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RESEARCH ARTICLE

Herstein's theorem for prime ideals of semirings with involution

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Abstract. This article discusses differential identities involving Jordan and Lie products that connect an MA-semiring to its prime ideals. The main objective is to investigate how these identities impose various properties on the quotient MA-semiring determined by a prime ideal. In particular, we establish Herstein's theorem for MA-semirings, which was previously proved for rings.

Introduction and preliminaries

The notion of ideals has several interesting features that attract researchers working on associative algebras and ring theory. Several ideals have been introduced and investigated for various sorts of algebraic structures, such as quasi-ideals, k-ideals, Q-ideals, prime ideals, Jordan ideals, and Lie ideals, one can see [9,10,13,19,25,26,29] for their details. In general, Lie theory and other related topics cannot be explored in semirings; however in 2012, Javed et al. [20] introduced a class of semirings known as MA-semirings, in which these topics are permitted. MA-Semiring is defined as an additively commutative additive inverse semiring S with absorbing zero $'0'$ satisfying A_2 condition of Bandlet and Petrich [14] that is $v + v^{'} \in Z(S)$, for all $v \in S$, where $Z(S)$ is the center and $v^{'}$ is pseudo inverse of v. Every ring is an MA-semiring but the converse may not hold in general (see [3,20,22,30]). Differential identities invol-

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ving different ideals leading to some interesting aspects of rings has been investigated by several authors (see [15–17, 21, 27]); these types of differential identities along with ideals have been responsible to identify some features of MA-semirings, (see [1–7, 20, 22, 30]).

We now state some necessary definitions and preliminary concepts. An ideal I of a semiring S is said to be a Q-ideal if there exists a subset Q of S such that $S = \bigcup\{q + I : q \in Q\}$ and if $q_1, q_2 \in Q$, then $(q_1 + I) \bigcap (q_2 + I) \neq \phi$ if and only if $q_1 = q_2$. For more on Q-ideals one can see [9]. An ideal I of a semiring S is said to be k-ideal if $a + b \in I$ and $b \in I$, then $a \in I$. In fact every Q-ideal is k-ideal but the converse may not be true in general, for detail we refer [10]. Moreover for a k-ideal I, one can conclude that $a \in I$ if and only if $a^{'} \in I$, and we use it frequently in the sequel without mentioning it. A proper ideal P of S is prime if for $a, b \in S$, $aSb \subseteq P$ implies either $a \in P$ or $b \in P$. For $u, v, w \in S$, the binary operations $[u, w] = uw + w^{'}u$ and $w \circ v = wv + vw$ are respectively known as commutator (Lie product) and anticommutator (Jordan product). A derivation is an additive mapping $\rho : S \longrightarrow S$ such that $\rho(wt) = \rho(w)t + w\rho(t)$, for all $w, t \in S$. An additive mapping $* : S \longrightarrow S$ is said to be an involution if $u^{**} = u$ and $(uv)^* = v^*u^*$, for all $u, v \in S$. An element u of S is said to be Hermitian if $u^* = u$ and skew Hermititian if $u^* = u^{'}$. Further more the set of all Hermitian elements of S is denoted by $H(S)$ and the set of all skew Hermitian elements of S is denoted by $K(S)$. We now recall and prove some fundamental results of quotient MA-semirings which will be helpful not only for this paper but can also be used for establishing more results on this topic. In this section, Several results of [25] have been proved for MA-semirings which play a fundamental role in the main results. Following lemma contains some very useful identities related to MA-semirings.

Lemma 1 ([20]). *Let S be an MA-semiring. Then for all $u, v, w \in S$, $z \in Z(S)$*

(i) $[w, wu] = w[w, u]$;

(ii) $[w, uv] = [w, u]v + u[w, v]$;

(iii) $[wu, v] = w[u, v] + [w, v]u$;

(iv) $(wu)^{'} = w^{'}u = wu^{'}$;

(v) $[w, u] + [u, w] = u(w + w^{'}) = w(u + u^{'})$;

(vi) $[w, u]^{'} = [w, u^{'}] = [w^{'}, u] = [u, w]$;

(vii) $w \circ (u+v) = w \circ u + w \circ v$;

(viii) $[w, uz] = z[w, u] = [w, u]z$;

(ix) $[w, w] = [w, w]'$;

(x) $w + u = 0 \ \Rightarrow w = u'$, *however the converse may not hold in general. For more such identities, one can see [30, 31].*

It is well known that if I is a Q-ideal of a semiring $(S, +, .)$, then $S/I = \{q + I : q \in S\}$ forms a semiring with respect to $\oplus$ and $\odot$ defined as $w + P \oplus v + P = w + v + P$ and $w + P \odot v + P = w.v + P$ respectively, which is referred as quotient semiring. For more detail, we refer [11, 12]. However, if S is an MA-semiring and I is a Q-ideal of S, then we have the following observation.

Theorem 1 ([24])**.** *Let $(S, +, .)$ be an MA-semiring and P be a Q-ideal of S. Then $(S/P, \oplus, \odot)$ is an MA-semiring.*

Theorem 2 ([24])**.** *Let S be an MA-semiring and P be a prime Q-ideal of S. Then following implications are true:*

(i) If $[v, w] \in P$ for all $v, w \in S$, then S/P is a commutative MA-semiring.

(ii) If $v \circ w \in P$ for all $v, w \in S$, then S/P is a commutative MA-semiring.

Moreover S/P has no nonzero zero divisors.

Theorem 3 ([24])**.** *Let S be an MA-semiring and P be a prime Q-ideal of S. If ρ is a derivation such that $[\rho(u), u] \in P$ for all $u \in S$, then S/P is commutative or $\rho(S) \subseteq P$.*

Definition 1. Let P be a prime ideal of an MA-semiring S. An involution $*$ of S is of P-second kind if $K(S) \cap Z(S) \nsubseteq P$ otherwise it is of P-first kind. If we take $P = \{0\}$, then we obtain involution of second kind and first kind.

Lemma 2. *Let P be a semiprime ideal of an MA-semiring S with involution $*$. If $K(S) \cap Z(S) \nsubseteq P$, then $H(S) \cap Z(S) \nsubseteq P$.*

Proof. Suppose that $H(S) \cap Z(S) \subseteq P$. Let $k_z \in K(S) \cap Z(S)$ and $k_z \notin P$. Then $k_z^2 \in H(S) \cap Z(S) \subseteq P$. Therefore $k_z^2 \in P$, which further implies $k_z S k_z \subseteq P$. As P is prime, so $k_z \in P$, a contradiction. Hence we conclude that $H(S) \cap Z(S) \nsubseteq P$. □

Lemma 3. *Let P be a semiprime Q-ideal of an MA-semiring and S/P be 2-torsion free. $\rho : S \longrightarrow S$ be a derivation such that $\rho^2(u) \in P$ for all $u \in S$, then $\rho(u) \in P$.*

Proof. According to the hypothesis, we have

$$\rho^2(u) \in P.$$

In the last expression substituting uv for u, we get $\rho^2(u)v + 2\rho(u)\rho(v) + u\rho^2(v) \in P$. As P is semiprime Q-ideal and S/P is 2-torsion free, using the hypothesis again, we get $\rho(u)\rho(v) \in P$. Further changing u by us and using the similar arguments, we get $\rho(u)S\rho(v) \subseteq P$ which shows that $\rho(S) \subset P$. □

Lemma 4. *Let P be a prime Q-ideal of an MA-semiring and S/P be 2-torsion free. $\rho_1, \rho_2 : S \longrightarrow S$ be derivations such that $\rho_2\rho_1(u) \in P$ for all $u \in S$. Then $\rho_1(S) \subseteq P$ or $\rho_2(S) \subseteq P$.*

Proof. Hypothesis indicates that for all $u \in S$

$$\rho_2\rho_1(u) \in P. \tag{1}$$

In (1), changing u by uv, we obtain

$$\rho_2\rho_1(u)v + \rho_1(u)\rho_2(v) + u\rho_2\rho_1(v) + \rho_2(u)\rho_1(v) \in P$$

for all $u, v \in S$. As P is a prime Q-ideal, using (1), we obtain

$$\rho_1(u)\rho_2(v) + \rho_2(u)\rho_1(v) \in P \tag{2}$$

for all $u, v \in S$. Substituting v by $\rho_1(v)$ in (2), we get

$$\rho_1(u)\rho_2(\rho_1(v)) + \rho_2(u)\rho_1^2(v) \in P$$

for all $u, v \in S$. Using (1) again, we further get

$$\rho_2(u)\rho_1^2(v) \in P \tag{3}$$

for all $u, v \in S$. In (3), substituting uw by u, and using (3), we obtain $\rho_2(u)S\rho_1^2(v) \subseteq P$ for all $u, v \in S$. As P is prime, we obtain $\rho_2(S) \subseteq P$ or $\rho_1^2(S) \subseteq P$. If $\rho_1^2(S) \subseteq P$, then by Lemma 2, we have $\rho_1(S) \subseteq P$. □

Lemma 5. *Let P be a prime Q-ideal of an MA-semiring and S/P be 2-torsion free. $\rho_1, \rho_2 : S \longrightarrow S$ be derivations such that $\rho_1(u) \circ \rho_2(v) \in P$ for all $u, v \in S$. Then one of the following holds* *(i)* $\rho_1(S) \subseteq P$;

(ii) $\rho_2(S) \subseteq P$;

(iii) S/P *is commutative.*

Proof. According to the hypothesis, for all $u \in S$ we have

$$\rho_1(u) \circ \rho_2(v) \in P. \tag{4}$$

From (4), we have $\rho_1(u)\rho_2(v) + \rho_2(v)\rho_1(u) \in P$ for all $u, v \in S$ and therefore

$$\rho_1(u)\rho_2(v) + P = (\rho_2(v)\rho_1(u))' + P \tag{5}$$

for all $u, v \in S$. In (4), changing u by us, we get

$$\rho_1(u)s\rho_2(v) + \rho_2(v)\rho_1(u)s + u\rho_1(s)\rho_2(v) + \rho_2(v)u\rho_1(s) \in P$$

for all $s, u, v \in S$ and therefore

$$\begin{gathered}\rho_1(u)s\rho_2(v) + P \oplus \rho_2(v)\rho_1(u)s + P \oplus u\rho_1(s)\rho_2(v) \\ + P \oplus \rho_2(v)u\rho_1(s) + P = P\end{gathered} \tag{6}$$

for all $s, u, v \in S$. Using (5) into (6), we get

$$\begin{gathered}\rho_1(u)s\rho_2(v) + P \oplus \rho_1(u)(\rho_2(v))'s + P \oplus \rho_2(v)u\rho_1(s) \\ + P \oplus u\rho_2(v)(\rho_1(s))' + P = P\end{gathered}$$

for all $s, u, v \in S$ and using Lemma 1, we can write

$$\rho_1(u)[s, \rho_2(v)] + [\rho_2(v), u]\rho_1(s) + P = P$$

for all $s, u, v \in S$ and this further implies

$$\rho_1(u)[s, \rho_2(v)] + [\rho_2(v), u]\rho_1(s) \in P \tag{7}$$

for all $s, u, v \in S$. In (7), changing u by $\rho_2(v)$, we get

$$(\rho_1\rho_2(v)[s, \rho_2(v)] + P) \oplus ([\rho_2(v), \rho_2(v)]\rho_1(s) + P) = P \tag{8}$$

for all $s, v \in S$. By the Jacobian identities (c.f Lemma 1), $[u, u] = [u, u]'$, therefore

$$(\rho_1\rho_2(v)[s, \rho_2(v)] + P) \oplus ([\rho_2(v), \rho_2(v)]'\rho_1(s) + P) = P$$

for all $s, v \in S$, which further implies

$$\rho_1\rho_2(v)[s, \rho_2(v)] + P = [\rho_2(v), \rho_2(v)]\rho_1(s) + P \tag{9}$$

for all $s, v \in S$. As S/P is 2-torsion free, using (9) into (8), we get $\rho_1\rho_2(v)[s, \rho_2(v)] + P = P$ and therefore

$$\rho_1\rho_2(v)[s, \rho_2(v)] \in P \tag{10}$$

for all $s, v \in S$. In (13), substituting rs for s, and using Lemma 1, we obtain

$$\rho_1\rho_2(v)r[s, \rho_2(v)] + \rho_1\rho_2(v)[r, \rho_2(v)]s \in P$$

for all $r, s, v \in S$. As P is prime Q-ideal, using (13), we obtain

$$\rho_1\rho_2(v)S[s, \rho_2(v)] \subseteq P$$

for all $s, v \in S$ and by the primeness of P, we have $\rho_1\rho_2(v) \in P$ or $[s, \rho_2(v)] \in P$ for all $s, v \in S$. Firstly if $\rho_1\rho_2(v) \in P$ for all $v \in S$, then by Lemma 4 $\rho_1(S) \subseteq P$ or $\rho_2(S) \subseteq P$. Secondly if $[s, \rho_2(v)] \in P$ for all $s, v \in S$, then by [8, Theorem 1], we have S/P is commutative or $\rho_2(S) \subseteq P$ and this completes the proof. $\square$

Lemma 6. *Let P be a prime Q-ideal of an MA-semiring and S/P be 2-torsion free. $\rho_1, \rho_2 : S \longrightarrow S$ be derivations such that $[\rho_1(u), \rho_2(v)] \in P$ for all $u, v \in S$. Then $\rho_1(S) \subseteq P$ (and $\rho_2(S) \subseteq P$) or S/P is commutative.*

Proof. By the hypothesis,

$$[\rho_1(u), \rho_2(v)] \in P \tag{11}$$

for all $u, v \in S$. In (11), changing v by vs and using Lemma 1, we get

$$\rho_2(v)[\rho_1(u), s] + [\rho_1(u), \rho_2(v)]s + v[\rho_1(u), \rho_2(s)] + [\rho_1(u), v]\rho_2(s) \in P$$

for all $s, v \in S$. As P is prime Q-ideal, using (11) again, we obtain

$$\rho_2(v)[\rho_1(u), s] + [\rho_1(u), v]\rho_2(s) \in P \tag{12}$$

for all $s, u, v \in S$. In (12), taking $v = \rho_2(v)$, we can write

$$\rho_2^2(v)[\rho_1(u), s] + [\rho_1(u), \rho_2(v)]\rho_2(s) \in P$$

for all $s, u, v \in S$ and using the above arguments, we have

$$\rho_2^2(v)[\rho_1(u), s] \in P$$

for all $s, u, v \in S$. Changing s by rs in the last expression and using it again, we have

$$\rho_2^2(v)S[\rho_1(u), s] \subseteq P$$

for all $s, u, v \in S$. By the primeness of P, we have either $\rho_2^2(S) \subseteq P$ or $[\rho_1(u), s] \in P$ for all $s, u \in S$. Firstly, if $\rho_2^2(S) \subseteq P$, then by Lemma 3 $\rho_2(S) \subseteq P$. Secondly, if $[\rho_1(u), s] \in P$ for all $s, u \in S$, then by [8, Theorem 1], we have either $\rho_1(S) \subseteq P$ or S/P is commutative and this completes the proof. □

Lemma 7. *Let P be a prime Q-ideal of S and S/P be 2-torsion free. If $aub + bua \in P$, then $a \in P$ or $b \in P$.*

Proof. According to the hypothesis, for all $u \in S$ we have

$$aub + bua \in P. \tag{13}$$

Using Lemma 1, from (13), we can write

$$aub + P = bu^{'}a + P \tag{14}$$

for all $u \in S$. In (13), changing u by ubv, we obtain

$$aubvb + bubva \in P$$

for all $u, v \in S$. Using (14) in the last expression twice, we obtain

$$aubvb + buav^{'}b \in P$$

for all $u, v \in S$ and it further implies

$$2aubvb = aubvb + aubvb \in P$$

for all $u, v \in S$. By the 2-torsion freeness of S/P, we have

$$aSbvb \subseteq P$$

for all $v \in S$. As P is prime, we have either $a \in P$ or $bvb \in P$ implying that $b \in P$. □

Lemma 8. *Let P be a prime Q-ideal of an MA-semiring and S/P be 2-torsion free. $\rho_1, \rho_2 : S \longrightarrow S$ be derivations such that*

$$[\rho_1(v), \rho_2(u)] + [\rho_1(v), u] + [v, \rho_2(u)] \in P \tag{15}$$

for all $u, v \in S$. Then S/P is commutative.

Proof. In (15), substituting vu for v, we obtain

$$\begin{aligned}([\rho_1(v), \rho_2(u)] + [\rho_1(v), u] + [v, \rho_2(u)])u + v([\rho_1(u), \rho_2(u)] + [\rho_1(u), u]\\ + [u, \rho_2(u)]) + \rho_1(v)[u, \rho_2(u)] + [v, \rho_2(u)]\rho_1(u) + [v, u]\rho_1(u) \in P\end{aligned}$$

for all $u, v \in S$ and using (15), we obtain

$$\rho_1(v)[u, \rho_2(u)] + [v, \rho_2(u)]\rho_1(u) + [v, u]\rho_1(u) \in P \tag{16}$$

for all $u, v \in S$. In (16), replacing v by vs, we obtain

$$\begin{aligned}\rho_1(v)s[u, \rho_2(u)] + [v, \rho_2(u)]s\rho_1(u) + [v, u]s\rho_1(u)\\ + v(\rho_1(s)[u, \rho_2(u)] + [s, \rho_2(u)]\rho_1(u) + [s, u]\rho_1(u)) \in P\end{aligned}$$

for all $s, u, v \in S$ and using (16) again we get

$$\rho_1(v)s[u, \rho_2(u)] + [v, \rho_2(u)]s\rho_1(u) + [v, u]s\rho_1(u) \in P \tag{17}$$

for all $s, u, v \in S$. In (17), changing v by u and using the fact that $[u, u] = [u, u]'$, we obtain

$$\rho_1(u)s[u, \rho_2(u)] + [u, \rho_2(u)]s\rho_1(u) + [u, u]'s\rho_1(u) \in P$$

for all $s, u \in S$, which further implies

$$\rho_1(u)s[u, \rho_2(u)] + [u, \rho_2(u)]s\rho_1(u) \in P.$$

By Lemma 7, we have $[u, \rho_2(u)] \in P$ for all $u \in S$ or $\rho_1(S) \subseteq P$. Firstly, if $[u, \rho_2(u)] \in P$ for all $u \in S$, then by Theorem 3, S/P is commutative. Secondly, if $\rho_1(S) \subseteq P$, then from (15), we have $[v, \rho_2(u)] \in P$ which again by Theorem 3 indicates that S/P is commutative. □

1. Main results

Throughout the sequel by S, we mean an MA-semiring and by h_z, k_z, h_o, and k_o, we mean the elements of $Z(S) \cap H(S)$, $Z(S) \cap K(S)$, $H(S)$, and $K(S)$ respectively, unless mentioned otherwise.

Khan et al. [23] proved some results for rings and prime ideals and in this article these results have been proved for a special class of semirings known as MA-semirings which is a generalized framework of rings. For instance: for rings, it makes no difference whether we define commutativity as $[v, w] = 0$ or $wv = vw$; however, for MA-semirings, $[v, w] = 0$ entails $wv = vw$ (c.f Lemma 1(x)), but the reverse may not hold in general. For example the set $R = \mathbb{Z} \times \mathbb{Z}_0^+$ is an MA-semiring with respect to addition $+$ and multiplication . defined by $(a, m) + (x, n) = (a + x, max\{m, n\})$ and $(a, m).(x, n) = (ax, mn)$. The set

$$S = \left\{ \begin{bmatrix} (a,m) & (c,0) & (e,l) \\ (0,0) & (b,n) & (d,0) \\ (0,0) & (0,0) & (a,m) \end{bmatrix} : (a,m),(b,n),(c,0),(d,0),(e,l) \in R \right\}$$

forms an MA-semiring which is not a ring. Let

$$A = \begin{bmatrix} (a,m) & (c,0) & (e,l) \\ (0,0) & (b,n) & (d,0) \\ (0,0) & (0,0) & (a,m) \end{bmatrix}, \ X = \begin{bmatrix} (x,p) & (0,0) & (z,r) \\ (0,0) & (y,q) & (0,0) \\ (0,0) & (0,0) & (x,p) \end{bmatrix} \in S.$$

Then

$$AX = \begin{bmatrix} (ax,mp) & (0,0) & (az+ex, mr+lp) \\ (0,0) & (by,nq) & (0,0) \\ (0,0) & (0,0) & (ax,mp) \end{bmatrix}$$

and

$$XA = \begin{bmatrix} (xa,pm) & (0,0) & (za+xe, rm+pl) \\ (0,0) & (yb,qn) & (0,0) \\ (0,0) & (0,0) & (xa,pm) \end{bmatrix}.$$

It is obvious that $AX = XA$ but then by adding XA' on both the sides, we can find

$$[A,X] = AX + XA' = X(A + A') = \begin{bmatrix} (0,pm) & (0,0) & (0, rm+pl) \\ (0,0) & (0,qn) & (0,0) \\ (0,0) & (0,0) & (0,pm) \end{bmatrix} \neq \mathbf{0},$$

where $A' = \begin{bmatrix} (-a,m) & (-c,0) & (-e,l) \\ (0,0) & (-b,n) & (-d,0) \\ (0,0) & (0,0) & (-a,m) \end{bmatrix}$. Therefore $AX = XA$ does not imply $[A, X] = \mathbf{0}$.

If ρ is a derivation of a ring S, then by the cancellation law, it follows that $\rho(Z(S)) \subseteq Z(S)$. In [23], the above relation is frequently used. However in an additive inverse semiring (MA-semiring) S may not be cancellative, therefore in general the statement $\rho(Z(S)) \subseteq Z(S)$ is not valid, and thus there is no logic to use this containment in this paper.

Following is an extension of Lemma 2.2 of [24].

Lemma 9. *Let P be a prime Q-ideal of S with involution $*$ of P-second kind and S/P be 2-torsion free. If $[t, t^*] \in P$ for all $t \in S$, then S/P is commutative.*

Proof. According to the hypothesis, for all $t \in S$ we have

$$[t, t^*] \in P. \tag{18}$$

After the linearization of (18), we have

$$[t, u^*] + [u, t^*] \in P \tag{19}$$

for all $t, u \in S$. We can write from (19)

$$[t, u^*] + P \oplus [u, t^*] + P = P \tag{20}$$

for all $t, u \in S$. In (19), substituting uk_z for u, we obtain $([t, u^*]' + [u, t^*])k_z \in P$ for all $t, u \in S$. As P is prime, therefore $[t, u^*]' + [u, t^*] \in P$ for all $t, u \in S$ which implies that $[t, u^*]' + P \oplus [u, t^*] + P = P$ for all $t, u \in S$. By Lemma 1, we have

$$[t, u^*] + P = [u, t^*] + P \tag{21}$$

for all $t, u \in S$. Using (21) into (20), we obtain $2[t, u^*] + P = P$ for all $t, u \in S$. By the 2-torsion freeness of S/P, we further get $[t, u^*] + P = P$ and therefore $[t, u^*] \in P$ for all $t, u \in S$. Hence $[t, u] \in P$ for all $t, u \in S$. and by Theorem 2, S/P is commutative. □

Lemma 10. *Let P be a prime Q-ideal of S with involution $*$ of P-second kind and S/P be 2-torsion free. If $\rho(t \circ t^*) \in P$ for all $t \in S$, then $\rho(S) \subseteq P$.*

Proof. According to the hypothesis

$$\rho(t \circ t^*) \in P \tag{22}$$

for all $t \in S$. Linearizing (22), we obtain

$$\rho(t \circ u^*) + \rho(u \circ t^*) \in P \tag{23}$$

for all $t, u \in S$. In (23), changing u by $uh_z, h_z \in H(S) \cap Z(S)$, we obtain

$$\rho(t \circ u^*)h_z + (t \circ u^*)\rho(h_z) + \rho(u \circ t^*)h_z + (u \circ t^*)\rho(h_z) \in P \tag{24}$$

for all $t, u \in S$. As P is prime Q-ideal, using (23) again, we get

$$[(u \circ t^*) + (u \circ t^*)]\rho(h_z) \in P$$

for all $t, u \in S$, which further gives

$$[(u \circ t) + (u^* \circ t^*)]\rho(h_z) \in P \tag{25}$$

for all $t, u \in S$. In (25), changing u by uk_z, we obtain

$$[(u \circ t) + (u^* \circ t^*)^{'}]\rho(h_z)k_z \in P$$

for all $t, u \in S$ and by the primeness of P,

$$[(u \circ t) + (u^* \circ t^*)^{'}]\rho(h_z) \in P,$$

which further implies

$$(u \circ t)\rho(h_z) + P = (u^* \circ t^*)\rho(h_z) \in P \tag{26}$$

for all $t, u \in S$. Using (26) into (25), we obtain $2(u \circ t)\rho(h_z) + P = P$ and hence

$$(u \circ t)\rho(h_z) \in P \tag{27}$$

for all $t, u \in S$. In (27), taking $u = k_z$, we obtain $2k_zS\rho(h_z) \subseteq P$ and 2-torsion freeness implies $k_zS\rho(h_z) \subseteq P$. Since $K(S) \cap Z(S) \nsubseteq P$, therefore $\rho(h_z) \in P$. Further more, as for every $k_z \in K(S) \cap Z(S)$, we have $k_z^2 \in H(S) \cap Z(S)$. Therefore $2k_z\rho(k_z) = \rho(k_z^2) \in P$ which further indicates $k_zS\rho(k_z) \subseteq P$ and by the primeness, we have $\rho(k_z) \in P$. As P is prime Q-ideal, from (24), we obtain

$$[\rho(t \circ u^*) + \rho(t^* \circ u)]h_z \in P$$

for all $t, u \in S$ and by the primeness, it further indicates

$$\rho(t \circ u^*) + \rho(t^* \circ u) \in P$$

for all $t, u \in S$. Replacing u by u^*, we obtain

$$\rho(t \circ u) + \rho(t^* \circ u^*) \in P \tag{28}$$

for all $t, u \in S$. In (28), changing u by uk_z, we obtain

$$\rho(t \circ u) + \rho(t^* \circ u^*) \in P \tag{29}$$

for all $t, u \in S$ and therefore

$$\rho(t \circ u) + P \oplus \rho(t^* \circ u^*) + P = P \tag{30}$$

for all $t, u \in S$. In (29), substituting u for uk_z, we get

$$\rho(t \circ u)k_z + (t \circ u)\rho(k_z) + \rho(t^* \circ u^*)k_z^{'} + (t^* \circ u^*)^{'}\rho(k_z) \in P$$

for all $t, u \in S$ and therefore

$$\rho(t \circ u)k_z + \rho(t^* \circ u^*)k_z^{'} + [(t \circ u) + (t^* \circ u^*)^{'}]\rho(k_z) \in P$$

for all $t, u \in S$. As P is prime Q-ideal, using the fact that $\rho(k_z) \in P$, we obtain

$$\rho(t \circ u)k_z + \rho(t^* \circ u^*)k_z^{'} \in P$$

for all $t, u \in S$ and hence

$$[\rho(t \circ u) + \rho(t^* \circ u^*)^{'}]Sk_z \subseteq P$$

which indicates

$$\rho(t \circ u) + \rho(t^* \circ u^*)^{'} \in P$$

for all $t, u \in S$ and therefore

$$\rho(t \circ u) + P \oplus \rho(t^* \circ u^*)^{'} + P = P$$

for all $t, u \in S$. Using Lemma 1, we obtain

$$\rho(t \circ u) + P = \rho(t^* \circ u^*) + P \tag{31}$$

for all $t, u \in S$. Using (31) into (30) and using 2-torsion freeness of S/P, we obtain $\rho(t \circ u) + P = P$ for all $t, u \in S$ and therefore

$$\rho(t \circ u) \in P \tag{32}$$

for all $t, u \in S$. In (32), taking $u = k_z$, we obtain $2\rho(tk_z) \in P$ and so $\rho(tk_z) = t\rho(k_z) + \rho(t)k_z \in P$ for all $t \in S$. Since $\rho(k_z) \in P$, therefore $\rho(t)k_z \in P$ and $\rho(t)Sk_z \subseteq P$ for all $t \in S$. Thus $\rho(t) \in P$ for all $t \in S$ and hence $\rho(S) \subseteq P$. $\square$

Khan et al. [23] proved Herstein's result [18], for rings with involution. In the following we prove this results for semirings with involution.

Theorem 4. *Let P be a prime Q-ideal of S with involution $*$ of P-second kind, and S/P be 2-torsion free. If ρ_1 and ρ_2 are derivations of S satisfying the condition $[\rho_1(u), \rho_2(u^*)] \in P$ for all $u \in S$, then one of the following holds:*

(i) $\rho_1(S) \subseteq P$;

(ii) $\rho_2(S) \subseteq P$;

(iii) S/P *is commutative.*

Proof. According to the hypothesis, we have

$$[\rho_1(u), \rho_2(u^*)] \in P \tag{33}$$

for all $u \in S$. Linearizing (33) and reusing (33), we obtain

$$[\rho_1(u), \rho_2(w^*)] + [\rho_1(w), \rho_2(u^*)] \in P \tag{34}$$

for all $u, w \in S$. Substituting uh_z for u in (34), we get

$$\begin{aligned}[\rho_1(u), \rho_2(w^*)]h_z + [u\rho_1(h_z), \rho_2(w^*)] + [\rho_1(w), \rho_2(u^*)]h_z \\ + [\rho_1(w), u^*\rho_2(h_z)] \in P\end{aligned}$$

for all $u, w \in S$ and therefore

$$\begin{aligned}([\rho_1(u), \rho_2(w^*)] + [\rho_1(w), \rho_2(u^*)])h_z + [u\rho_1(h_z), \rho_2(w^*)] \\ + [\rho_1(w), u^*\rho_2(h_z)] \in P\end{aligned}$$

for all $u, w \in S$. As P is Q-ideal, using (34), we get

$$[u\rho_1(h_z), \rho_2(w^*)] + [\rho_1(w), u^*\rho_2(h_z)] \in P \tag{35}$$

for all $u, w \in S$. In (35), changing h_z by k_z^2 and using the 2-torsion freeness, we obtain

$$[u\rho_1(k_z), \rho_2(w^*)] + [\rho_1(w), u^*\rho_2(k_z)]k_z \in P$$

for all $u, w \in S$ and using the similar arguments as above, we can write

$$[u\rho_1(k_z), \rho_2(w^*)] + [\rho_1(w), u^*\rho_2(k_z)] \in P \tag{36}$$

for all $u, w \in S$. In (36), changing u by uk_o and using (36) again, we get

$$[u\rho_1(k_z), \rho_2(w^*)] + [\rho_1(w), u^*\rho_2(k_z)]' \in P \tag{37}$$

for all $u, w \in S$. In (34), changing u by uk_z, we get

$$\begin{aligned}[\rho_1(u), \rho_2(w^*)]k_z + [\rho_1(w), \rho_2(u^*)]' k_z + [u\rho_1(k_z), \rho_2(w^*)] \\ + [\rho_1(w), u^*\rho_2(k_z)]' \in P\end{aligned} \tag{38}$$

for all $u, w \in S$. As P is prime Q-ideal, using (37) into (38), we obtain

$$[\rho_1(u), \rho_2(w^*)]k_z + [\rho_1(w), \rho_2(u^*)]' k_z \in P$$

for all $u, w \in S$ and therefore

$$([\rho_1(u), \rho_2(w^*)] + [\rho_1(w), \rho_2(u^*)]') \in P$$

for all $u, w \in S$. Using Lemma 1, we can write

$$[\rho_1(u), \rho_2(w^*)] + P = [\rho_1(w), \rho_2(u^*)] + P \tag{39}$$

for all $u, w \in S$. From (34), we can write

$$[\rho_1(u), \rho_2(w^*)] + P \oplus [\rho_1(w), \rho_2(u^*)] + P = P \tag{40}$$

for all $u, w \in S$. Using (39) into (40), we obtain $2[\rho_1(u), \rho_2(w^*)] + P = P$ for all $u, w \in S$ and by the 2-torsion freeness of S/P, we obtain $[\rho_1(u), \rho_2(w^*)] \in P$ for all $u, w \in S$, which further gives

$$[\rho_1(u), \rho_2(w)] \in P \tag{41}$$

for all $u, w \in S$. By Lemma 6, we conclude that $\rho_1(S) \subseteq P$ (or $\rho_2(S) \subseteq P$) or S/P is commutative. $\square$

Theorem 4 has the following direct consequences.

Corollary 1. *Let S be a 2-torsion free prime MA-semiring with involution $*$ of the second kind. If ρ_1 and ρ_2 are nonzero derivations of S satisfying $[\rho_1(u), \rho_2(u^*)] = 0$ for all $u \in S$, then S is commutative.*

Proof. In Theorem 4, taking $P = \{0\}$, we obtain the required result. $\square$

Corollary 2. *Let P be a prime Q-ideal of S with involution $*$ of the P-second kind and S/P be 2-torsion free. If ρ_1 and ρ_2 are derivations of S satisfying the condition $[\rho_1(u), \rho_2(v^*)] \in P$ for all $u, v \in S$, then one of the following holds:*

(i) $\rho_1(S) \subseteq P$;

(ii) $\rho_2(S) \subseteq P$;

(iii) S/P *is commutative.*

Corollary 3. *Let* P *be a prime* Q*-ideal of* S *with involution* $*$ *of the* P*-second kind and* S/P *be 2-torsion free. If* ρ *is a derivation of* S *satisfying the condition* $[\rho(u), \rho(u^*)] \in P$ *for all* $u \in S$*, then either* $\rho(S) \subseteq P$ *or* S/P *is commutative.*

Corollary 4. *Let* S *be a 2-torsion free prime MA-semiring with involution* $*$ *of the second kind. If* ρ *is a nonzero derivation of* S *satisfying* $[\rho(u), \rho(u^*)] = 0$ *for all* $u \in S$*, then* S *is commutative.*

Proof. In Corollary 3, taking $P = \{0\}$ and considering the case when $\rho \neq 0$, we obtain the required result. □

Corollary 5. *Let* P *be a prime* Q*-ideal of* S *with involution* $*$ *of the* P*-second kind and* S/P *be 2-torsion free. If* ρ_1 *and* ρ_2 *are derivations of* S *satisfying the condition* $[\rho_1(u), \rho_1(u^*)] + \rho_2(u \circ u^*) \in P$ *for all* $u \in S$*, then either (*$\rho_1(S) \subseteq P$ *and* $\rho_2(S) \subseteq P$*) or* S/P *is commutative.*

Proof. According to the hypothesis, for all $u \in S$ we have

$$[\rho_1(u), \rho_1(u^*)] + \rho_2(u \circ u^*) \subseteq P. \tag{42}$$

In (42), substituting u^* for u, we obtain

$$[\rho_1(u), \rho_1(u^*)]' + \rho_2(u \circ u^*) \subseteq P$$

and using Lemma 1, the last expression further indicates

$$[\rho_1(u), \rho_1(u^*)] + P = \rho_2(u \circ u^*) + P \tag{43}$$

for all $u \in S$. Again from (42), we can write

$$[\rho_1(u), \rho_1(u^*)] + P \oplus \rho_2(u \circ u^*) + P = P \tag{44}$$

for all $u \in S$. As S/P is 2-torsion free, using (43) into (44), we obtain $[\rho_1(u), \rho_1(u^*)] + P = P$ for all $u \in S$, which further indicates $[\rho_1(u), \rho_1(u^*)] \in P$ for all $u \in S$ and employing Corollary 3, we conclude that $\rho_1(S) \subseteq P$ or S/P is commutative. Using this information, from (42), we have $\rho_2(u \circ u^*) \in P$ for all $u \in S$ and by Lemma 10 we have $\rho_2(S) \subseteq P$ or S/P is commutative. □

Theorem 5. *Let P be a prime Q-ideal of S with involution $*$ of the P-second kind and S/P be 2-torsion free. If ρ_1 and ρ_2 are derivations of S such that for all $u \in S$*

$$\rho_1(u) \circ \rho_2(u^*) \in P, \tag{45}$$

then $\rho_1(S) \subseteq P$ or $\rho_2(S) \subseteq P$.

Proof. From (45), after the linearization we have

$$\rho_1(u) \circ \rho_2(v^*) + \rho_1(v) \circ \rho_2(u^*) \in P \tag{46}$$

for all $u, v \in S$. In (46), substituting uh_z for u and using Lemma 1, we obtain

$$\rho_1(uh_z) \circ \rho_2(v^*) + \rho_1(v) \circ \rho_2(u^* h_z) \in P$$

for all $u, v \in S$, which further indicates

$$(\rho_1(u)\circ\rho_2(v^*)+\rho_1(v)\circ\rho_2(u^*))h_z+(u\rho_1(h_z))\circ\rho_2(v^*)+\rho_1(v)\circ(u^*\rho_2(h_z))\in P$$

for all $u, v \in S$. Using (46) again, we obtain

$$(u\rho_1(h_z)) \circ \rho_2(v^*) + \rho_1(v) \circ (u^*\rho_2(h_z)) \in P \tag{47}$$

for all $u, v \in S$. In (47), writing k_z^2 for h_z and using the same reasoning as above, we obtain

$$(u\rho_1(k_z)) \circ \rho_2(v^*) + \rho_1(v) \circ (u^*\rho_2(k_z)) \in P \tag{48}$$

for all $u, v \in S$. In (48), changing u by uk_z, we get

$$(u\rho_1(k_z)) \circ \rho_2(v^*) + (\rho_1(v) \circ (u^*\rho_2(k_z)))' \in P \tag{49}$$

for all $u, v \in S$. In (45), changing u by uk_z, we get

$$[(\rho_1(u) \circ \rho_2(v^*)) + (\rho_1(v) \circ \rho_2(u^*))']k_z + [(u\rho_1(k_z)) \circ \rho_2(v^*) \\ + (\rho_1(v) \circ (u^*\rho_2(k_z))'] \in P$$

for all $u, v \in S$. As P is prime Q-ideal, using (49) again, we get

$$[(\rho_1(u) \circ \rho_2(v^*)) + (\rho_1(v) \circ \rho_2(u^*))']k_z \in P$$

for all $u, v \in S$ and therefore by the primeness of P

$$[(\rho_1(u) \circ \rho_2(v^*)) + (\rho_1(v) \circ \rho_2(u^*))'] \in P$$

and by Lemma 1, we can write

$$(\rho_1(u) \circ \rho_2(v^*)) + P = (\rho_1(v) \circ \rho_2(u^*)) + P \tag{50}$$

for all $u, v \in S$. From (48), we can write $(u\rho_1(k_z)) \circ \rho_2(v^*) + P \oplus \rho_1(v) \circ (u^*\rho_2(k_z)) + P = P$ and using (50) we get

$$2(\rho_1(u) \circ \rho_2(v^*)) + P = P$$

for all $u, v \in S$. As S/P is 2-torsion free, therefore

$$(\rho_1(u) \circ \rho_2(v^*)) + P = P$$

for all $u, v \in S$. This means $\rho_1(u) \circ \rho_2(v^*) \in P$ for all $u, v \in S$, which further indicates that

$$\rho_1(u) \circ \rho_2(v) \in P \tag{51}$$

for all $u, v \in S$, and use of Lemma 5 completes the proof. □

Theorem 6. *Let P be a prime Q-ideal of S with involution $*$ of the P-second kind and S/P be 2-torsion free. If ρ_1 and ρ_2 are derivations of S satisfying*

$$[\rho_1(u), \rho_2(u^*)] + [u, u^*]' \in P \tag{52}$$

for all $u \in S$, then S/P is a commutative.

Proof. If $\rho_1(S) \subseteq P$ or $\rho_2(S) \subseteq P$, or $\rho_1(S) \subseteq P$ and $\rho_2(S) \subseteq P$, then (52) gives $[u, u^*]' \in P$ for all $u \in S$ and by Lemma 9, S/P is commutative. We consider the other case when $\rho_1(S) \nsubseteq P$ and $\rho_2(S) \nsubseteq P$. Linearizing (52) and using (52) again, we get

$$[\rho_1(u), \rho_2(v^*)] + [u, v^*]' + [\rho_1(v), \rho_2(u^*)] + [v, u^*]' \in P \tag{53}$$

for all $u, v \in S$. In (53), changing u by uh_z, we obtain

$$[\rho_1(uh_z), \rho_2(v^*)] + [\rho_1(v), \rho_2(u^*h_z)] + [u, v^*]'h_z + [v, u^*]'h_z \in P$$

for all $u, v \in S$. Using Lemma 1 and then rearranging the terms, we obtain

$$\begin{gathered}[u\rho_1(h_z), \rho_2(v^*)] + [\rho_1(v), u^*\rho_2(h_z)] + ([\rho_1(u), \rho_2(v^*)] \\ + [\rho_1(v), \rho_2(u^*)] + [u, v^*]' + [v, u^*]')h_z \in P\end{gathered}$$

for all $u, v \in S$ and since P is prime Q-ideal, using (53) again, we get

$$[u\rho_1(h_z), \rho_2(v^*)] + [\rho_1(v), u^*\rho_2(h_z)] \in P \tag{54}$$

for all $u, v \in S$. Writing uk_z for u in (54) and using primeness of P, we obtain

$$[u\rho_1(h_z), \rho_2(v^*)] + [\rho_1(v), u^*\rho_2(h_z)]^{'} \in P \tag{55}$$

for all $u, v \in S$. In (55), changing h_z by k_z^2, we obtain

$$[u\rho_1(k_z), \rho_2(v^*)] + [\rho_1(v), u^*\rho_2(k_z)]^{'} \in P \tag{56}$$

for all $u, v \in S$. In (53), changing u by uk_z, we obtain

$$\begin{aligned} [u\rho_1(k_z), \rho_2(v^*)] + [\rho_1(u), \rho_2(v^*)]k_z + [u, v^*]^{'}k_z + [\rho_1(v), u^*\rho_2(k_z)]^{'} \\ + [\rho_1(v), \rho_2(u^*)]^{'}k_z + [v, u^*]k_z \in P \end{aligned} \tag{57}$$

for all $u, v \in S$. Using (56) into (57), we obtain

$$[\rho_1(u), \rho_2(v^*)]k_z + [u, v^*]^{'}k_z + [\rho_1(v), \rho_2(u^*)]^{'}k_z + [v, u^*]k_z \in P$$

for all $u, v \in S$ and therefore

$$[\rho_1(u), \rho_2(v^*)] + [u, v^*]^{'} + [\rho_1(v), \rho_2(u^*)]^{'} + [v, u^*] \in P$$

for all $u, v \in S$. Using Lemma 1, we obtain

$$[\rho_1(u), \rho_2(v^*)] + [u, v^*]^{'} + P \oplus [\rho_1(v), \rho_2(u^*)]^{'} + [v, u^*] + P = P \tag{58}$$

for all $u, v \in S$. From (53), we can write

$$[\rho_1(u), \rho_2(v^*)] + [u, v^*]^{'} + P = [\rho_1(v), \rho_2(u^*)]^{'} + [v, u^*] + P \tag{59}$$

for all $u, v \in S$. As S/P is 2-torsion free, using (59) into (58), we obtain

$$[\rho_1(u), \rho_2(v^*)] + [u, v^*]^{'} + P = P$$

for all $u, v \in S$ and therefore

$$[\rho_1(u), \rho_2(v^*)] + [u, v^*]^{'} \in P$$

for all $u, v \in S$ and changing v by v^*, we obtain

$$[\rho_1(u), \rho_2(v)] + [u, v]^{'} \in P \tag{60}$$

for all $u, v \in S$. In (60), replacing v by vs and using Lemma 1, we obtain

$$[\rho_1(u), \rho_2(v)s] + [\rho_1(u), v\rho_2(s)] + v[u, s]^{'} + [u, v]^{'}s \in P$$

for all $u, v, s \in S$, which further implies

$$\rho_2(v)[\rho_1(u), s] + [\rho_1(u), \rho_2(v)]s + v[\rho_1(u), \rho_2(s)] \\ + [\rho_1(u), v]\rho_2(s) + v[u, s]' + [u, v]'s \in P$$

for all $u, v, s \in S$. Using (60), we obtain

$$\rho_2(v)[\rho_1(u), s] + [\rho_1(u), v]\rho_2(s) \in P \tag{61}$$

for all $u, v, s \in S$. The expression (61) is similar to the (7) of Lemma 5, therefore by the same reasoning, remaining part of the required proof follows. □

Theorem 7. *Let P be a prime Q-ideal of S with involution $*$ of the P-second kind and S/P be 2-torsion free. If ρ_1 and ρ_2 are derivations of S satisfying*

$$[\rho_1(u), \rho_2(u^*)] + [\rho_1(u), u^*] + [u, \rho_2(u^*)] \in P \tag{62}$$

for all $u \in S$, then $\rho_1(S) \subseteq P$ and $\rho_2(S) \subseteq P$, or S/P is a commutative.

Proof. Linearizing (62), we obtain

$$[\rho_1(u), \rho_2(v^*)] + [\rho_1(v), \rho_2(u^*)] + [\rho_1(u), v^*] + [\rho_1(v), u^*] \\ + [u, \rho_2(v^*)] + [v, \rho_2(u^*)] \in P \tag{63}$$

for all $u, v \in S$. In (63), changing u by uh_z, we obtain

$$[\rho_1(u), \rho_2(v^*)]h_z + [u\rho_1(h_z), \rho_2(v^*)] + [\rho_1(v), \rho_2(u^*)]h_z + [\rho_1(v), u^*\rho_2(h_z)] \\ + [\rho_1(u), v^*]h_z + [u\rho_1(h_z), v^*] + [\rho_1(v), u^*]h_z + [u, \rho_2(v^*)]h_z \\ + [v, \rho_2(u^*)]h_z + [v, u^*\rho_2(h_z)] \in P$$

for all $u, v \in S$, and therefore

$$([\rho_1(u), \rho_2(v^*)] + [\rho_1(v), \rho_2(u^*)] + [\rho_1(u), v^*] + [\rho_1(v), u^*] + [u, \rho_2(v^*)] \\ + [v, \rho_2(u^*)])h_z + [u\rho_1(h_z), \rho_2(v^*)] + [\rho_1(v), u^*\rho_2(h_z)] \\ + [u\rho_1(h_z), v^*] + [v, u^*\rho_2(h_z)] \in P$$

for all $u, v \in S$. As P is prime Q-ideal, using (63) in the last relation, we get

$$[u\rho_1(h_z), \rho_2(v^*)] + [\rho_1(v), u^*\rho_2(h_z)] + [u\rho_1(h_z), v^*] + [v, u^*\rho_2(h_z)] \in P \tag{64}$$

for all $u, v \in S$. In (64), changing h_z by k_z^2 and using the primeness of P, we obtain

$$[u\rho_1(k_z), \rho_2(v^*)]+[\rho_1(v), u^*\rho_2(k_z)]+[u\rho_1(k_z), v^*]+[v, u^*\rho_2(k_z)] \in P. \quad (65)$$

In (65), changing u by uk_o and using same reasoning as above, we obtain

$$[u\rho_1(k_z), \rho_2(v^*)]+[\rho_1(v), u^*\rho_2(k_z)]^{'}+[u\rho_1(k_z), v^*]+[v, u^*\rho_2(k_z)]^{'} \in P \quad (66)$$

for all $u, v \in S$. In (62), changing u by uk_z and using the same reasoning as above, we obtain

$$\begin{aligned}[\rho_1(u), &\rho_2(v^*)]k_z + [\rho_1(v), \rho_2(u^*)]^{'}k_z + [\rho_1(u), v^*]k_z + [\rho_1(v), u^*]^{'}k_z \\ &+ [u, \rho_2(v^*)]k_z + [v, \rho_2(u^*)]^{'}k_z + [u\rho_1(k_z), \rho_2(v^*)] \\ &+ [\rho_1(v), u^*\rho_2(k_z)]^{'} + [u\rho_1(k_z), v^*] + [v, u^*\rho_2(k_z)]^{'} \in P\end{aligned} \quad (67)$$

for all $u, v, s \in S$. As P is Prime Q-ideal, using (66) into (67), we get

$$\begin{aligned}[\rho_1(u), \rho_2(v^*)]k_z + [\rho_1(v), \rho_2(u^*)]^{'}k_z + [\rho_1(u), v^*]k_z + [\rho_1(v), u^*]^{'}k_z \\ + [u, \rho_2(v^*)]k_z + [v, \rho_2(u^*)]^{'}k_z \in P\end{aligned}$$

for all $u, v \in S$ and primeness of P it further leads to

$$\begin{aligned}[\rho_1(u), \rho_2(v^*)] + [\rho_1(v), \rho_2(u^*)]^{'} + [\rho_1(u), v^*] + [\rho_1(v), u^*]^{'} \\ + [u, \rho_2(v^*)] + [v, \rho_2(u^*)]^{'} \in P\end{aligned} \quad (68)$$

for all $u, v, s \in S$. By Lemma 1, from (68), we have

$$\begin{aligned}[\rho_1(u), &\rho_2(v^*)] + [\rho_1(v), \rho_2(u^*)] + [\rho_1(u), v^*] + [\rho_1(v), u^*] \\ &+ [u, \rho_2(v^*)] + [v, \rho_2(u^*)] + 2[\rho_1(v), \rho_2(u^*)]^{'} \\ &+ 2[\rho_1(v), u^*]^{'} + 2[v, \rho_2(u^*)]^{'} \in P\end{aligned} \quad (69)$$

for all $u, v \in S$. Using (63) into (69) and by the 2-torsion freeness of S/P, we obtain

$$[\rho_1(v), \rho_2(u^*)] + [\rho_1(v), u^*] + [v, \rho_2(u^*)] \in P \quad (70)$$

for all $u, v \in S$. In (70) u by u^*, we obtain

$$[\rho_1(v), \rho_2(u)] + [\rho_1(v), u] + [v, \rho_2(u)] \in P \quad (71)$$

for all $u, v \in S$. Employing Lemma 8, (71) infer that S/P is commutative. □

Next we state some interesting consequences of Theorem 7 and can be followed easily as particular cases.

Corollary 6. *Let S be a 2-torsion free prime MA-semiring with involution $*$ of the second kind. If ρ_1 and ρ_2 are nonzero derivations of S satisfying*

$$[\rho_1(u), \rho_2(u^*)] + [\rho_1(u), u^*] + [u, \rho_2(u^*)] = 0 \quad \textit{for all } u \in S,$$

then S is commutative.

Corollary 7. *Let P be a prime Q-ideal of S with involution $*$ of the P-second kind and S/P be 2-torsion free. If ρ is a derivation of S satisfying*

$$[\rho(u), \rho(u^*)] + [\rho(u), u^*] + [u, \rho(u^*)] \in P \quad \textit{for all } u \in S,$$

then $\rho(S) \subseteq P$, or S/P is a commutative.

Corollary 8. *Let S be a 2-torsion free prime MA-semiring with involution $*$ of the second kind. If ρ is a nonzero derivation of S satisfying*

$$[\rho(u), \rho(u^*)] + \rho[u, u^*] = 0 \quad \textit{for all } u \in S,$$

then S is a commutative.

Theorem 8. *Let P be a prime Q-ideal of S with involution $*$ of the P-second kind and S/P be 2-torsion free. If ρ_1 and ρ_2 are derivations of S satisfying*

$$\rho_1(u) \circ \rho_2(u^*) + (\rho_1(u) \circ u^*)^{'} + (u \circ \rho_2(u^*))^{'} \in P, \quad \forall u \in S, \tag{72}$$

then $\rho_1(S) \subseteq P$ and $\rho_2(S) \subseteq P$.

Proof. After the linearization of (72), we obtain

$$\begin{aligned}\rho_1(u) \circ \rho_2(v^*) + \rho_1(v) \circ \rho_2(u^*) + (\rho_1(u) \circ v^*)^{'} + (\rho_1(v) \circ u^*)^{'} \\ + (u \circ \rho_2(v^*))^{'} + (v \circ \rho_2(u^*))^{'} \in P\end{aligned} \tag{73}$$

for all $u, v \in S$. In (73), changing u by uh_z, we get after the rearrangements of terms

$$\begin{aligned}(\rho_1(u) \circ \rho_2(v^*) + (\rho_1(v) \circ \rho_2(u^*)) + (\rho_1(u) \circ v^*)^{'} + (\rho_1(v) \circ u^*)^{'} \\ + (u \circ \rho_2(v^*))^{'} + (v \circ \rho_2(u^*))^{'})h_z + (u\rho_1(h_z)) \circ \rho_2(v^*) \\ + \rho_1(v) \circ (u^*\rho_2(h_z)) + ((u\rho_1(h_z)) \circ v^*)^{'} + (v \circ (u^*\rho_2(h_z)))^{'} \in P\end{aligned} \tag{74}$$

for all $u, v \in S$. As P is Prime Q-ideal, using (73) into (74), we obtain

$$\begin{gathered}(u\rho_1(h_z)) \circ \rho_2(v^*) + \rho_1(v) \circ (u^*\rho_2(h_z)) + ((u\rho_1(h_z)) \circ v^*)' \\ + (v \circ (u^*\rho_2(h_z)))' \in P\end{gathered} \tag{75}$$

for all $u, v \in S$. In (75), putting $h_z = k_z^2$ and using the same reasoning as above, we get

$$\begin{gathered}(u\rho_1(k_z)) \circ \rho_2(v^*) + \rho_1(v) \circ (u^*\rho_2(k_z)) + ((u\rho_1(k_z)) \circ v^*)' \\ + (v \circ (u^*\rho_2(k_z)))' \in P\end{gathered} \tag{76}$$

for all $u, v \in S$. In (73), substituting uk_z for u, we get

$$\begin{gathered}(\rho_1(u) \circ \rho_2(v^*) + (\rho_1(v) \circ \rho_2(u^*))' + (\rho_1(u) \circ v^*)' + (\rho_1(v) \circ u^*) \\ + (u \circ \rho_2(v^*))' + (v \circ \rho_2(u^*)))k_z + (u\rho_1(k_z)) \circ \rho_2(v^*) \\ + \rho_1(v) \circ (u^*\rho_2(k_z))' + ((u\rho_1(k_z)) \circ v^*)' + (v \circ (u^*\rho_2(k_z))) \in P\end{gathered} \tag{77}$$

for all $u, v \in S$. In (76), changing u by uk_o, we obtain

$$\begin{gathered}(u\rho_1(k_z)) \circ \rho_2(v^*) + \rho_1(v) \circ (u^*\rho_2(k_z))' + ((u\rho_1(k_z)) \circ v^*)' \\ + (v \circ (u^*\rho_2(k_z))) \in P\end{gathered} \tag{78}$$

for all $u, v \in S$. As P is prime Q-ideal, using (78) into (77), we obtain

$$\begin{gathered}\rho_1(u) \circ \rho_2(v^*) + (\rho_1(v) \circ \rho_2(u^*))' + (\rho_1(u) \circ v^*)' + (\rho_1(v) \circ u^*) \\ + (u \circ \rho_2(v^*))' + (v \circ \rho_2(u^*)) \in P\end{gathered} \tag{79}$$

for all $u, v \in S$. Using Lemma 1, we can write

$$\begin{aligned}&[\rho_1(u) \circ \rho_2(v^*) + (\rho_1(u) \circ v^*)' + (u \circ \rho_2(v^*))'] + P \\ &= [(\rho_1(v) \circ \rho_2(u^*)) + (\rho_1(v) \circ u^*)' + (v \circ \rho_2(u^*))'] + P\end{aligned} \tag{80}$$

for all $u, v, s \in S$. From (73), we can write

$$\begin{aligned}&[\rho_1(u) \circ \rho_2(v^*) + (\rho_1(u) \circ v^*)' + (u \circ \rho_2(v^*))'] + P \\ &\oplus [(\rho_1(v) \circ \rho_2(u^*)) + (\rho_1(v) \circ u^*)' + (v \circ \rho_2(u^*))'] + P = P\end{aligned} \tag{81}$$

for all $u, v, s \in S$. Using (80) into (81) and then using 2-torsion freeness of S/P, we obtain

$$\rho_1(u) \circ \rho_2(v^*) + (\rho_1(u) \circ v^*)' + (u \circ \rho_2(v^*))' \in P$$

for all $u, v \in S$ and changing v by v^*, we obtain

$$\rho_1(u) \circ \rho_2(v) + (\rho_1(u) \circ v)^{'} + (u \circ \rho_2(v))^{'} \in P \tag{82}$$

for all $u, v \in S$. In (82), changing v by vk, we obtain

$$\begin{gathered}(\rho_1(u) \circ \rho_2(v))k_z + \rho_1(u) \circ (v\rho_2(k_z)) + (\rho_1(u) \circ v)^{'}k_z \\ + (u \circ \rho_2(v))^{'}k_z + (u \circ (v\rho_2(k_z)))^{'} \in P\end{gathered}$$

for all $u, v \in S$. Using (82) again, we obtain

$$\rho_1(u) \circ (v\rho_2(k_z)) + (u \circ (v\rho_2(k_z))^{'}) \in P$$

for all $u, v \in S$. Taking $v = h_z$ and using primeness of P, we can write

$$\rho_1(u) \circ \rho_2(k_z) + (u \circ \rho_2(k_z))^{'} \in P \tag{83}$$

for all $u \in S$. For $u = h_z$, we obtain from (83)

$$\rho_1(h_z) \circ \rho_2(k_z) + (h_z \circ \rho_2(k_z))^{'} \in P. \tag{84}$$

In (83), changing u by uh_z, we obtain

$$u\rho_1(h_z) \circ \rho_2(k_z) + \rho_1(u) \circ \rho_2(k_z)h_z + (u \circ \rho_2(k_z))^{'}h_z \in P$$

for all $u \in S$ and using (83) again, we get

$$u\rho_1(h_z) \circ \rho_2(k_z) \in P$$

and taking $u = k_z$ in particular, and using primeness, we obtain

$$\rho_1(h_z) \circ \rho_2(k_z) \in P. \tag{85}$$

As P is prime Q-ideal and S/P is 2-torsion free, using (85) into (84), we get $\rho_2(k_z) \in P$. Therefore taking $v = k_z$ in (82), we obtain $\rho_1(S) \subseteq P$ which further implies $\rho_1(k_z) \in P$. Taking $u = k_z$ in (82), we obtain $\rho_2(S) \subseteq P$ and this completes the proof. □

We now present some interesting consequences of Theorem 8.

Corollary 9. *Let P be a prime Q-ideal of S with involution $*$ of the P-second kind and S/P be 2-torsion free. If ρ is derivation of S such that for all for all $u \in S$,*

$$\rho(u) \circ \rho(u^*) + (\rho(u) \circ u^*)^{'} + (u \circ \rho(u^*))^{'} \in P \quad \textit{for all } u \in S,$$

then $\rho(S) \subseteq P$.

Corollary 10. *Let S be a 2-torsion free prime MA-semiring with involution $*$ of the second kind. If ρ_1 and ρ_2 are derivations of S satisfying*

$$\rho_1(u) \circ \rho_2(u^*) + (\rho_1(u) \circ u^*)' + (u \circ \rho_2(u^*))' = 0 \quad \text{for all } u \in S,$$

then $\rho_1 = 0 = \rho_2$.

Remark 1. Last corollary can also be stated as:
Let S be a 2-torsion free prime MA-semiring with involution $*$ of the second kind. Then no nonzero derivations ρ_1 and ρ_2 can satisfy

$$\rho_1(u) \circ \rho_2(u^*) + (\rho_1(u) \circ u^*)' + (u \circ \rho_2(u^*))' = 0 \quad \text{for all } u \in S.$$

Corollary 11. *Let S be a 2-torsion free prime MA-semiring with involution $*$ of the second kind. Then no nonzero derivations ρ_1 and ρ_2 of S can satisfy*

$$\rho_1(u) \circ \rho_2(u^*) + (\rho_1(u \circ u^*))' = 0 \quad \text{for all } u \in S.$$

Concluding remarks

In this research, we studied MA-semiring S equipped with P-second kind involution, where P represents a prime ideal of S. In this context, we established some differential identities that ensure various characteristics of the quotient MA-semirings S/P.

References

[1] Ali, L., Aslam, M., Khan, Y.A.: Commutativity of semirings with involution. Asian-Eur. J. Math. **14**(01), 2050153 (2021). https://doi.org/10.1142/S1793557120501533.

[2] Ali, L., Khan, Y.A., Aslam, M.: On Posner's second theorem for semirings with involution. J. Discrete Math. Sci. Crypt. **23**(6), 1195–1202 (2020). https://doi.org/10.1080/09720529.2020.1809113

[3] Ali, L., Aslam, M., Khan, Y.A.: Some commutativity conditions on $*$-prime semirings. JP J. Algebra, Number Theory Appl. **46**(02), 109–121 (2020). http://dx.doi.org/10.17654/NT046020109

[4] Ali, L., Aslam, M., Khan, Y.A.: On additive maps of MA-semirings with involution. Proyecciones (Antofagasta, On line) **39**(04), 1097–1112 (2020). https://doi.org/10.22199/issn.0717-6279-2020-04-0067

[5] Ali, L., Aslam, M., Farid, G., Khalek, S.A.: On differential identities of Jordan ideals of semirings. AIMS Math. **6**(7), 6833–6844 (2020). https://www.aimspress.com/article/doi/10.3934/math.2021400

[6] Ali, L., Khan, Y.A., Mousa, A.A., Khalek, S.A., Farid, G.: Some differential identities of MA-semirings with involution. AIMS Math. **6**(3), 2304–2314 (2020). https://www.aimspress.com/article/doi/10.3934/math.2021139

[7] Ali, L., Aslam, M., Khan, Y.A., Farid, G., Qureshi, S.I., Rehman, S.: Commutativity of MA-semirings with involution through generalized derivations. J. Math. **2020**, 8867247 (2020). https://doi.org/10.1155/2020/8867247

[8] Ali, L., Aslam, M., Farid, G., Mahmood, T.: On differential identities involving partitioning ideals of semirings. Comm. Korean Math. Soc. **39**(3), 595–609 (2024). https://doi.org/10.4134/CKMS.c230288

[9] Atani, S.E., Atani, R.E.: Some remarks on partitioning semirings. An. St. Univ. Ovidius Constanta **18**(1), 49–62 (2010)

[10] Atani, S.E.: The zero-divisor graph with respect to ideals of a commutative semiring. Glas. Math. **43**(63), 309–320 (2008). https://doi.org/10.3336/gm.43.2.06

[11] Atani, S.E., Atani, R.E.: Ideal theory in commutative semirings. Bul. Acad. Ştiinte Repub. Mold. Mat. **2**(57), 14–23 (2008)

[12] Atani, S.E.: The ideal theory in quotients of commutative semirings. Glas. Mat. **42**(62), 301–308 (2007). https://doi.org/10.3336/gm.42.2.05

[13] Awtar, R.: Lie and Jordan structure in prime rings with derivations. Proc. Amer. Math. Soc. **41**, 67–74 (1973). https://doi.org/10.1090/S0002-9939-1973-0318233-5

[14] Bandlet, H.J., Petrich, M.: Subdirect products of rings and distrbutive lattices. Proc. Edinburgh Math. Soc. **25**(2), 135–171 (1982). https://doi.org/10.1017/S0013091500016643

[15] Bell, H.E., Martindale, W.S.: Centralizing mappings of semiprime rings. Canadian Math. Bull. **30**(1), 92–101 (1987). https://doi.org/10.4153/CMB-1987-014-x

[16] Berger, J., Herstein, I.N., Kerr, J.W.: Lie ideals and derivations of prime rings. J. Algebra **71**(1), 259–267 (1981). https://doi.org/10.1016/0021-8693(81)90120-4

[17] Bresar, M.: On the distance of the composition of two derivations to the generalized derivations. Glasgow Math. J. **33**(1), 89–93 (1991). https://doi.org/10.1017/S0017089500008077

[18] Herstein, I.N.: A note on derivations. Canadian Math. Bull. **21**(3), 369–370 (1978). https://doi.org/10.4153/CMB-1978-065-x

[19] Iséki, K.: Quasiideals in semirings without zero. Proc. Japan Acad. **34**(2), 79–81 (1958). https://doi.org/10.3792/pja/1195524783

[20] Javed, M.A., Aslam, M., Hussain, M.: On condition (A_2) of Bandlet and Petrich for inverse semirings. Int. Math. Forum **7**(59), 2903–2914 (2012)

[21] Jordan, D.A.: On the ideals of a Lie algebra of derivations. J. Lond. Math. Soc. **2**(33), 33–39 (1986). https://doi.org/10.1112/jlms/s2-33.1.33

[22] Khan, Y.A., Aslam, M., Ali, L.: Commutativity of additive inverse semirings through $f(xy) = [x, f(y)]$. Thai J. Math. Special Issue. 288–300 (2018). https://thaijmath.com/index.php/thaijmath/article/view/743

[23] Khan, M.S., Ali, S., Ayedh, M.: Herstein's theorem for prime ideals in rings with involution involving pair of derivations. Comm. Algebra **50**(6), 2592–2603 (2022). https://doi.org/10.1080/00927872.2021.2014513

[24] Mahmood, T., Ali, L., Aslam, M., Farid, G.: On commutativity of quotient semirings through generalized derivations. AIMS Math. **8**(11), 25729–25739 (2023). https://www.aimspress.com/article/doi/10.3934/math.20231312

[25] Mamouni, A., Zerra, L., Oukhtite, L.: On derivations involving prime ideals and commutativity in rings. Sao Paulo J. Math. Sci. **14**, 675–688 (2020). https://doi.org/10.1007/s40863-020-00187-z

[26] Mir, H.E., Mamouni, A., Oukhtite, L.: Commutativity with algebraic identities involving prime ideals. Comm. Korean Math. Soc. **35**(3), 723–731 (2020). https://doi.org/10.4134/CKMS.c190338

[27] Posner, E.C.: Derivations in prime rings. Proc. Amer. Math. Soc. **8**, 1093–1100 (1957). https://doi.org/10.1090/S0002-9939-1957-0095863-0

[28] Rehman, N.U., Alnoghashi, H.M.: Action of prime ideals on generalized derivations-I. Preprint at https://doi.org/10.48550/arXiv.2107.06769 (2021).

[29] Sen, M.K., Adhikari, M.R.: On k-ideals of semirings. Int. J. Math. Sci. **15**, 347–350 (1992). https://doi.org/10.1155/S0161171292000437

[30] Shafiq, S., Aslam, M., Javed, M.A.: On centralizer of semiprime inverse semiring. Discuss. Math. Gen. Algebra Appl. **36**, 71–84 (2016). http://dx.doi.org/10.7151/dmgaa.1252

[31] Shafiq, S., Aslam, M.: On Jordan mappings of inverse semirings. Open Math. **15**(1), 1123–1131 (2017). https://doi.org/10.1515/math-2017-0088

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