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Abelianization of the Cartwright-Steger lattice*

G. Blachar, O. Sela–Ben-David, and U. Vishne

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Abstract. The Cartwright-Steger lattice is a group whose Cayley graph can be identified with the Bruhat-Tits building of PGL_d over a local field of positive characteristic. We give a lower bound on the abelianization of this lattice, and report that the bound is tight in all computationally accessible cases.

Introduction

Arithmetic lattices acting on trees provide a linkage between group theory, arithmetic and dynamics, paving the way to applications of representation theory to combinatorics, as beautifully demonstrated by Margulis and Lubotzky-Philips-Sarnak in the construction of Ramanujan graphs (see [4]). The lattice constructed by Cartwright and Steger [3] is a higher-dimensional analog, acting on a Bruhat-Tits building associated with PGL_d over a local field in arbitrary rank. Acting simply transitively, this remarkable lattice can be identified with the building, giving the building a structure of a Cayley complex. This identification allows for an explicit construction of Ramanujan complexes [7], as well as the construction of isospectral but noncommensurable complexes of any dimension $d \geqslant 5$ ($d \neq 6$) [8]. These are obtained as quotients of the lattice with respect to congruence subgroups.

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The epimorphism from the lattice to $\mathbb{Z}/d\mathbb{Z}$ is a d-coloring of the building, essentially defined by the quotient $\mathrm{PGL}_d(F)/\mathrm{PSL}_d(F)$, which result in colored Laplacians generating the Hecke algebra. The purpose of this short note is to describe a larger abelian quotient of each of the Cartwright-Steger lattices, which could be used to refine the colored Laplacians. Computer-aided verification suggest that our quotient is the full abelianization.

The recent construction of lattices acting simply transitively on a product of trees [9] is quite similar in nature, and one may expect the abelianization of these lattices to be amenable to the same analysis.

We define the building and the Cartwright-Steger lattice in section 1. In section 2 we introduce an extension $\tilde{\Gamma}$ of Γ, obtained by removing one of the defining relations. The abelian quotient of $\tilde{\Gamma}$ is given in section 3, leading to a closely related abelian quotient of Γ in section 4. Finally in section 5 we describe the computation of $\Gamma/[\Gamma,\Gamma]$ for the cases where the relation matrix has up to 2^{28} entries. In all cases, the abelianization is identical with the quotient described in Conjecture 1. Part of this work is based on [10].

1. The affine Bruhat-Tits building

Fix an integer $d \geqslant 2$ and a prime power q. Let $\mathbb{F}_q$ denote the finite field of order q. Let $F = \mathbb{F}_q((\pi))$ be the local field of Laurent series over $\mathbb{F}_q$. Let $\mathcal{O} = \mathbb{F}_q[[\pi]]$ be the ring of integers in F with respect to the π-adic valuation. We refer the reader to [6] and references therein for more details.

1.1. The building

The Bruhat-Tits building associated to the group $G = \mathrm{PGL}_d(F)$ is a simply connected simplicial complex of dimension d. The group G acts transitively on the vertex set, and the stabilizer of a vertex is a maximal compact subgroup, conjugate to $K = \mathrm{PGL}_d(\mathcal{O})$. In this sense the building can be described as the quotient G/K. A quotient $\Gamma\backslash G/K$ with respect to a discrete cocompact subgroups $\Gamma \leqslant G$ is a finite simplicial complex.

Let us now define the building as a simplicial complex. The vertices are the $\mathcal{O}$-submodules of full rank of the vector space $V = F^d$, up to similitude: a submodule M is equivalent to all the multiples $c \cdot M$, for any $c \in F^{\times}$. Every two submodules of full rank are commensurable. Distinct vertices $[M_0], \dots, [M_i]$ compose an i-cell if, after reordering, the representatives can be chosen so that $\pi M_0 \subset M_i \subset M_{i-1} \subset \cdots \subset M_1 \subset M_0$. Since the quotients $0 \subset M_i/\pi M_0 \subset M_{i-1}/\pi M_0 \subset \cdots \subset M_1/\pi M_0 \subset M_0/\pi M_0 = (\mathbb{F}_q)^d$ would then compose a flag of subspaces in $(\mathbb{F}_q)^d$, the maximal cells all have dimension d, and the links are isomorphic to the projective flag complex of $(\mathbb{F}_q)^d$. The vertices can be colored by taking the index of a submodule in a fixed pivot, such as $[\mathcal{O}^d]$. This leads to a coloring of the directed edges of the complex: we color the edge from $[M]$ to $[M']$ by color k if (up to choice of representatives) $M' \subseteq M$ and $\dim_{\mathbb{F}_q}(M/M') = k$. This coloring of the edges gives rise to $d-1$ "colored Laplacians", generating the Hecke algebra of G. We say that $[M']$ is an *immediate neigbor* of $[M]$ if the color of the edge from $[M]$ to $[M']$ is 1.

1.2. The division algebra

In order to define an arithmetic lattice in $\mathrm{PGL}_d(F)$, let $k = \mathbb{F}_q(\pi)$ be a global field endowed with the π-adic valuation, whose completion is F. Let $\phi: \mathbb{F}_{q^d} \to \mathbb{F}_{q^d}$ denote the Frobenius automorphism of the finite field, extended to $\bar{k} = \mathbb{F}_{q^d}(\pi)$ by acting trivially on π. Let D denote the algebra generated over k by $\bar{k}$ and z, subject to the relations $zf = \phi(f)z$ for every $f \in \bar{k}$, and $z^d = 1 + \pi$. Thus defined, D is a division algebra of dimension d^2 over its center k. Moreover, extension of scalars to F splits the algebra, namely $F \otimes_k D \cong \mathrm{M}_d(F)$. We then have an embedding $D^\times \subseteq (F \otimes_k D)^\times = \mathrm{M}_d(F)^\times = \mathrm{GL}_d(F)$, and so $D^\times/k^\times$ embeds in $G = \mathrm{PGL}_d(F)$.

1.3. The lattice

Now consider the special element $b_1 = 1 - z^{-1} \in D$. It has reduced norm $\pi/(1+\pi)$, which is equivalent to π up to units of $\mathcal{O}$. The immediate neighbors of the special vertex $[\mathcal{O}^d]$ are the vertices $[b_u \mathcal{O}^d]$, where $b_u = u b_1 u^{-1}$ are the conjugates of b_1 by scalars $u \in \mathbb{F}_{q^d}^\times / \mathbb{F}_q^\times$.

Cartwright and Steger proved that the subgroup of $D^\times/k^\times$ generated by the conjugates b_u ($u \in \mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times$), which we denote henceforth by Γ, acts simply transitively on the vertices of the building. Embedded through $\Gamma \subseteq D^\times/k^\times \subseteq \mathrm{PGL}_d(F)$, this is indeed a cocompact discrete subgroup of $\mathrm{PGL}_d(F)$.

Notice that $b_u = u(1 - z^{-1})u^{-1} = 1 - \frac{u}{\phi^{-1}(u)} z^{-1}$. Let $\mathbb{F}_{q^d}^{(1)}$ denote the set of elements of norm 1 in the extension $\mathbb{F}_{q^d}/\mathbb{F}_q$. The map $u \mapsto \frac{u}{\phi^{-1}(u)}$ is an isomorphism $\mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times \to \mathbb{F}_{q^d}^{(1)}$ by Hilbert's theorem 90. For our purposes it will be more convenient to write the generators of Γ as $b_{(r)} = 1 - rz^{-1}$, ranging over $r \in \mathbb{F}_{q^d}^{(1)}$.

1.4. The relations

We need some easy facts on products of generators of Γ. First, when $d > 2$, we have the equality

$$b_{(r)}b_{(r')} = b_{(s)}b_{(s')} \tag{1.1}$$

as elements of D, if and only if

$$r + r' = s + s' \tag{1.2}$$
$$r\phi^{-1}(r') = s\phi^{-1}(s'), \tag{1.3}$$

by comparing the elements in D.

Furthermore, let us say that a series of scalars $r_1, \dots, r_d \in \mathbb{F}_{q^d}^{(1)}$ forms a *flag* if $b_{(r_1)} \cdots b_{(r_d)}$ is central in D; norm considerations then show that $b_{(r_1)} \cdots b_{(r_d)} = \frac{\pi}{1+\pi}$. Indeed, such series are in one-to-one correspondence with maximal flags of subspaces in $\mathbb{F}_q^d$. For each flag we have that

$$b_{(r_1)} \cdots b_{(r_d)} = 1 \tag{1.4}$$

in $\Gamma \leqslant D^\times/k^\times$, and it is easy to show that the relations resulting from various flags are all equivalent modulo (1.1).

Finally, it is shown in [7, Theorem 5.2] that the relations (1.1) (for every r, r', s, s' satisfying (1.2)–(1.3)), together with a single relation of the form (1.4), compose a presentation of Γ. From now on, we view Γ as the group generated by the $b_{(r)}$, subject to the defining relations (1.1) and (1.4).

1.5. The case $d = 2$

Remark 1. If $d = 2$, the conditions (1.2)–(1.3) imply that $r' = -r$ and $s' = -s$.

Indeed, in this case $\phi^{-1}(r') = \phi(r') = r'^{-1}$ since $\mathrm{N}(r') = 1$ by assumption, so (1.3) implies $sr' = s'r$, and then $s(s + s') = s(r + r') = r(s + s')$, but $r \neq s$.

When $d = 2$ the building is a tree, and it is now easy to show that Γ is a free group of rank $(q + 1)/2$ when q is odd, and a free product of q cyclic groups of order two when q is even [7, Cor. 5.4]. The abelianization is easily computed in each case, so our main focus is on the case $d > 2$.

2. Covering Γ

There is a length function $\Gamma \to \mathbb{Z}/d\mathbb{Z}$, defined by sending each generator to 1; indeed this is the coloring mentioned above. It would be more convenient to work with the group $\tilde{\Gamma}$, formally generated by the $b_{(r)}$ ($r \in \mathbb{F}_{q^d}^{(1)}$), subject only to the relations (1.1). The length function is now defined in the same manner as $\tilde{\Gamma} \to \mathbb{Z}$.

Remark 2. For every $r \neq s$, there are unique $r', s' \in \mathbb{F}_{q^d}^{(1)}$ for which the relation (1.1) holds. Solving the equations we find that

$$r' = \frac{r-s}{\phi(r)-\phi(s)}\phi(s) \qquad \text{and} \qquad s' = \frac{r-s}{\phi(r)-\phi(s)}\phi(r).$$

We thus have a presentation of $\tilde{\Gamma}$ with $N = \frac{q^d-1}{q-1}$ generators, the $b_{(r)}$, and $\binom{N}{2}$ relations.

This fact implies an interesting property of $\tilde{\Gamma}$, namely that the submonoid $\tilde{\Gamma}_0$ generated by the $b_{(r)}$ satisfies the Ore condition. Indeed, given $r \neq s$, we can write $b_{(s)}^{-1} b_{(r)} = b_{(s')} b_{(r')}^{-1}$, so by induction on the length, every element of $\tilde{\Gamma}$ can be expressed in the form uw^{-1} for $u, w \in \tilde{\Gamma}_0$. This property holds in Γ as well, by projection.

Remark 3. The pullback $\hat{\Gamma}$ of the diagram

$$\begin{array}{ccc} \hat{\Gamma} & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \end{array} \quad ,$$

is an intermediate group in the sense that there are projections $\tilde{\Gamma} \to \hat{\Gamma} \to \Gamma$. Choose a flag $r_1, \dots, r_d \in \mathbb{F}_{q^d}^{(1)}$, and let $t = b_{(r_1)} \cdots b_{(r_d)} \in \tilde{\Gamma}$. Then Γ is the quotient of $\tilde{\Gamma}$ obtained by imposing the relation $t = 1$, and $\hat{\Gamma}$ is the quotient obtained by imposing that t is cental. (A presentation of the pullback is discussed in [1]).

3. An abelian quotient of $\tilde{\Gamma}$

We produce an abelian quotient of $\tilde{\Gamma}$, which is the full abelianization in all the cases we computed (see section 5).

Let $\mu_d(\mathbb{F}_q) = \left\{a \in \mathbb{F}_q^{\times} : a^d = 1\right\}$ denote the multiplicative group of roots of unity of order d in $\mathbb{F}_q$, which has order $|\mu_d(\mathbb{F}_q)| = (d, q-1)$, the greatest common divisor of d and $q-1$. Since the Galois norm of elements in $\mathbb{F}_q$ in the extension $\mathbb{F}_{q^d}/\mathbb{F}_q$ is exponentiation by d, we have that $\mu_d(\mathbb{F}_q) = \mathbb{F}_q^\times \cap \mathbb{F}_{q^d}^{(1)}$. Let

$$\theta = [\mathbb{F}_{q^d}^{(1)} : \mu_d(\mathbb{F}_q)] = \frac{q^d - 1}{(q-1)(d, q-1)}$$

be the index, which is necessarily an integer. Let $\phi : \mathbb{F}_{q^d} \to \mathbb{F}_{q^d}$ denote the Frobenius automorphism of exponentiation by q .

Lemma 1. *For $\alpha \in \mathbb{F}_{q^d}^{(1)}$ we have that $(\phi\alpha)^\theta = \alpha^\theta$.*

Proof. Since $\mu_d(\mathbb{F}_q) \leqslant \mathbb{F}_{q^d}^{(1)}$ are cyclic groups and the index is θ, we have that $\alpha^\theta \in \mu_d(\mathbb{F}_q)$ for every $\alpha \in \mathbb{F}_{q^d}^{(1)}$. Now $(\phi\alpha)^\theta = \phi(\alpha^\theta) = \alpha^\theta$. □

Clearly θ is minimal with this property. Now we can prove:

Proposition 1. *There is a homomorphism*

$$\tilde{\psi} : \tilde{\Gamma} \to \mathbb{Z} \times (\mathbb{F}_{q^d}, +) \times \mu_d(\mathbb{F}_q).$$

Proof. Define $\tilde{\psi}$ on the generators of Γ through the components

$$\tilde{\psi}_1(b_{(r)}) = 1; \qquad \tilde{\psi}_2(b_{(r)}) = r; \qquad \tilde{\psi}_3(b_{(r)}) = r^\theta.$$

We need to verify that each $\tilde{\psi}_i$ is well-defined, namely that the maps respect the defining relations (1.1). Assume $b_{(r)}b_{(r')} = b_{(s)}b_{(s')}$. The map $\tilde{\psi}_1$ maps both sides to

$$1 + 1 = 2. \tag{3.1}$$

The map $\tilde{\psi}_2$ maps the products to $r + r'$ and $s + s'$, respectively, which are equal by (1.2). For $\tilde{\psi}_3$, notice that the scalars $r, r', s, s' \in \mathbb{F}_{q^d}^{(1)}$ have norm 1, so by Lemma 1, $(rr')^\theta = (r\phi^{-1}(r'))^\theta = (s\phi^{-1}(s'))^\theta = (ss')^\theta$ because of (1.3). □

Surjectivity requires the following lemmas:

Lemma 2. *Let K/F be a finite dimensional extension of fields. Let $1 \neq A \subseteq B \leqslant K^\times$ be multiplicative subgroups. Then* $\operatorname{span}_F((A - A)B) = F[B]$.

Here $A - A = \{a - a' :\ a, a' \in A\}$.

Proof. Let V be the subspace spanned over F by $(A-A)B$. The inclusion $V \subseteq \mathrm{span}_F(AB) = \mathrm{span}_F(B) = F[B]$ is trivial. Since $((A-A)B)^2 = (A-A)(A-A)B \subseteq (A(A-A) - A(A-A))B = ((A^2-A^2)-(A^2-A^2))B = ((A-A)-(A-A))B \subseteq V$, we get that V is a subring, and thus contains $F[(A-A)B]$. Take arbitrary $a \neq a'$ in A, and let $b \in B$. Then $a-a', (a-a')b \in V$, but since V is a domain of finite dimension over F, it is a subfield, so $b \in V$. It follows that $F[B] \subseteq V$. □

Lemma 3. *Let $\mathbb{F}_{q^d}/\mathbb{F}_q$ be a proper extension of finite fields. The elements of norm 1 in $\mathbb{F}_{q^d}$ span $\mathbb{F}_{q^d}$ over the prime field.*

We first prove a stronger statement:

Lemma 4. *Let $\mathbb{F}_{q^d}/\mathbb{F}_q$ be a proper extension of finite fields. The subgroup of order $\theta = \frac{q^d-1}{(q-1)(d,q-1)}$ of $\mathbb{F}_{q^d}^\times$ spans $\mathbb{F}_{q^d}$ over the prime field, with two exceptions: $\mathbb{F}_9/\mathbb{F}_3$ and $\mathbb{F}_{64}/\mathbb{F}_4$.*

Proof. In the extension $\mathbb{F}_9/\mathbb{F}_3$ ($q=3$ and $d=2$) the subgroup of order $\theta = \frac{9-1}{(3-1)(2,3-1)} = 2$ spans $\mathbb{F}_3 < \mathbb{F}_9$, and in the extension $\mathbb{F}_{64}/\mathbb{F}_4$ ($q=4$ and $d=3$) the subgroup of order $\theta = \frac{64-1}{(4-1)(3,4-1)} = 7$ spans $\mathbb{F}_8 < \mathbb{F}_{64}$.

Let $\mathbb{F}_{q^d}/\mathbb{F}_q$ be any other extension of finite fields. A subgroup spans a subalgebra, which, as a finite domain, is necessarily a subfield of $\mathbb{F}_{q^d}$. We show that θ does not divides the order of the multiplicative group of any proper subfield.

1) For $d \geqslant 4$ we have that $q^d - 1 > q^{d/2+1}(q-1) > q^{d/2}(q-1)^2$, so $\theta = \frac{q^d-1}{(q-1)(d,q-1)} \geqslant \frac{q^d-1}{(q-1)^2} > q^{d/2}$.
2) Assume $d=3$. We have that $\theta = \frac{q^2+q+1}{3} > q-1$, so if the subgroup spans a proper subfield it has to have codimension 2; so write $q=p^2$ where p is a prime power. Now $\theta - (\sqrt{q^3}-1) \geqslant \frac{p^4+p^2+1}{3} - (p^3-1) = \frac{p^2+p+1}{3}(p-2)^2 > 0$ unless $q=4$, which was ruled out.
3) Assume $d=2$. If q is even then $\theta = q+1$ is larger than the order of any subfield. Assume q is odd, then $\theta = \frac{q+1}{2}$ is always larger than $q^{2/3}-1$, so if the subgroup is contained in a proper subfield it has to have codimension 2. But $\frac{q+1}{2} \mid q-1$ only when $q=3$, which was also ruled out.

□

Proof of Lemma 3. The group of elements of norm 1 has order $\frac{q^d-1}{q-1}$, a multiple of θ, so the claim follows from Lemma 4, except for the two exceptional cases, which we now verify: In $\mathbb{F}_9/\mathbb{F}_3$ there are four elements of norm 1, and in $\mathbb{F}_{64}/\mathbb{F}_4$ there are 21 such elements; both groups are larger than any subfield. □

Theorem 1. *The map $\tilde{\psi}$ of Proposition 1 is surjective.*

Proof. Let $\tilde{\Gamma}_i$ denote the kernel of $\tilde{\psi}_i$ and let $\tilde{\Gamma}_{ij} = \tilde{\Gamma}_i \cap \tilde{\Gamma}_j$.

We prove three claims:

1) $\tilde{\psi}_1$ is onto. This is trivial, as $\tilde{\psi}_1(b_{(r)}) = 1$ for every $r \in \mathbb{F}_{q^d}^{(1)}$.
2) The restriction of $\tilde{\psi}_3$ to $\tilde{\Gamma}_1$ is onto. Indeed, the image of $b_{(r)}b_{(1)}^{-1} \in \tilde{\Gamma}_1$ is r^θ, and exponentiation by θ is onto $\mu_d(\mathbb{F}_q)$ because $\theta = [\mathbb{F}_{q^d}^{(1)} : \mu_d(\mathbb{F}_q)]$.
3) The restriction of $\tilde{\psi}_2$ to $\tilde{\Gamma}_{13}$ is onto. Denote $B = \mathbb{F}_{q^d}^{(1)}$ and let A be the group of elements whose order divides θ; this is a subgroup of B, of order θ. Notice that we always have $\theta > 1$. Choose any $\alpha, \alpha' \in A$, and any $r \in B$. Then $\tilde{\psi}_3(b_{(\alpha' r)}b_{(\alpha r)}^{-1}) = (\alpha'/\alpha)^\theta = 1$, so that $b_{(\alpha r)}b_{(r)}^{-1} \in \tilde{\Gamma}_{13}$, and its image under $\tilde{\psi}_2$ is $(\alpha' - \alpha)r$. Letting $\mathbb{F}_p$ be the prime subfield underlying $\mathbb{F}_q$, it follows that $\mathrm{span}_{\mathbb{F}_p}((A - A)B) \subseteq \tilde{\psi}_2(\tilde{\Gamma}_{13})$. Taking K/F to be extension $\mathbb{F}_{q^d}/\mathbb{F}_p$, we obtain from Lemma 2 that $\tilde{\psi}_2(\tilde{\Gamma}_{13})$ contains $\mathbb{F}_p[B]$, which is all of $\mathbb{F}_{q^d}$ by Lemma 3.

To complete the proof, write A_1, A_2, A_3 for the summands of the range of $\tilde{\psi}$. The claims just proved translate, respectively, to the inclusions $A_1A_2A_3 \subseteq \mathrm{Im}(\tilde{\psi})A_2A_3 \subseteq \mathrm{Im}(\tilde{\psi})A_2 \subseteq \mathrm{Im}(\tilde{\psi})$. □

4. Abelianization of Γ

The map $\tilde{\psi}$ defined in section 3 does not always induce a map from Γ to the same abelian group; sometimes we need to fold up the third component by a factor of 2. Recall that $\theta = \frac{q^d-1}{(q-1)(d,q-1)}$. Let

$$\mu'_d(\mathbb{F}_q) = \mu_d(\mathbb{F}_q)/\left\langle (-1)^{(d-1)\theta} \right\rangle,$$

which is equal to $\mu_d(\mathbb{F}_q)$ unless q and θ are odd and d is even (see Proposition 3 below for complete details).

Proposition 2. *There is a commutative diagram*

$$\begin{array}{ccc} \tilde{\Gamma} & \xrightarrow{\quad\tilde{\psi}\quad} & \mathbb{Z} \times (\mathbb{F}_{q^d}, +) \times \mu_d(\mathbb{F}_q) \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow{\quad\psi\quad} & \mathbb{Z}/d\mathbb{Z} \times (\mathbb{F}_{q^d}, +) \times \mu'_d(\mathbb{F}_q) \end{array}$$

Proof. We need to show that the components $\tilde{\psi}_1$, $\tilde{\psi}_2$ and $\tilde{\psi}_3$ induce well-defined maps from Γ to the respective components. Since the relation (1.1) holds in $\tilde{\Gamma}$, it remains to verify the relation (1.4). So assume $r_1, \dots, r_d$ form a flag. We need to compute that each product $t_i = \psi_i(b_{(r_1)}) \cdots \psi_i(b_{(r_d)})$ is the identity element in the respective component. We are done with $t_1 = d \equiv 0 \in \mathbb{Z}/d\mathbb{Z}$ by counting. The fact that (1.4) holds means that

$$(1 - r_1 z^{-1}) \cdots (1 - r_d z^{-1})$$

is central in the division algebra D. Opening parentheses, this product is equal to $1 - (r_1 + \cdots + r_d)z^{-1} + \cdots + (-1)^d \rho z^{-d}$ where

$$\rho = r_1 \phi^{-1}(r_2)\phi^{-2}(r_3) \cdots \phi^{-(d-1)}(r_d).$$

Since this element is assumed to be central in D, we have that

$$r_1 + \cdots + r_d = 0,$$

which proves that $t_2 = 0$ in $\mathbb{F}_{q^d}$. The coefficients of each z^{-i} $(i = 1, \dots, d-1)$ must be zero, so the product is $(1 - r_1 z^{-1}) \cdots (1 - r_d z^{-1}) = 1 + (-1)^d \frac{\rho}{1+\pi} = \frac{1+(-1)^d \rho + \pi}{1+\pi}$ since $z^d = 1 + \pi$ by definition. Taking the reduced norm, we now obtain $(\frac{\pi}{1+\pi})^d = (\frac{1+(-1)^d \rho + \pi}{1+\pi})^d$ since the right-hand side is a scalar, so necessarily $\rho = (-1)^{d-1}$. But now, by Lemma 1, we have that

$$t_3 = r_1^\theta \cdots r_d^\theta = (r_1 \phi^{-1}(r_2)\phi^{-2}(r_3) \cdots \phi^{-(d-1)}(r_d))^\theta = \rho^\theta = (-1)^{(d-1)\theta},$$

which is the identity element in $\mu'_d(\mathbb{F}_q)$ by definition. □

Corollary 1. *There is an epimorphism from the abelianization of* Γ *to* $\mathbb{Z}/d\mathbb{Z} \times (\mathbb{F}_{q^d}, +) \times \mu'_d(\mathbb{F}_q)$.

Conjecture 1. When $d > 2$ we have that

$$\tilde{\Gamma}/[\tilde{\Gamma}, \tilde{\Gamma}] \cong \mathbb{Z} \times (\mathbb{F}_{q^d}, +) \times \mu_d(\mathbb{F}_q)$$

and

$$\Gamma/[\Gamma, \Gamma] \cong \mathbb{Z}/d\mathbb{Z} \times (\mathbb{F}_{q^d}, +) \times \mu'_d(\mathbb{F}_q).$$

We conclude with a number-theoretic observation concerning the kernel of the map $\mu_d(\mathbb{F}_q) \to \mu'_d(\mathbb{F}_q)$.

Proposition 3. *Let* q *be a prime power and* $d \geqslant 2$. *Let* $\lambda = (d, q-1)$ *be the greatest common divisor. For an integer* m, *let* $\nu_2(m)$ *denote the highest* j *for which* 2^j *divides* m. *Let* (#) *denote the condition* $\max\{2, \nu_2(d)\} \leqslant \nu_2(q-1)$.

(1) $\frac{q^d-1}{q-1}$ *is odd if and only if: d is odd; or q is even.*
(2) $\theta = \frac{q^d-1}{(q-1)\lambda}$ *is odd if and only if: d is odd; or q is even; or (#) holds.*
(3) $\theta(d-1)$ *is odd if and only if: d and q are even; or d is even and (#).*
(4) $(-1)^{\theta(d-1)} \neq 1$ *in* $\mathbb{F}_q$ *if and only if: d is even and (#).*

Proof. (1) is easy. For (2) notice that if $\lambda = (d, q-1)$ is odd we are back in (1), and otherwise substitute $q = 1 + 2q'$ and expand. Then (3) follows by imposing the condition that d is odd, and (4) by adding that q is odd as well. □

In particular $\mu'_d(\mathbb{F}_q) = \mu_d(\mathbb{F}_q)$, unless d is even and $\max\{2, \nu_2(d)\} \leqslant \nu_2(q-1)$.

5. Computational results

Given a presentation of a group G, the abelianization $G/[G,G]$ is a $\mathbb{Z}$-module, generated by the same generators with the same relations, viewed as equations over the integers. In our case, fixing $d > 2$ and a prime power q, the generators correspond to the scalars in $\mathbb{F}_{q^d}^{(1)}$, and the relations of $\tilde{\Gamma}$ are given in (1.1). As stated above, there are $N = \left|\mathbb{F}_{q^d}^{(1)}\right| = \frac{q^d-1}{q-1}$ generators, and $\binom{N}{2}$ relations. The number of entries in the matrix is therefore $N\binom{N}{2}$.

We used a standard matrix reduction algorithm, written in `sage` [11], to bring the matrix to the Smith normal form, $\mathrm{diag}(d_1, \ldots, d_N)$ where the fundamental invariants satisfy $d_1 \mid d_2 \mid \cdots \mid d_N$. The module in this case is $(\mathbb{Z}/d_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/d_N\mathbb{Z})$. We did not try to employ any scarce matrix techniques.

We carried out the computation for all the cases where $N\binom{N}{2} < 2^{28}$, namely: $d = 3, 4, 5, 6, 7, 8, 9$ for $q = 2$; $d = 3, 4, 5, 6$ for $q = 3$; $3 \leqslant d \leqslant 5$ for $q = 4, 5$; $d = 3, 4$ for $q = 7, 8$; and $d = 3$ for $q = 9, 11, 13, 16, 17, 19, 23, 25, 27$. In all cases, the abelianization of $\tilde{\Gamma}$ coincides with the prediction of Conjecture 1.

References

[1] M.R. Bridson, J. Howie, Ch.F. Miller III and H. Short, *On the finite presentation of subdirect products and the nature of residually free groups*, Amer J. Math **135**(4), 891–933, (2013).

[2] D.I. Cartwright, *Groups Acting Simply Transitive on the Verticies of a Building of Type* $\tilde{A}_n$, in Proceedings of the Conference "Groups of Lie Type and Their Geometries", Como 1993 (W.N. Kantor, ed.), Cambridge University Press.

[3] D.I. Cartwright and T. Steger, *A family of $\tilde{A}_n$ groups*, Israel J.Math. 103, 125-140, (1998).

[4] A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Progress in Math. **125**, Birkhäuser, (1994).

[5] A. Lubotzky, R. Philips and P. Sarnak, *Ramanujan graphs*, Combinatorica **8**, 261-277, (1988).

[6] A. Lubotzky, B. Samuels and U. Vishne, *Ramanujan complexes of type $\tilde{A}_d$*, Israel Journal Math, **149**, 267-299, (2005).

[7] A. Lubotzky, B. Samuels and U. Vishne, *Explicit construction of Ramanujan complexes*, European Journal of Combinatorics, **26**, 965-993, (2005).

[8] A. Lubotzky, B. Samuels and U. Vishne, *Isospectral Cayley graphs of some finite simple groups*, Duke J. Math., **135**, 381–393, (2006).

[9] N. Rungtanapirom, J. Stix and A. Vdovina, *Infinite series of quaternionic 1-vertex cube complexes, the doubling construction, and explicit cubical Ramanujan complexes*, International J of Algebra and Computation, **29**(6), 951–1007, (2019).

[10] O. Sela–Ben-David, "Lattices and their action on buildings and hyperbolic Riemannian surfaces", PhD dissertation, Bar Ilan Univerity, 2011.

[11] *SageMath, the Sage Mathematics Software System (Version 9.2)*, The Sage Developers, 2020, `https://www.sagemath.org`.

Contact information

Guy Blachar, Uzi Vishne
Department of Mathematics, Bar Ilan University, Israel
E-Mail(s): guy.blachar@gmail.com, vishne@math.biu.ac.il

Orit Sela–Ben-David
Kinneret Academic College, Jordan Valley, Israel
E-Mail(s): bndvd7@gmail.com

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