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RESEARCH ARTICLE

# Note on cyclic doppelsemigroups

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Abstract. A doppelsemigroup $(G, \dashv, \vdash)$ is called *cyclic* if $(G, \dashv)$ is a cyclic group. In the paper, we describe up to isomorphism all cyclic (strong) doppelsemigroups. We prove that up to isomorphism there exist $\tau(n)$ finite cyclic (strong) doppelsemigroups of order $n$, where $\tau$ is the number of divisors function. Also there exist infinite countably many pairwise non-isomorphic infinite cyclic (strong) doppelsemigroups.

## Introduction

Given a semigroup $(S, \dashv)$, consider a semigroup $(S, \vdash)$ defined on the same set. We say that the semigroups $(S, \vdash)$ and $(S, \dashv)$ are *interassociative* provided

$$(x \dashv y) \vdash z = x \dashv (y \vdash z) \quad \text{and} \quad (x \vdash y) \dashv z = x \vdash (y \dashv z)$$

for all $x, y, z \in S$. When this occurs, we say that $(S, \vdash)$ is an *interassociate* of $(S, \dashv)$, or that the semigroups are interassociates of each other. If the semigroups $(S, \vdash)$ and $(S, \dashv)$ are interassociative, then rearranging the parentheses in an expression that contains only operations $\vdash$, $\dashv$ and elements of $S$ will not change the result. In 1971, Zupnik [25] coined the term interassociativity in a general groupoid setting. However, he required only one of the two defining equations to hold. The present concept of interassociativity for semigroups originated in 1986 in Drouzy [3], where

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it is noted that every group is isomorphic to each of its interassociates. In 1983, Gould and Richardson [6] introduced *strong interassociativity*, defined by the above equations along with $x \dashv (y \vdash z) = x \vdash (y \dashv z)$. J. B. Hickey in 1983 [7] and 1986 [8] dealt with the special case of interassociativity in which the operation $\vdash$ is defined by specifying $a \in S$ and stipulating that $x \vdash y = x \dashv a \dashv y$ for all $x, y \in S$. Clearly $(S, \vdash)$, which Hickey calls a *variant* of $(S, \dashv)$, is a semigroup that is an interassociate of $(S, \dashv)$. It is easy to show that if $(S, \dashv)$ is a monoid, every interassociate $(S, \vdash)$ must satisfy the condition $x \vdash y = x \dashv a \dashv y$ for some fixed element $a \in S$ and for all $x, y \in S$, that is $(S, \vdash)$ is a variant of $(S, \dashv)$. Methods of constructing interassociates were developed, for semigroups in general and for specific classes of semigroups, in 1997 by Boyd, Gould and Nelson [1]. The description of all interassociates of finite monogenic semigroups was presented by Gould, Linton and Nelson in 2004, see [5].

This paper is devoted to study of doppelsemigroups which are sets with two associative binary operations satisfying axioms of interassociativity. More accurately, a *doppelsemigroup* is an algebraic structure $(D, \dashv, \vdash)$ consisting of a non-empty set $D$ equipped with two associative binary operations $\dashv$ and $\vdash$ satisfying the following axioms:

$$(\mathrm{D}_1) \qquad (x \dashv y) \vdash z = x \dashv (y \vdash z),$$
$$(\mathrm{D}_2) \qquad (x \vdash y) \dashv z = x \vdash (y \dashv z).$$

Thus, we can see that in any doppelsemigroup $(D, \dashv, \vdash)$, $(D, \vdash)$ is an interassociate of $(D, \dashv)$, and conversely, if a semigroup $(D, \vdash)$ is an interassociate of a semigroup $(D, \dashv)$, then $(D, \dashv, \vdash)$ is a doppelsemigroup. A doppelsemigroup $(D, \dashv, \vdash)$ is called *commutative* [18] if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative. A doppelsemigroup $(D, \dashv, \vdash)$ is said to be *strong* [20] if it satisfies the axiom $x \dashv (y \vdash z) = x \vdash (y \dashv z)$.

The study of doppelsemigroups was initiated by A. Zhuchok. The idea of doppelsemigroups bases on the study of dimonoids in the sense of Loday [11]. Doppelalgebras introduced by Richter [13] in the context of algebraic $K$-theory are linear analogs of doppelsemigroups and commutative dimonoids are examples of doppelsemigroups. Consequently, doppelsemigroup theory has connections to doppelalgebra theory and dimonoid theory. A doppelsemigroup can also be determined by using the notion of a duplex [12]. Free duplexes were constructed in [12]. Doppelsemigroups are closely related to bisemigroups considered in the work of Schein [15]. The latter algebras have applications in the theory of binary relations [16]. If operations of a doppelsemigroup coincide, we obtain the notion of a semigroup.

Many classes of doppelsemigroups were studied by A. Zhuchok and his coauthors. The free product of doppelsemigroups, the free (strong) doppelsemigroup, the free commutative (strong) doppelsemigroup, the free $n$-nilpotent (strong) doppelsemigroup and the free rectangular doppelsemigroup were constructed in [18, 20, 24]. Relatively free doppelsemigroups were studied in [22]. The free $n$-dinilpotent (strong) doppelsemigroup was constructed in [17, 20]. In [19], A. Zhuchok described the free left $n$-dinilpotent doppelsemigroup. Representations of ordered doppelsemigroups by binary relations were studied by Yu. Zhuchok and J. Koppitz, see [23].

One of the main tasks in the study of algebraic structures is their classification up to isomorphism. In [4], the task of describing all pairwise non-isomorphic (strong) doppelsemigroups of order 3 has been solved. It was proved that there exist 75 pairwise non-isomorphic three-element doppelsemigroups among which 41 doppelsemigroups are commutative. Non-commutative doppelsemigroups are divided into 17 pairs of dual doppelsemigroups. Also up to isomorphism there are 65 strong doppelsemigroups of order 3, and all non-strong doppelsemigroups are not commutative.

In this paper, we concentrate on studying cyclic doppelsemigroups. A doppelsemigroup $(G, \dashv, \vdash)$ said to be *cyclic* if $(G, \dashv)$ is a cyclic group. Of course, the task of classifying all cyclic doppelsemigroups up to isomorphism is quite interesting. We prove that up to isomorphism there exist $\tau(n)$ finite cyclic (strong) doppelsemigroups of order $n$, where $\tau$ is the number of divisors function. Also we show that there exist infinite countably many pairwise non-isomorphic infinite cyclic (strong) doppelsemigroups.

## 1. Preliminaries

First, recall some useful facts which we shall use in this paper. In fact, each semigroup $(S, \dashv)$ can be consider as a (strong) doppelsemigroup $(S, \dashv, \dashv)$, and we denote this *trivial* doppelsemigroup by $S$. We denote by $(S, \dashv_a)$ a variant of a semigroup $(S, \dashv)$, where $x \dashv_a y = x \dashv a \dashv y$, $a \in S$.

Proposition 3.1 and Lemma 6.1 of [20] imply the following corollary.

**Corollary 1.** *Let $(S, \dashv)$ be a commutative semigroup and let $a, b \in S$. Then $(S, \dashv_a, \dashv_b)$ is a commutative strong doppelsemigroup.*

A bijective map $\psi : D_1 \to D_2$ is called an *isomorphism of doppelsemigroups* $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ if

$$\psi(a \dashv_1 b) = \psi(a) \dashv_2 \psi(b) \quad \text{and} \quad \psi(a \vdash_1 b) = \psi(a) \vdash_2 \psi(b)$$

for all $a, b \in D_1$.

If there exists an isomorphism between the doppelsemigroups $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ then $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are said to be *isomorphic*, denoted $(D_1, \dashv_1, \vdash_1) \cong (D_2, \dashv_2, \vdash_2)$.

A doppelsemigroup $(G, \dashv, \vdash)$ is called *a group doppelsemigroup* if $(G, \dashv)$ is a group. Since every group is isomorphic to each of its interassociates, it follows that $(G, \vdash)$ is a group isomorphic to $(G, \dashv)$.

**Proposition 1.** *Let $(G, \dashv)$ be a group and let $(G, \dashv, \vdash)$ be a doppelsemigroup. Then $\vdash = \dashv_a$ for some $a \in G$.*

*Proof.* Let $e$ be identity of a group $(G, \dashv)$. It follows that

$$x \vdash y = (x \dashv e) \vdash (e \dashv y) = x \dashv (e \vdash e) \dashv y,$$

and we conclude that $\vdash = \dashv_a$ for $a = e \vdash e$. □

Proposition 1 and Corollary 1 imply the following corollary.

**Corollary 2.** *Let $(G, \dashv)$ be an Abelian group and let $(G, \dashv, \vdash)$ be a doppelsemigroup. Then $(G, \dashv, \vdash)$ is a commutative strong doppelsemigroup and $\vdash = \dashv_a$ for some $a \in G$.*

## 2. Characterization of cyclic doppelsemigroups

In this section, we characterize up to isomorphism all finite cyclic doppelsemigroups. First, let us recall some known information concerning finite cyclic groups. If $H$ is a subgroup of a finite cyclic group $G$, then each group automorphism $\psi : H \to H$ can be extended to a group automorphism $\overline{\psi} : G \to G$, see [10, Lemma 3.1]. Let $a$ and $b$ be elements of a finite cyclic group $G$. Then there exists a group automorphism $\psi$ of $G$ such that $\psi(a) = b$ if and only if $|a| = |b|$. For more information about group theory, see for example [14].

Recall that the number of divisors function $\tau(n)$ is defined as the number of natural divisors of a natural number $n$. It is well-known that $\tau(n) = (a_1 + 1) \cdot (a_2 + 1) \cdot \ldots \cdot (a_t + 1)$, where $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_t^{a_t}$ is the prime factorization of $n$.

**Theorem 1.** *There exist $\tau(n)$ pairwise non-isomorphic finite cyclic doppelsemigroups of order $n$. All finite cyclic doppelsemigroups are strong and commutative.*

*Proof.* Let us describe up to isomorphism all cyclic (strong) doppelsemigroups of finite order $n$. Let $(\mathbb{Z}_n, +, \vdash)$ be arbitrary doppelsemigroup of order $n$. According to Proposition 1, we can assume without loss of generality that $(\mathbb{Z}_n, +, \vdash) = (\mathbb{Z}_n, +, +_a)$ for some $a \in \{0, \ldots, n-1\}$.

Consider two arbitrary doppelsemigroups $(\mathbb{Z}_n, +, +_a)$ and $(\mathbb{Z}_n, +, +_b)$. Taking into account that

$$(n-a) +_a x = n - a + a + x = x \pmod{n}$$

and

$$x +_a (n-a) = x + a + n - a = x \pmod{n}$$

for any $x \in \mathbb{Z}_n$, we conclude that $n-a$ is a neutral element of the group $(\mathbb{Z}_n, +_a)$.

Consider the following two cases:

**Case 1**. Let $|a| \neq |b|$. Since $|n-a| = |a|$ and $|n-b| = |b|$, it follows that also $|n-a| \neq |n-b|$. Let us show by contradiction that in this case the doppelsemigroups $(\mathbb{Z}_n, +, +_a)$ and $(\mathbb{Z}_n, +, +_b)$ are not isomorphic. Assume conversely that $\psi : \mathbb{Z}_n \to \mathbb{Z}_n$ is a doppelsemigoup isomorphism from $(\mathbb{Z}_n, +, +_a)$ to $(\mathbb{Z}_n, +, +_b)$. Taking into account that isomorphisms preserve the neutral elements, we conclude that $\psi(n-a) = n-b$. Therefore, $|n-a| = |n-b|$, and we get a contradiction.

**Case 2**. Let $|a| = |b|$. Let us show that in this case the doppelsemigroups $(\mathbb{Z}_n, +, +_a)$ and $(\mathbb{Z}_n, +, +_b)$ are isomorphic. Let $\psi : \mathbb{Z}_n \to \mathbb{Z}_n$ be an automorphism of the group $(\mathbb{Z}_n, +)$ such that $\psi(a) = b$. It follows that $\psi$ is also a doppelsemgroup isomorphism from $(\mathbb{Z}_n, +, +_a)$ to $(\mathbb{Z}_n, +, +_b)$. Indeed,

$$\begin{aligned}\psi(x +_a y) &= \psi(x + a + y) = \psi(x) + \psi(a) + \psi(y) \\ &= \psi(x) + b + \psi(y) = \psi(x) +_b \psi(y)\end{aligned}$$

for any $x, y \in \mathbb{Z}_n$.

Taking into account that for each positive divisor $d$ of $|G|$, there exists an element of order $d$ (see [14]), we conclude that the number of pairwise non-isomorphic doppelsemigroups of order $n$ is equal to number of natural divisors of $n$, and hence is equal to $\tau(n)$.

According to Corollary 2, all finite cyclic doppelsemigroups are strong and commutative. □

It is well-known that each infinite cyclic group is isomorphic to the additive group of integers.

**Theorem 2.** *Let $a, b \in \mathbb{Z}$. The doppelsemigroups $(\mathbb{Z}, +, +_a)$ and $(\mathbb{Z}, +, +_b)$ are isomorphic if and only if $b \in \{a, -a\}$. Consequently, there exist countably infinite many pairwise non-isomorphic infinite cyclic (commutative strong) doppelsemigroups.*

*Proof.* It is well-known that there are exactly two automorphisms of the group $(\mathbb{Z},+)$: the identity automorphism $\mathrm{id}_{\mathbb{Z}}:\mathbb{Z}\to\mathbb{Z},\ \mathrm{id}_{\mathbb{Z}}(x)=x$ and $\mathrm{inv}_{\mathbb{Z}}:\mathbb{Z}\to\mathbb{Z},\ \mathrm{inv}_{\mathbb{Z}}(x)=-x$.

It is clear that $\mathrm{id}_{\mathbb{Z}}$ is an automorhism of a doppelsemigroup $(\mathbb{Z},+,+_a)$.

Taking into account that

$$\begin{aligned}\mathrm{inv}_{\mathbb{Z}}(x +_a + y) &= \mathrm{inv}_{\mathbb{Z}}(x+a+y) = -(x+a+y) = -x+(-a)-y\\ &= \mathrm{inv}_{\mathbb{Z}}(x)+(-a)+\mathrm{inv}_{\mathbb{Z}}(y) = \mathrm{inv}_{\mathbb{Z}}(x) +_{-a} + \mathrm{inv}_{\mathbb{Z}}(y),\end{aligned}$$

we conclude that $\mathrm{inv}_{\mathbb{Z}}$ is an isomorphism of the doppelsemigroups $(\mathbb{Z},+,+_a)$ and $(\mathbb{Z},+,+_{-a})$.

Let $\psi:\mathbb{Z}\to\mathbb{Z}$ be arbitrary doppelsemigoup isomorphism from $(\mathbb{Z},+,+_a)$ to $(\mathbb{Z},+,+_b)$. It follows that $\psi$ is a group automorphism of $(\mathbb{Z},+)$ and a group isomorphism from the group $(\mathbb{Z},+_a)$ with neutral element $-a$ to the group $(\mathbb{Z},+_b)$ with neutral element $-b$. It follows that $\psi(-a)=-b$, and hence $\psi(a)=b$. If $\psi=\mathrm{id}_{\mathbb{Z}}$, then $a=b$, and in the case $\psi=\mathrm{inv}_{\mathbb{Z}}$, we get $a=-b$.

We conclude that the doppelsemigroups $(\mathbb{Z},+,+_a)$ and $(\mathbb{Z},+,+_b)$ are isomorphic if and only if $b=a$ or $b=-a$.

According to Corollary 2, all infinite cyclic doppelsemigroups are strong and commutative. □

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