, this lex-exact sequence splits, so

$$G(T_t) = \mathbb{Z} \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z}) = B_t.$$

We have constructed $T_1, T_2, ..., T_q$ such that the valuation domains $V_1, V_2, ..., V_q$ are independent. Let $R_3 = \bigcap_{t=1}^{q} B_t$. Then by Theorem 1,

$$\begin{aligned} G(R_3) &= G(T_1) \times_c G(T_2) \times_c ... \times_c G(T_q) \\ &= B_1 \times_c B_2 \times_c ... \times_c B_q \\ &= G_3. \end{aligned}$$

Let $R = R_1 \cap R_2 \cap R_3 = \left(\bigcap_{i=1}^{n} S_i\right) \cap \left(\bigcap_{j=1}^{m} W_j\right) \cap \left(\bigcap_{t=1}^{q} B_t\right)$.

Since the S_i and W_j are distinct and each of them has rank one, the valuation rings in $\{S_1, S_2, ..., S_n, W_1, W_2, ..., W_m\}$ are independent. Also, for any $t \in \{1, 2, ..., q\}, x_1 + e_t$ belongs to the height one prime ideal of B_t and is a unit in S_i and W_j for all $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., m\}$. Thus the V_{ti} are independent with the S_i and W_j. Then by Theorem 1,

$$\begin{aligned} G(R) &= G(R_1) \times_c G(R_2) \times_c G(R_3) \\ &= G_1 \times_c G_2 \times_c G_3 \\ &\cong G. \end{aligned}$$

This shows G is realizable.

Conversely, suppose G is nonzero and realizable over $k[x_1, x_2]$. Let R be a semilocal Bézout overring of $k[x_1, x_2]$ such that $G(R) = G$. Let $R = \bigcap_{m \in M} R_m$, where M is the collection of all maximal ideals of R. Since R is semilocal, we may write $M = \{m_1, m_2, ..., m_n\}$. For each $i = 1, 2, ..., n$, let $V_i = R_{m_i}$. Denote the set $\{V_1, V_2, ..., V_n\}$ by $\mathscr{F}$.

From Proposition 3, there exist three types of valuation overrings of $k[x_1, x_2]$. Then by Lemma 2,

- If each V_i has rank two, then G is isomorphic to a finite cardinal product of groups of the form $\mathbb{Z} \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z})$.
- If each V_i has value group a subgroup of $\mathbb{Q}$, then G is isomorphic to a finite cardinal product of subgroups of $\mathbb{Q}$.
- If each V_i has a finitely generated value group which is a subgroup of $\mathbb{R}$ having rational rank two, then G is isomorphic to a finite cardinal product of finitely generated subgroups of $\mathbb{R}$ having rational rank two.

Now assume that the collection $\{V_1, V_2, ..., V_n\}$ contains valuation domains appearing in Proposition 3. We relabel the V_i so that $V_1, V_2, ..., V_p$ have rational value group, $V_{p+1}, V_{p+2}, ..., V_{p+m}$ have real value group having rational rank two, and $V_{p+m+1}, ..., V_n$ have discrete value group of rank two. Since distinct valuation rings of rank one are independent, $V_1, V_2, ..., V_{p+m}$ are independent. Moreover, $V_1, V_2, ..., V_{p+m}$ are independent with $V_{p+m+1}, ..., V_n$ since $V_1, V_2, ..., V_n$ are incomparable and $V_1, V_2, ..., V_{p+m}$ have dimension one. Then by Theorem 1,

$$\begin{aligned} G(R) = {} & (G(V_1) \times_c G(V_2) \times_c ... \times_c G(V_p)) \times_c (G(V_{p+1}) \times_c G(V_{p+2}) \times_c \\ & ... \times_c G(V_{p+m})) \times_c (G(V_{p+m+1} \cap V_{p+m+2} \cap ... \cap V_n)), \end{aligned}$$

where the groups appearing in the first component are subgroups of $\mathbb{Q}$, the groups in the second component are subgroups of $\mathbb{R}$ and the groups in the third component are a finite cardinal product of groups of the form $\mathbb{Z} \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z})$.

If $\mathscr{F}$ contains only two types of valuation rings appearing in Proposition 3, then we have the following cases.

Case I: If each $V_i \in \mathscr{F}$ has rank one, then we relabel the V_i so that $\{V_1, V_2, ..., V_r\}$ have rational value group and $\{V_{r+1}, V_{r+2}, ..., V_n\}$ have real value group of rational rank two. Since the V_i are distinct and of rank one, the V_i are independent. Then by Proposition 1,

$$G(R) = (G(V_1) \times_c G(V_2) \times_c ... \times_c G(V_r)) \times_c (G(V_{r+1}) \times_c ... \times_c G(V_n)),$$

where the groups appearing in the first component are subgroups of $\mathbb{Q}$, and the groups in the second component are subgroups of $\mathbb{R}$.

Case II: If some elements in $\mathscr{F}$ have rational value group and some elements in $\mathscr{F}$ have discrete value group of rank two, then we relabel the V_i so that $V_1, V_2, ..., V_q$ have rational value group and $V_{q+1}, V_{q+2}, ..., V_n$ have discrete value group of rank two. Since distinct valuation rings of rank one are independent, $V_1, V_2, ..., V_q$ are independent. Moreover, $V_1, V_2, ..., V_q$ are independent with $V_{q+1}, V_{q+2}, ..., V_n$ since V_i are incomparable. Then by Theorem 1,

$$\begin{aligned} G(R) &= (G(V_1) \times_c G(V_2) \times_c ... \times_c G(V_q)) \times_c \\ &\quad (G(V_{q+1}) \cap G(V_{q+2}) \cap ... \cap G(V_n)), \end{aligned}$$

where the groups appearing in the first component are subgroups of $\mathbb{Q}$, and the groups in the second component are a finite cardinal product of groups of the form $\mathbb{Z} \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z})$.

Case III: If some elements in $\mathscr{F}$ have real value group of rational rank two and some elements in $\mathscr{F}$ have discrete value group of rank two, then we relabel the V_i so that $V_1, V_2, ..., V_s$ have real value group and $V_{s+1}, V_{s+2}, ..., V_n$ have discrete value group of rank two. Since distinct valuation rings of rank one are independent, $V_1, V_2, ..., V_s$ are independent. Moreover, $V_1, V_2, ..., V_s$ are independent with $V_{s+1}, V_{s+2}, ..., V_n$ since V_i are incomparable. Then by Theorem 1,

$$\begin{aligned} G(R) &= (G(V_1) \times_c G(V_2) \times_c ... \times_c G(V_s)) \times_c \\ &\quad (G(V_{s+1}) \cap G(V_{s+2}) \cap ... \cap G(V_n)), \end{aligned}$$

where the groups appearing in the first component are subgroups of $\mathbb{R}$, and the groups in the second component are a finite cardinal product of groups of the form $\mathbb{Z} \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z})$. □

2.3. Valuation overrings of $\mathbf{k[x_1, x_2, x_3]}$

In this section we discuss the semilocal ℓ-groups which can be realized over $k[x_1, x_2, x_3]$.

The following lemma describes the value group of a valuation overring V of $k[x_1, x_2, x_3]$ having Krull dimension greater than one in terms of the value group of a nontrivial valuation overring of V.

Lemma 3. *Let V and $V^{'}$ be two nontrivial valuation overrings of $k[x_1, x_2, x_3]$ with $V \subsetneq V^{'}$.*

(1) If rank $G(V^{'}) = 1$, then $G(V^{'})$ is isomorphic to a subgroup of $\mathbb{Q}$ or $G(V^{'})$ is isomorphic to $\mathbb{Z} + \alpha\mathbb{Z}$, where α is an irrational number.

(2) If rank $G(V^{'}) = 2$, then $G(V^{'}) \cong \mathbb{Z} \times_\ell \mathbb{Z}$.

(3) If rank $G(V) =$ rat.rank $G(V) = 2$, then either $G(V) \cong \mathbb{Z} \times_\ell H$, where $H \subseteq \mathbb{Q}$, or $G(V)$ is a lex-extension of $\mathbb{Z}$ by H_1, where $H_1 \subseteq \mathbb{Q}$ and H_1 is not finitely generated.

(4) If rank $G(V) = 2$ and rat.rank $G(V) = 3$, then $G(V) \cong (\mathbb{Z}+r\mathbb{Z}) \times_\ell \mathbb{Z}$ or $G(V) \cong \mathbb{Z} \times_\ell (\mathbb{Z} + r\mathbb{Z})$, where r is an irrational number.

(5) If rank $G(V) = 3$, then $G(V) \cong \mathbb{Z} \times_\ell \mathbb{Z} \times_\ell \mathbb{Z}$.

Proof. (1) Since $V \subsetneq V^{'}$, [14, Proposition 1.11] implies there exists a nontrivial convex subgroup $H^{'}$ of $G(V)$ such that $G(V^{'}) \cong G(V)/H^{'}$. If rat.rank $G(V^{'}) = 3$, then this implies rat.rank $G(V) > 3$, which is not possible by [7, Theorem 6.6.7]. Thus rat.rank $G(V^{'}) = 1$ or 2.

If rat.rank $G(V^{'}) = 1$, then $G(V^{'}) \cong H$, where $H \subseteq \mathbb{Q}$.

Suppose rat.rank $G(V^{'}) = 2$. Since $G(V^{'}) \cong G(V)/H^{'}$, and $H^{'}$ is a nontrivial convex subgroup, this implies rat.rank $G(V) = 3$. Then by [7, Theorem 6.6.7], $G(V)$ is finitely generated, so $G(V^{'})$ is finitely generated. Since rat.rank $G(V^{'}) = 2$, and rank $G(V^{'}) = 1$, $G(V^{'}) \cong \mathbb{Z} + \alpha\mathbb{Z}$, where α is an irrational number.

(2) Since $V \subsetneq V^{'}$, [14, Proposition 1.11] implies there exists a nontrivial convex subgroup B_1 of $G(V)$ such that $G(V^{'}) \cong G(V)/B_1$. Since rank $G(V^{'}) = 2$, this implies rank $G(V) = 3$ and hence by [7, Theorem 6.6.7], $G(V) \cong \mathbb{Z} \times_\ell \mathbb{Z} \times_\ell \mathbb{Z}$. Since nontrivial convex subgroups of $G(V)$ have rank either one or two and $G(V^{'})$ has rank two, B_1 has rank one by [14, p. 9], and hence $B_1 \cong \mathbb{Z}$. This implies $G(V^{'}) \cong \mathbb{Z} \times_\ell \mathbb{Z}$.

(3) Suppose rank $G(V) =$ rat.rank $G(V) = 2$. Then by [4, Lemma 2.3.1], there exists a nonzero nonmaximal prime ideal P of V such

that the Krull dimension of V_P is one. If there exists a nontrivial valuation overring of V different from V_P, then the Krull dimension of V will be three, which is not possible since rank $G(V) = 2$. Moreover, V/P is a valuation ring of V_P/PV_P and $G(V/P)$ has rank one by [14, Proposition 1.11]. Thus by Lemma 1, $G(V)$ is a lex-extension of $G(V/P)$ by $G(V_P)$. By [14, p. 9], $G(V_P)$ and $G(V/P)$ both have rational rank one. Then by [14, Proposition 1.11], $G(V_P)$ and $G(V/P)$ both are isomorphic to subgroups of the group of real numbers. Since rat.rank $G(V_P) = 1$, $G(V_P)$ is cyclic or $G(V_P)$ is isomorphic to a noncyclic subgroup of $\mathbb{Q}$. Then we have the following two cases.

Case I: $G(V_P) \cong \mathbb{Z}$. In this case, $G(V)$ is a lex-extension of $G(V/P)$ by $\mathbb{Z}$. Since $\mathbb{Z}$ is a projective $\mathbb{Z}$-module, by Lemma 1 a lex-extension of $G(V/P)$ by $G(V_P) \cong \mathbb{Z}$ splits. Thus $G(V) \cong \mathbb{Z} \times_\ell H$, where $G(V/P) \cong H$ and $H \subseteq \mathbb{Q}$.

Case II: $G(V_P)$ is isomorphic to a subgroup of $\mathbb{Q}$ and is not finitely generated. By Theorem 2, $G(V_P)$ is realizable over $k[x_1, x_2]$. Then the transcendence degree of the quotient field of V_P over k is two. Here, rank $G(V/P) = 1$, and rat.rank $G(V/P) = 1$. We claim $G(V/P) \cong \mathbb{Z}$. Suppose not. Then $G(V/P)$ is isomorphic to a noncyclic subgroup of $\mathbb{Q}$ and hence the transcendence degree of the quotient field of V/P over k is at least two since any subgroup of $\mathbb{Q}$ which is not finitely generated cannot be realizable over $k[y]$, where y is transcendental over k [12, Lemma 4.3]. By using the composition of valuations from [14, Proposition 1.12], the transcendence degree of the quotient field of V over k is four, which is not possible since the transcendence degree of the quotient field of V over k is three. Thus $G(V/P) \cong \mathbb{Z}$ and $G(V)$ is a lex-extension of $\mathbb{Z}$ by $G(V_P)$.

(4) Suppose rat.rank $G(V) = 3$. Then by [7, Theorem 6.6.7], $G(V)$ is finitely generated and by [3, Corollary 9], we can write $G(V)$ as the lexicographic product of finitely generated subgroups of the group of real numbers. Since rank $G(V) = 2$, $G(V) \cong G_1 \times_\ell G_2$, where G_1 and G_2 are finitely generated totally ordered groups of rank one.

If rat.rank $G_1 = 1$, then rat.rank $G_2 = 2$. Since G_1 and G_2 are finitely generated, $G_1 \cong \mathbb{Z}$ and $G_2 \cong \mathbb{Z} + r\mathbb{Z}$, where r is an irrational number. Thus $G(V) \cong \mathbb{Z} \times_\ell (\mathbb{Z} + r\mathbb{Z})$.

Similarly, if rat.rank $G_1 = 2$, then rat.rank $G_2 = 1$, and $G(V) \cong (\mathbb{Z} + r\mathbb{Z}) \times_\ell \mathbb{Z}$.

(5) If rank $G(V) = 3$, then by [7, Theorem 6.6.7], $G(V) \cong \mathbb{Z} \times_\ell \mathbb{Z} \times_\ell \mathbb{Z}$. □

The proposition below shows that the group of divisibility of the intersection of a finite collection of dependent valuation overrings of $k[x_1, x_2, x_3]$ depends on the group of divisibility of a valuation ring which contains all others valuation rings.

Proposition 4. *Let V be a valuation overring of $k[x_1, x_2, x_3]$, and let $\mathscr{F} = \{V_1, V_2, ..., V_n\}$ be a finite collection of incomparable dependent valuation overrings of $k[x_1, x_2, x_3]$ such that for each $i = 1, 2, ..., n, V_i \subsetneq V$. Let $R = \bigcap_{i=1}^{n} V_i$.*

(1) *If for some i, there is a valuation ring W_i such that $V_i \subsetneq W_i \subsetneq V$, then $G(R) \cong \mathbb{Z} \times_\ell A$, where A is a finite cardinal product of one or more of the following groups, which are realizable over $k[x_1, x_2]$.*

- $(\mathbb{Z} + r_1\mathbb{Z}) \times_c (\mathbb{Z} + r_2\mathbb{Z}) \times_c ... \times_c (\mathbb{Z} + r_q\mathbb{Z})$, *where for each $i = 1, 2, ..., q, r_i$ is an irrational number.*
- $H_1 \times_c H_2 \times_c ... \times_c H_p$, *where for each $i = 1, 2, ..., p, H_i \subseteq \mathbb{Q}$.*
- $\mathbb{Z} \times_\ell (\mathbb{Z} \times_c ... \times_c \mathbb{Z})$.

(2) *Suppose that for each i, there are no valuation rings properly between V_i and V. Then*

(a) *If $G(V) \cong \mathbb{Z}$, then $G(R) \cong \mathbb{Z} \times_\ell C$, where C is a finite cardinal product of one or more of the following groups, which are realizable over $k[x_1, x_2]$.*

- $(\mathbb{Z} + r_1\mathbb{Z}) \times_c (\mathbb{Z} + r_2\mathbb{Z}) \times_c ... \times_c (\mathbb{Z} + r_q\mathbb{Z})$, *where for each $i = 1, 2, ..., q, r_i$ is an irrational number.*
- $H_1 \times_c H_2 \times_c ... \times_c H_p$, *where for each $i = 1, 2, ..., p, H_i \subseteq \mathbb{Q}$.*

(b) *If $G(V) \cong \mathbb{Z} + \alpha\mathbb{Z}$, where α is an irrational number, then $G(R) \cong (\mathbb{Z} + \alpha\mathbb{Z}) \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c \cdots \times_c \mathbb{Z})$.*

(c) *If rat.rank $G(V) = 1$ and $G(V)$ is not finitely generated, then $G(R)$ is order isomorphic to a group of the form*

$$\text{lex-extension of } (\mathbb{Z} \times_c \mathbb{Z} \times_c \cdots \times_c \mathbb{Z}) \text{ by } H.$$

(d) *If $G(V) \cong \mathbb{Z} \times_\ell \mathbb{Z}$, then $G(R) \cong (\mathbb{Z} \times_\ell \mathbb{Z}) \times_\ell (\mathbb{Z} \times_c ... \times_c \mathbb{Z})$.*

Proof. (1) Let $W_1, W_2, ..., W_m \notin \mathscr{F}, m \leq n$ be distinct valuation overrings of $k[x_1, x_2, x_3]$ such that for each $j = 1, 2, ..., m, W_j$ contains at least one of the V_i with $W_j \subsetneq V$. Now by Theorem 1, the group of divisibility $G(R)$ is order isomorphic to a group of the form

$$\text{lex-extension of } \left(\prod_{j=1}^{m}\Bigg(\text{ lex-extension of } (\prod_{S\in\mathscr{F}, S\subsetneq W_j} H_{W_j,S})\right.$$
$$\left.\text{by } H_{V,W_j}\Bigg) \times_c \prod_{T\in\mathscr{F}, T\not\subseteq W_j} H_{V,T}\right) \text{ by } G(V), \tag{3}$$

where $H_{V,W_j} = \ker(G(W_j) \to G(V))$, $H_{V,T} = \ker(G(T) \to G(V))$, $H_{W_j,S} = \ker(G(S) \to G(W_j))$, $H_{k(x_1,x_2,x_3),V} = \ker(G(V) \to G(k(x_1,x_2,x_3)))$ and $S, T \in \mathscr{F}$.

Here, by [7, Theorem 6.67], if $S \in \mathscr{F}$ with $S \subsetneq W_j$ for some $j = 1, 2, ..., m$, then S has value group $\mathbb{Z}\times_\ell\mathbb{Z}\times_\ell\mathbb{Z}$, and by Lemma 3, $G(W_j) \cong \mathbb{Z}\times_\ell\mathbb{Z}$. Since $0\times_\ell\mathbb{Z}$ is the only nontrivial convex subgroup of $G(W_j)$ and $W_j \subsetneq V$, we have $G(V) \cong G(W_j)/(0\times_\ell\mathbb{Z}) \cong \mathbb{Z}$. Then for each $T \in \mathscr{F}$ with $T \not\subseteq W_j$ for all j, rank $G(T) = 2$. Since $T \subsetneq V$, there exists a nontrivial convex subgroup B_t of $G(T)$ such that $G(T)/B_t \cong G(V) \cong \mathbb{Z}$. Then by Lemma 1, $G(T) \cong \mathbb{Z} \times_\ell B_t$. Since rank $G(T) = 2$, either $B_t \cong \mathbb{Z}$, B_t is isomorphic to a non-finitely generated subgroup of $\mathbb{Q}$, or $B_t \cong \mathbb{Z} + \gamma\mathbb{Z}$ for some irrational number γ.

Now $H_{V,W_j} \cong \mathbb{Z}$, since $\mathbb{Z}$ is the only nonzero convex subgroup of $G(W_j)$, $H_{V,T} \cong B_t$, since B_t is the only nonzero convex subgroup of $G(T)$, $H_{W_j,S} \cong \mathbb{Z}$, and $H_{k(x_1,x_2,x_3),V} = G(V) \cong \mathbb{Z}$, since $G(k(x_1,x_2,x_3)) = 0$. Since $\mathbb{Z}$ is projective by Lemma 1, the lex-exact sequences in (3) split and hence $G(R)$ is order isomorphic to

$$G(V) \times_\ell \left(\prod_{j=1}^{m}\left(H_{V,W_j} \times_\ell (\prod_{S\in\mathscr{F}, S\subsetneq W_j} H_{W_j,S})\right) \times_c \prod_{T\in\mathscr{F}, T\not\subseteq W_j} H_{V,T}\right) \tag{4}$$

(2)(a) Since $G(V) \cong \mathbb{Z}, V$ is a DVR. Thus there are no valuation rings properly between V and $k(x_1, x_2, x_3)$. Then each valuation ring in $\mathscr{F}$ has Krull dimension two since there are no valuation rings properly between V_i and V. By Theorem 1, the group of divisibility $G(R)$ is order isomorphic to a group of the form

$$\text{lex-extension of } (\prod_{i=1}^{n} H_{V,V_i}) \text{ by } G(V), \tag{5}$$

where $H_{V,V_i} = \ker(G(V_i) \to G(V))$.

Since each $G(V_i)$ has rank two and $V_i \subsetneq V$, there exists a nontrivial convex subgroup B_i of $G(V_i)$ such that $G(V_i)/B_i \cong G(V) \cong \mathbb{Z}$. Then by Lemma 1, $G(V_i) \cong \mathbb{Z} \times_\ell B_i$. Since rank $G(V_i) = 2$, either $B_i \cong \mathbb{Z}$, B_i is isomorphic to a non-finitely generated subgroup of $\mathbb{Q}$, or $B_i \cong \mathbb{Z} + \beta\mathbb{Z}$ for some irrational number β. Also $H_{V,V_i} = B_i$ since B_i is the only nonzero convex subgroup of $G(V_i)$.

By Lemma 1, since $\mathbb{Z}$ is projective, the lex-exact sequence (5) splits and hence $G(R)$ is order isomorphic to $\mathbb{Z} \times_\ell (\prod_{i=1}^{n} H_{V,V_i})$.

(b) Suppose $G(V) \cong \mathbb{Z} + \alpha\mathbb{Z}$, where α is an irrational number. Then by Lemma 3 (4), for each $i = 1, 2, ..., n$, $G(V_i) \cong (\mathbb{Z}+\alpha\mathbb{Z}) \times_\ell \mathbb{Z}$. Let $H_{V,V_i} = \ker(G(V_i) \to G(V))$ and let $H_{k(x_1,x_2,x_3),V} = \ker(G(V) \to G(k(x_1,x_2,x_3)))$. Then $H_{V,V_i} \cong \mathbb{Z}$, since the only nonzero convex subgroup of $G(V_i)$ is cyclic, and $H_{k(x_1,x_2,x_3),V} = G(V) \cong \mathbb{Z} + \alpha\mathbb{Z}$ since $G(k(x_1,x_2,x_3)) = 0$. Then by Theorem 1, the group of divisibility $G(R)$ is order isomorphic to a group of the form

$$\text{lex-extension of } (H_{V,V_1} \times_c H_{V,V_2} \times_c \cdots \times_c H_{V,V_n}) \text{ by } H_{k(x_1,x_2,x_3),V}.$$

The group $G(R)$ is then order isomorphic to a group of the form

$$\text{lex-extension of } (\mathbb{Z} \times_c \mathbb{Z} \times_c \cdots \times_c \mathbb{Z}) \text{ by } (\mathbb{Z} + \alpha\mathbb{Z}) \qquad (b').$$

Since $\mathbb{Z} + \alpha\mathbb{Z}$ is projective, Lemma 1 implies that the lex-exact sequence (b') splits and hence $G(R) \cong (\mathbb{Z} + \alpha\mathbb{Z}) \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z})$.

(c) Suppose rat.rank $G(V) = 1$ and $G(V)$ is not finitely generated. Then $G(V_i)$ is not finitely generated, since otherwise $G(V)$ is finitely generated by [14, Proposition 1.11]. Moreover, rank $G(V_i)$ cannot be three, since otherwise $G(V_i)$ will be finitely generated by Lemma 3. Thus each $G(V_i)$ has rank two. By [14, Proposition 1.11], there exists a nonzero convex subgroup H_i' of $G(V_i)$ such that $G(V) \cong G(V_i)/H_i'$. As in the proof of case II of (3) of Lemma 3, $H_i' \cong \mathbb{Z}$. Lemma 1 implies that $G(V_i)$ is order isomorphic to a group of the form

$$\text{lex-extension of } \mathbb{Z} \text{ by } H.$$

Let $H_{V,V_i} = \ker(G(V_i) \to G(V))$, and let $H_{k(x_1,x_2,x_3),V} = \ker(G(V) \to G(k(x_1,x_2,x_3)))$. Then $H_{V,V_i} \cong \mathbb{Z}$ since the only nontrivial convex subgroup of $G(V_i)$ is cyclic, and $H_{k(x_1,x_2,x_3),V} = H$ since $G(k(x_1,x_2,x_3)) =$

0. Now, by Theorem 1, the group of divisibility $G(R)$ is order isomorphic to a group of the form

$$\text{lex-extension of } (H_{V,V_1} \times_c H_{V,V_2} \times_c \cdots \times_c H_{V,V_n}) \text{ by } H_{k(x_1,x_2,x_3),V}.$$

The group $G(R)$ is then order isomorphic to a group of the form

$$\text{lex-extension of } (\mathbb{Z} \times_c \mathbb{Z} \times_c \cdots \times_c \mathbb{Z}) \text{ by } H.$$

(d) Suppose $G(V) \cong \mathbb{Z} \times_\ell \mathbb{Z}$. Since $V_i \subsetneq V$, rank $G(V_i) = 3$. Then by [7, Theorem 6.6.7], $G(V_i) \cong \mathbb{Z} \times_\ell \mathbb{Z} \times_\ell \mathbb{Z}$. Let $H_{V,V_i} = \ker(G(V_i) \to G(V))$ and let $H_{k(x_1,x_2,x_3),V} = \ker(G(V) \to G(k(x_1,x_2,x_3)))$. Then $H_{V,V_i} \cong \mathbb{Z}$ since $\ker(G(V_i) \to G(V))$ is cyclic, and $H_{k(x_1,x_2,x_3),V} \cong \mathbb{Z} \times_\ell \mathbb{Z}$ since $G(k(x_1,x_2,x_3)) = 0$. Now, by Theorem 1, the group of divisibility $G(R)$ is order isomorphic to a group of the form

$$\text{lex-extension of } (H_{V,V_1} \times_c H_{V,V_2} \times_c \cdots \times_c H_{V,V_n}) \text{ by } H_{k(x_1,x_2,x_3),V}.$$

The group $G(R)$ is then order isomorphic to a group of the form

$$\text{lex-extension of } (\mathbb{Z} \times_c \mathbb{Z} \times_c \cdots \times_c \mathbb{Z}) \text{ by } (\mathbb{Z} \times_\ell \mathbb{Z}).$$

Since the group $\mathbb{Z} \times_\ell \mathbb{Z}$ is projective, Lemma 1 implies that the lex-exact sequence splits and hence $G(R) \cong (\mathbb{Z} \times_\ell \mathbb{Z}) \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c \cdots \times_c \mathbb{Z})$. □

Definition 1. *Let $G(R)$ be the group of divisibility of a semilocal Bézout domain R. Then $G(R)$ is called completely determined by the lexico-cardinal decomposition form of $G(R)$ if each lex-exact sequence appearing in the lexico-cardinal decomposition form of $G(R)$ splits.*

Definition 2. *Let $\mathscr{F} = \{V_1, V_2, ..., V_n\}$ be a finite collection of valuation rings with the same quotient field K and let $\mathscr{N}(\mathscr{F}) = \{(V,W) : V, W \in \mathscr{F}\} \cup \{K\}$. Let $\mathscr{T}(\mathscr{F}; K) := (\mathscr{N}(\mathscr{F}), \{([\sigma,\tau], H_{\sigma,\tau}); \sigma, \tau \in \mathscr{N}(\mathscr{F})\})$, where τ immediate successor of σ be the weighted dependency tree of $\mathscr{F}$ and d the dependency dimension of $\mathscr{F}$. Let $R = \bigcap_{i=1}^{n} V_i$ and let $G(R)$ be the group of divisibility of R. Then $G(R)$ is called completely determined by the weighted dependency tree of $\mathscr{F}$ if $G(R)$ can be expressed as a finite product of sequences of lexicographic product and cardinal product of the totally ordered groups $H_{\sigma,\tau}$.*

The following proposition shows that $G(R)$ being completely determined by the lexico-cardinal decomposition form of $G(R)$ implies $G(R)$ is completely determined by the weighted dependency tree of valuation rings in $\mathscr{F}$.

Proposition 5. *Let $\mathscr{F} = \{V_1, V_2, ..., V_n\}$ be a finite collection of valuation rings with the same quotient field K. Let $R = \bigcap_{i=1}^{n} V_i$. If $G(R)$ is completely determined by the lexico-cardinal decomposition form of $G(R)$, then $G(R)$ is completely determined by the weighted dependency tree of valuation rings in $\mathscr{F}$.*

Proof. Let $R = \bigcap_{i=1}^{n} V_i$ and let $G(R)$ be completely determined by the lexico-cardinal decomposition form of $G(R)$.

Let $\mathscr{N}(\mathscr{F}) = \{(V, W) : V, W \in \mathscr{F}\} \cup \{K\}$. Then by Theorem 1, let

$$\mathscr{T}(\mathscr{F}; K) := (\mathscr{N}(\mathscr{F}), \{([\sigma, \tau], H_{\sigma,\tau}); \sigma, \tau \in \mathscr{N}(\mathscr{F})\sigma\}),$$

τ immediate successor of be the weighted dependency tree of $\mathscr{F}$ and d the dependency dimension of $\mathscr{F}$. For every node $\sigma \in \mathscr{N}(\mathscr{F})$, let $\mathscr{S}(\sigma) := \{\tau \in \mathscr{N}(\mathscr{F}); \tau \text{ is an immediate successor of } \sigma\}$. Then, $G(R)$ is order isomorphic to a group of the form

$$\prod_{\sigma_1 \in \mathscr{S}(K)} \Bigg(\text{lex-extension of } {}_1\Bigg[\prod_{\sigma_2 \in \mathscr{S}(\sigma_1)} \Bigg(\text{lex-extension of } {}_2\Bigg[\prod_{\sigma_3 \in \mathscr{S}(\sigma_2)} \Bigg(... \prod_{\sigma_d \in \mathscr{S}(\sigma_{d-1})} \Bigg(\text{lex-extension of } {}_d\Bigg[\prod_{\sigma_{d+1} \in \mathscr{S}(\sigma_d)} H_{\sigma_d,\sigma_{d-1}}\Bigg]_d \text{ by } H_{\sigma_{d-1},\sigma_d}\Bigg)...\Bigg)\Bigg]_2 \text{ by } H_{\sigma_1,\sigma_2}\Bigg)\Bigg]_1 \text{ by } H_{K,\sigma_1}\Bigg),$$

which is the lexico-cardinal decomposition of $G(R)$.

Since each of the lex-exact sequence splits, then the group $G(R)$ is order isomorphic to

$$\prod_{\sigma_1 \in \mathscr{S}(K)} \Bigg(H_{K,\sigma_1} \times_\ell {}_1\Bigg[\prod_{\sigma_2 \in \mathscr{S}(\sigma_1)} \Bigg(H_{\sigma_1,\sigma_2} \times_\ell {}_2\Bigg[\prod_{\sigma_3 \in \mathscr{S}(\sigma_2)} \Bigg(... \prod_{\sigma_d \in \mathscr{S}(\sigma_{d-1})} \Bigg(H_{\sigma_{d-1},\sigma_d} \times_\ell \Bigg[\prod_{\sigma_{d+1} \in \mathscr{S}(\sigma_d)} H_{\sigma_d,\sigma_{d-1}}\Bigg]_d\Bigg)...\Bigg)\Bigg]_2\Bigg)\Bigg]_1\Bigg). \quad (6)$$

The expression (6) shows that $G(R)$ is completely determined by the weighted dependency tree of $\mathscr{F}$. □

Corollary 1. *Let $\mathscr{F}$ be a finite collection of dependent valuation overrings of $k[x_1, x_2, x_3]$ and let W be a nontrivial valuation overring of $k[x_1, x_2, x_3]$ that contains each $V \in \mathscr{F}$. If $G(W)$ is finitely generated,* *then the group of divisibility* $G(\bigcap_{V\in\mathscr{F}} V)$ *is completely determined by the weighted dependency tree of valuations in* $\mathscr{F}$.

Proof. Let $\mathscr{F} = \{V_1, V_2, ..., V_n\}$. Let $R = \bigcap_{i=1}^{n} V_i$. Then by the proof of (1) of Proposition 4, the group of divisibility of R is order isomorphic to a group of the form

$$\text{lex-extension of } \Bigg(\prod_{j=1}^{m}\Bigg(\text{ lex-extension of } (\prod_{S\in\mathscr{F}, S\subsetneq W_j} H_{W_j,S})$$

$$\text{by } H_{V,W_j}\Bigg) \times_c \prod_{T\in\mathscr{F}, T\not\subseteq W_j} H_{V,T}\Bigg) \text{ by } G(V), \tag{7}$$

Here, W_j may or may not exist. If W_j exists for some j, then by the proof of (1) of Proposition 4, $G(W_j)$ is finitely generated and hence the convex subgroup H_{V,W_j} of $G(W_j)$ is finitely generated. This determines the group of divisibility of R, since the lex-exact sequences appearing in (7) split by Lemma 1.

If the W_j do not exist, then the short exact sequence appearing in (7) splits by Lemma 1, since $G(V)$ is finitely generated. This also determines the group of divisibility of R. □

Corollary 2. *Let* $\mathscr{F}$ *be a finite collection of valuation overrings of* $k[x_1, x_2, x_3]$. *Let* $\mathscr{F}_1, \mathscr{F}_2, ..., \mathscr{F}_r$ *be the dependency classes of* $\mathscr{F}$. *Let* W_i *be a nontrivial valuation overring of* $k[x_1, x_2, x_3]$ *which contains each of the valuation rings in* $\mathscr{F}_i$. *Then the group of divisibility* $G(\bigcap_{V\in\mathscr{F}} V)$ *is completely determined by the weighted dependency tree of the valuations in* $\mathscr{F}$ *if for each* $i = 1, 2, ..., r, G(W_i)$ *is finitely generated.*

Proof. Let $\mathscr{F}_1, \mathscr{F}_2, ..., \mathscr{F}_r$ be the dependency classes of valuation rings in $\mathscr{F}$. For each $i = 1, 2, ..., r$, let $S_i = \bigcap_{V\in\mathscr{F}_i} V$. Let $R = \bigcap_{i=1}^{r} S_i$. By [3, Theorem 3], the group of divisibility $G(R)$ is order isomorphic to $\prod_{i=1}^{r} G(S_i)$, where $G(S_i)$ can be determined as in the Corollary 1. Thus $G(R)$ can be determined. □

The following proposition gives an ℓ-group, which is a group of divisibility of the intersection of valuation overrings of $k[x_1, x_2, x_3]$ having Krull dimension greater than one.

Proposition 6. *Let $\mathscr{F}$ be a finite collection of valuation overrings of $k[x_1, x_2, x_3]$ such that each valuation ring in $\mathscr{F}$ has rank greater than one, and let $R = \bigcap_{V \in \mathscr{F}} V$. Then $G(R)$ is order isomorphic to a group of the form $G_1 \times_c G_2 \times_c G_3$, where*

(1) G_1 is a finite cardinal product of the groups of the form

$$\mathbb{Z} \times_\ell A, \tag{8}$$

where A is a finite cardinal product of one or more of the following groups, which are realizable over $k[x_1, x_2]$.

- $(\mathbb{Z} + r_1\mathbb{Z}) \times_c (\mathbb{Z} + r_2\mathbb{Z}) \times_c ... \times_c (\mathbb{Z} + r_q\mathbb{Z})$, *where $r_1, r_2, ..., r_q$ are irrational numbers.*
- $H_1 \times_c H_2 \times_c ... \times_c H_p$, *for each $i = 1, 2, ..., p, H_i \subseteq \mathbb{Q}$.*
- $\mathbb{Z} \times_\ell (\mathbb{Z} \times_c ... \times_c \mathbb{Z})$,

(2) G_2 is a finite cardinal product of the groups of the form

$$\textit{lex-extension of } (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z}) \textit{ by } H, \tag{9}$$

where $H \subseteq \mathbb{Q}$ and H is not finitely generated, and

(3) G_3 is a finite cardinal product of the groups of the form

$$(\mathbb{Z} + r\mathbb{Z}) \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z}). \tag{10}$$

Proof. Let $\mathscr{F} = \{V_1, V_2, ..., V_n\}$ and let $\mathscr{F}_1, \mathscr{F}_2, ..., \mathscr{F}_m, m \leq n$ denote the set of dependency classes of $\mathscr{F}$. For each $j = 1, 2, ..., m$, let $W_j \in \mathscr{N}(\mathscr{F}_j)$ be a nontrivial valuation overring of $k[x_1, x_2, x_3]$ that contains each $V \in \mathscr{F}_j$.

Then we have the following cases.

Case I: Suppose for each $h = 1, 2, ..., r, G(W_r) \cong \mathbb{Z}$. Then by (1) and (2)$(a)$ of Proposition 4, $G\left(\bigcap_{V \in \mathscr{F}_h} V\right)$ is order isomorphic to a group of the form $\mathbb{Z} \times_\ell A$. Let $R_h' = \bigcap_{V \in \mathscr{F}_h} V$ and let $R_1 = \bigcap_{h=1}^{r} R_h'$. Then by [3, Theorem 3], $G_1 := G(R_1)$ is order isomorphic to $G(R_1') \times_c G(R_2') \times_c \cdots \times_c G(R_r')$.

Case II: Suppose for each $p = r+1, r+2, ..., s, G(W_p) \cong H \subseteq \mathbb{Q}$ and $G(W_p)$ is not finitely generated. Let $R_p^{''} = \bigcap_{V \in \mathscr{F}_p} V$. Then by (2)($c$) of Proposition 4, $G(R_h^{''})$ is order isomorphic to a group of the form

$$\text{lex-extension of } (\mathbb{Z} \times_c \mathbb{Z} \times_c \cdots \times_c \mathbb{Z}) \text{ by } H.$$

Let $R_2 = \bigcap_{p=r+1}^{s} R_h^{''}$. Then by [3, Theorem 3], $G_2 := G(R_2)$ is order isomorphic to $G(R_{r+1}^{''}) \times_c G(R_{r+2}^{''}) \times_c \cdots \times_c G(R_s^{''})$ which is in the form given in (2).

Case III: Suppose for each $q = s+1, s+2, ..., m, G(W_q) \cong \mathbb{Z} \times_c \gamma_q \mathbb{Z}$, where γ_q is an irrational number. Let $R_q^{'''} = \bigcap_{V \in \mathscr{F}_q} V$. Then by (2)($b$) of Proposition 4, $G(R_q^{'''})$ is order isomorphic to the group $(\mathbb{Z} \times_c \gamma_q \mathbb{Z}) \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c \cdots \times_c \mathbb{Z})$.

Let $R_3 = \bigcap_{q=s+1}^{s} R_q^{'''}$. Then by [3, Theorem 3], $G(R_3)$ is order isomorphic to $G(R_{s+1}^{'''}) \times_c G(R_{s+2}^{'''}) \times_c \cdots \times_c G(R_m^{'''}) = G_3$.

Let $R = R_1 \cap R_2 \cap R_3$. Then $R = \bigcap_{V \in \mathscr{F}} V$. Now by [3, Theorem 3], $G(R)$ is order isomorphic to $G(R_1) \times_c G(R_2) \times_c G(R_3)$ and by Theorem 1, $G(R)$ is order isomorphic to a group of the form $G_1 \times_c G_2 \times_c G_3$. $\square$

The result below describes the group of divisibility of a finite intersection of valuation overrings of $k[x_1, x_2, x_3]$.

Theorem 3. *Let k be an infinite field. A semilocal ℓ-group G is weakly realizable over $k[x_1, x_2, x_3]$, where k is a field and x_1, x_2, x_3 are indeterminates over k if and only if G is order isomorphic to a group of the form $G_1 \times_c G_2 \times_c G_3 \times_c G_4 \times_c G_5 \times_c G_6$, where G_1, G_2, G_3 are as in Proposition 6 and each of G_4, G_5 and G_6 is isomorphic to a cardinal sum of subgroups of the real numbers of rational rank one, two and three respectively.*

Proof. Let G be an ℓ-group and suppose $G = G_1 \times_c G_2 \times_c G_3 \times_c G_4 \times_c G_5 \times_c G_6$, where each $G_i; i = 1, 2, ..., 6$ is zero or G_1, G_2, G_3, G_4, G_5 and G_6 as in the theorem.

First, we realize the group G_1. If $G_1 = 0$, let $V = k(x_1, x_2, x_3)$ so that $G(V) = 0 = G_1$. Assume that $G_1 \neq 0$ and write $G_1 = A_1 \times_c A_2 \times_c \cdots \times_c A_n$, where each $A_i \cong \mathbb{Z} \times_\ell A_i^{'}$ and where $A_i^{'}$ is a finite cardinal product of one or more of the following groups, which are realizable over $k[x_1, x_2]$.

- $(\mathbb{Z} + r_1\mathbb{Z}) \times_c (\mathbb{Z} + r_2\mathbb{Z}) \times_c ... \times_c (\mathbb{Z} + r_q\mathbb{Z})$, where $r_1, r_2, ..., r_q$ are irrational numbers.
- $H_1 \times_c H_2 \times_c ... \times_c H_p$, for each $i = 1, 2, ..., p$, $H_i \subseteq \mathbb{Q}$.
- $\mathbb{Z} \times_\ell (\mathbb{Z} \times_c ... \times_c \mathbb{Z})$.

First, we want to realize A_i. Let $a_1, a_2, ..., a_n$ be distinct elements of k.

Let $p_i = x_1 + a_i$, and let $V_i = k[x_1, x_2, x_3]_{(x_1+a_i)}$. Then by [15, Corollary 2, p. 42], V_i is a DVR. Let μ_{V_i} be the maximal ideal of V_i. Let $\bar{x}_2 = x_2 + \mu_{V_i}$ and $\bar{x}_3 = x_3 + \mu_{V_i}$. Then $k[\bar{x}_2, \bar{x}_3] \subseteq V_i/\mu_{V_i}$.

The group $A_i^{'}$ can be realized over $k[\bar{x}_2, \bar{x}_3]$ as in Theorem 2. Let $R_i^{'}$ be the corresponding domain. Let $D_i = \phi^{-1}(R_i^{'})$, where $\phi : V_i \to V_i/\mu_{V_i}$ be the canonical homomorphism. Then by [11, Theorem 3.2], the sequence

$$0 \to G(R_i^{'}) \to G(D_i) \to G(V_i) \to 0 \qquad (a)$$

is lexicographically exact. Since $G(V_i) \cong \mathbb{Z}$ by Lemma 1, the sequence (a) splits and hence

$$\begin{aligned} G(D_i) &\cong \mathbb{Z} \times_\ell G(R_i^{'}) \\ &\cong \mathbb{Z} \times_\ell A_i^{'} \\ &\cong A_i. \end{aligned}$$

Let $R_1 = \bigcap_{i=1}^{n} D_i$. We have constructed $D_1, D_2, ..., D_n$ such that the valuation domains $V_1, V_2, ..., V_n$ are independent. Then by [3, Theorem 3], $G(R_1)$ is order isomorphic to $A_1 \times_c A_2 \times_c \cdots \times_c A_n$. Thus $G(R_1) \cong G_1$.

Next, we show that G_2 can be weakly realized, where $G_2 \neq 0$. Suppose $G_2 \cong B_1 \times_c B_2 \times_c \cdots \times_c B_m$, where each B_j is in the form

$$\text{lex-extension of } (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z}) \text{ by } H,$$

where H is a subgroup of $\mathbb{Q}$, and H is not finitely generated.

Let $b_1, b_2, ..., b_m \in k - \{a_1, a_2, ..., a_n\}$. As in Theorem 2, we realize H over $k(x_3)[x_1 + b_j, x_2]$. Let W_j be the corresponding valuation domain. Let $\bar{x}_3 = x_3 + \mu_{W_j}$, where μ_{W_j} denotes the maximal ideal of W_j. Then $k[\bar{x}_3] \subsetneq W_j/\mu_{W_j}$. Let $e_1, e_2, ..., e_{r_1}$ be distinct elements of k. Let $T_i^{'} = k[\bar{x}_3]_{(\bar{x}_3+e_i)}$. Then $T_i^{'}$ is a DVR [15, Corollary 2, p. 42] and hence $G(T_i^{'}) = \mathbb{Z}$. Let $T_i = \psi^{-1}(T_i^{'})$, where $\psi : W_j \to W_j/\mu_{W_j}$ is the canonical homomorphism. Then by Lemma 1, $G(T_i)$ is order isomorphic to a lex-extension of $\mathbb{Z}$ by H. Let $R_{2j} = \bigcap_{i=1}^{r_j} T_i$, where r_j denotes the number of copies of $\mathbb{Z}$ in B_j which appear in the cardinal product. Let $H_{W_j,T_i} = \ker(G(T_i) \to G(W_j))$ and $H_{k(x_1,x_2,x_3),W_j} = \ker(G(W_j) \to G(k(x_1,x_2,x_3)))$. Then $H_{W_j,T_i} = \mathbb{Z}$, since H_{W_j,T_i} is a nontrivial convex subgroup of $G(T_i)$ and $H_{k(x_1,x_2,x_3),W_j} = H$, since $G(k(x_1,x_2,x_3)) = 0$. Thus by Theorem 1, the group of divisibility $G(R_{2j})$ is order isomorphic to a group of the form

$$\text{lex-extension of } (H_{W_j,T_1} \times_c H_{W_j,T_2} \times_c ... \times_c H_{W_j,T_{r_1}}) \text{ by } H_{k(x_1,x_2,x_3),W_j},$$

which is order isomorphic to a group of the form

$$\text{lex-extension of } (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z}) \text{ by } H.$$

Let $R_2 = \bigcap_{j=1}^{m} R_{2j}$. We have constructed $R_{21}, R_{22}, ..., R_{2m}$ such that the valuation domains $W_1, W_2, ..., W_m$ are independent. Then by [3, Theorem 3], the group of divisibility $G(R_2)$ is order isomorphic to a group of the form $G(B_1) \times_c G(B_2) \times_c \cdots \times_c G(B_m)$. Thus $G(R_2)$ is order isomorphic to a group of the form G_2. Hence G_2 is weakly realizable over $k[x_1, x_2, x_3]$.

Finally, we realize the group G_3, where $G_3 \neq 0$. Suppose $G_3 = C_1 \times_c C_2 \times_c \cdots \times_c C_p$, where for each $t = 1, 2, ..., p$, C_t is in the form $(\mathbb{Z}+\gamma\mathbb{Z}) \times_\ell (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z})$, where γ is an irrational number.

Let $c_1, c_2, ..., c_p \in k - \{a_1, a_2, ..., a_n, b_1, b_2, ..., b_m\}$. Let r_t be the number of copies of $\mathbb{Z}$ in C_t which are appearing in the cardinal product. As in Theorem 2, we realize $\mathbb{Z} + \gamma\mathbb{Z}$ over $k(x_2)[x_1 + a_t, x_3]$. Let N_t be the corresponding valuation ring. Let $\bar{x}_2 = x_2 + \mu_{N_t}$, where μ_{N_t} is the maximal ideal of N_t. Then $k[\bar{x}_2] \subsetneq N_t/\mu_{N_t}$. As in Proposition 2, we can realize $\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z}$ over $k[\bar{x}_2]$. For each $i = 1, 2, ..., r_t$, let N_i' be the corresponding valuation ring. Then $G(N_i') = \mathbb{Z}$ and the residue field of N_i' is k. Let $S_i = \eta^{-1}(N_i')$, where $\eta : N_t \to N_t/\mu_{N_t}$ is the canonical homomorphism. Then by Lemma 1, $G(S_i) \cong (\mathbb{Z} + \gamma\mathbb{Z}) \times_\ell \mathbb{Z}$.

Let $R_{3t} = \bigcap_{i=1}^{r_t} S_i$. Let $H_{N_t,S_i} = \ker(G(S_i) \to G(N_t))$ and let $H_{k(x_1,x_2,x_3),N_t}$ $= \ker((G(N_t) \to G(k(x_1,x_2,x_3)))$. Then $H_{N_t,N_i} = \mathbb{Z}$ since H_{N_t,N_i} is a nontrivial convex subgroup of $G(N_i)$, and $H_{k(x_1,x_2,x_3),N} = \mathbb{Z} + \gamma\mathbb{Z}$ since $G(k(x_1,x_2,x_3)) = 0$. Then by using Theorem 1, the group of divisibility $G(R_{3t})$ is order isomorphic to a group of the form

$$\text{lex-extension of } (H_{N_t,N_1} \times_c H_{N_t,N_2} \times_c ... \times_c H_{N_t,N_{r_2}}) \text{ by } H_{k(x_1,x_2,x_3),N_t}.$$

The group $G(R_{3t})$ is then order isomorphic to a group of the form

$$\text{lex-extension of } (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z}) \text{ by } (\mathbb{Z} + \gamma\mathbb{Z}).$$

Since $\mathbb{Z}+\gamma\mathbb{Z}$ is projective, the lex-exact sequence splits and hence the group $G(R_{3t})$ is order isomorphic to $(\mathbb{Z}+\gamma\mathbb{Z}) \times_{\ell} (\mathbb{Z} \times_c \mathbb{Z} \times_c ... \times_c \mathbb{Z})$ which is C_t. Let $R_3 = \bigcap_{t=1}^{p} R_{3t}$. We have constructed $R_{31}, R_{32}, ..., R_{3p}$ such that the valuation domains $N_1, N_2, ..., N_p$ are independent. Then by [3, Theorem 3], the group of divisibility $G(R_3)$ is order isomorphic to $C_1 \times_c C_2 \times_c \cdots \times_c C_p$. Thus $G(R_3) \cong G_3$.

The groups of the form G_4, G_5 and G_6 can be realized by [12, Theorem 5.8]. Let R_4, R_5 and R_6 be the semilocal Bézout domains associated with the ℓ-groups G_4, G_5 and G_6 respectively.

Let $R = \bigcap_{i=1}^{6} R_i$. We have constructed R_1, R_2 and R_3 such that for each i, j and t, the valuation domains V_i, T_j and N_t are independent. Since the valuation domains corresponding to R_4, R_5 and R_6 are distinct and of rank one, they are independent. By Theorem 1, the group of divisibility $G(R) = G(\bigcap_{i=1}^{6} R_i)$ is order isomorphic to $G(R_1) \times_c G(R_2) \times_c G(R_3) \times_c G(R_4) \times_c G(R_5) \times_c G(R_6)$. Since except for $i = 2, G(R_i) \cong G_i$ and for $i = 2, G(R_2)$ is order isomorphic to a group of the form G_2, then the group $G(R)$ is order isomorphic to a group of the form

$$G_1 \times_c G_2 \times_c G_3 \times_c G_4 \times_c G_5 \times_c G_6.$$

Thus the group G is weakly realizable over $k[x_1, x_2, x_3]$.

Conversely, let G be weakly realizable over $k[x_1, x_2, x_3]$, where k is a field and x_1, x_2, x_3 are indeterminates over k. Then there exists a semilocal Bézout overring R of $k[x_1, x_2, x_3]$ such that G and $G(R)$ admit a lexico-cardinal decomposition of the same form. By using the Proposition 6, G is order isomorphic to a group of the form $G_1 \times_c G_2 \times_c G_3 \times_c G_4 \times_c G_5 \times_c G_6$, where G_4, G_5 and G_6 are corresponding to the rank one valuation rings and G_4, G_5 and G_6 are isomorphic to a cardinal sum of subgroups of the real numbers of rational rank one, two and three respectively. □

Finally, we conclude the following result.

Theorem 4. *The semilocal ℓ-groups that can be realized over $k[x_1, x_2, ..., x_n]$ can be determined for $n = 1, 2$. For $n = 3$, if $V_1, V_2, ..., V_m$ are dependent valuation overrings of $k[x_1, x_2, ..., x_n]$ with finitely generated or divisible value groups then $G(V_1 \cap V_2 \cap \cdots \cap V_m)$ can be determined completely by the lexico-cardinal decomposition form of $G(V_1 \cap V_2 \cap \cdots \cap V_m)$.*

Proof. From Proposition 2 and Theorem 2, semilocal ℓ-groups which can be realized over $k[x_1, x_2, ..., x_n]$ can be determined for $n = 1$ and $n = 2$, respectively.

Let $V_1, V_2, ..., V_m$ be dependent the valuation overrings of $k[x_1, x_2, ..., x_n]$, where $n \geq 3$. By Theorem 1, the group of divisibility $G(V_1 \cap V_2 \cap \cdots \cap V_m)$ can be expressed in terms of the finite product of lex-exact sequences. If each $V_i, i = 1, 2, ..., m$ has finitely generated or each V_i has divisible value groups, then the lex-exact sequences split and the group $G(V_1 \cap V_2 \cap \cdots \cap V_m)$ can be determined completely. □

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