© Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **37** (2024). Number 2, pp. 160–170
DOI:10.12958/adm1958

Some properties of the commutators of special linear quantum groups

Mehsin Jabel Atteya

Communicated by A. Petravchuk

Abstract. This article is interested to a detailed computation of the commutators of the Hopf algebra $U_q(sl(n))$. It can be treated as a second way to computation the brackets of the Hopf algebra $U_q(sl(n))$ which could be introducing and understanding the $U_q(sl(n))$ for the researchers.

Introduction

In 1986, the concept of quantum groups initiated via Drinfeld [1], where Drinfeld forms a assured class of Hopf algebras U_q to date there is no rigorous, assume $k = \mathbb{C}$ and that q is not a root of unity. Then recall U_q is the algebra generated by symbols E, F, K, K^{-1} subject to the constraints $KE = q^2EK$, $KK^{-1} = K^{-1}K = 1$, $KF = q^{-2}FK$, and $[E, F] = \frac{K-K^{-1}}{q-q^{-1}}$, where $[E, F] = EF - FE$ denotes the commutator. In fact, universally accepted the definition, but it is mostly agreed that this term depends on assured deformations in one or more parameters of classical objects associated to algebraic groups. A good example to illustrated that the fact that, the enveloping algebras which has action

The author is grateful to Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq. Likewise, he is beholden to the mathematician experts(s) for (his/ their) precision with vocational reviewing this article.

2020 Mathematics Subject Classification: *16T20, 17B37, 20C08, 20C33, 20G43.*

Key words and phrases: *Hopf algebra, complex simple Lie algebra, quantized enveloping algebra.*

on semisimple Lie algebras or algebras of regular functions on the corresponding algebraic groups. It is possible for anyone to link the algebraic groups with commutative Hopf algebras via group schemes, in addition to that, it was agreed that the category of quantum groups should coincide with the opposite category of the category of Hopf algebras. This is one of the important reasons why some authors define quantum groups as non-commutative and non-cocommutative Hopf algebras. As a matter of fact, something of a misnomer concerns the name of the quantum group. They are not really groups at all. In a measure, the quantum groups seem almost to look like science fiction. Particularly given the weirdness surrounding the discoveries of quantum physics.

Consequently, just what are these exciting new structures called quantum groups? It's always nice to be honest at the outset of a significant undertaking. Nevertheless, the readers will probably be disappointed to know that there is no extremist, universally accepted definition of the concept of the quantum group. In spite of that, this has not restricted the development of a rich, powerful, and elegant theory with an ever-broadening horizon of application. Pleasurable. There is also a significant collection of patterns for which mathematicians in general can say that's a quantum group. One of the realities is that it began to be in the language of mathematics. It is not an unusual relation that a part of mathematics is development with no physical application in remembrance. Although this is not a one-way road, the theory of quantum groups happens to be an occasion for this. Their roots are in the work of theoretical and mathematical physics. It is no coincidence that the adjective quantum suggests a strong association with quantum mechanics in particular. Therefore, let us begin with an outline of the transition from classical to quantum mechanics. The quantum revolution began in the 1920s. Without wasting time plowing through the details of the several experiments that called for a complete modification of our understanding of reality, suffice it to say that the axiomatic heritage was built and handed down to us. For more information, see the references [1-6]. In this paper, we give a description of computation of the commutators of the Hopf algebra $U_q(sl(n))$.

1. The Lie algebra $sl(n)$

In any event, the complex Lie algebra g represents a vector space over $\mathbb{C}$ equipped with the concept of a non-associative product, ordinarily denoted by the Lie bracket. This is a linear map $[.,.]\colon g \otimes g \longrightarrow g$ which satisfies the following identities:

(i) $[e,g] = -[g,e]$ anti-symmetry,

(ii) $[e,[g,h]] = [[e,g],h] + [g,[e,h]]$ Jacobi identity,

where $e,g,h \in R$. During this article, $sl(2)$ denoted to the complex simple Lie algebra spanned as a vector space via three elements σ^+, σ^- and τ. The Lie bracket which write as $[\sigma^+,\sigma^-] = \tau, [\tau,\sigma^\pm] = \pm 2\sigma^\pm$.

At the same time with the previous relations anti-symmetry property and the bilinearity defined the Lie bracket on the whole algebra uniquely. Postulate V acts as a vector space and treat the algebra $End(V)$ consisting of an endomorphisms of V (with composition supplying the multiplication). Named that a (Lie algebra) representation of g on V is simply a linear map $\phi\colon g \to End(V)$ such that $\phi([g_1,g_2]) = \phi(g_1)\circ\phi(g_2) - \phi(g_2)\circ\phi(g_1)$, for all g_1 and g_2 in g. Automatically, the associated map $\phi\colon U_g \to End(V)$ which is an (algebra) representation of U_g on V and equivalently, we may do view V as a U_g-module. In categorical terms, the category of representations of g and the category of left U_g-modules are isomorphic.

The special linear Lie algebra of order n which denoted via $sl_n(\mathbb{F})$ or $sl(n,\mathbb{F})$ is the Lie algebra of $n \times n$ matrices. In fact, the influence point here is that $sl_n(\mathbb{F})$ has trace zero with the Lie bracket. $[\sigma,\tau] := \sigma\tau - \tau\sigma$.

2. The Quantized Enveloping Algebra $U_q(gl(n))$

We postulate an invertible element $q \in \mathbb{C}$, $q \neq 1$. Therefore, the fraction $\frac{1}{q-q^{-1}}$ is well defined. For any integer n define $[n] := \frac{q^n - q^{-n}}{q-q^{-1}} = q^{n-1} + q^{n-3} + ... + q^{-n+3} + q^{-n+1}$.

If q is not a root of unity, then $[n] \neq 0$ for any non-zero integer n. If q is a root of unity, then denote its order by d, i.e. $d \in \mathbb{N}$ is minimal such that $q^d = 1$.

We define $U_q(gl(n))$ as a unital associative complex algebra generated by $e_i, f_i, i = 1,2,\cdots,n-1, k_j, k_j^{-1}, j = 1,2,\cdots,n$ subject to the relations

$$k_i k_j = k_j k_i,\ k_i k_i^{-1} = k_i^{-1} k_i = 1,$$

$$k_i e_j k_i^{-1} = q^{\delta \frac{ij}{2}} q^{-\delta \frac{ij+1}{2}} e_j,$$

$$k_i f_j k_i^{-1} = q^{-\delta \frac{ij}{2}} q^{\delta \frac{ij+1}{2}} f_j,$$

$[e_i, f_j] = \delta_{ij}\dfrac{k_i^2 k_{i+1}^2 - k_{i+1}^2 k_i^2}{q + q^{-1}}$,

$[e_i, f_j] = [f_j, e_i] = 0, \mid i - j \mid \geq 2$,

$e_i^2 e_{i\pm1} - (q + q^{-1}) e_i e_{i\pm1} e_i + e_{i\pm1} e_i^2 = 0$,

$f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0$. The generators e_i and f_i correspond to the simple roots.

3. The Main Results

Proposition 1. *Assume that u_1, u_2 and u_3 are elements of $U_q(sl(n))$ with q, a and b are arbitrary parameters then*

(i) $[u_1, u_2]_q - [u_2, u_1]_q = (1 + q)(u_1u_2 - u_2u_1)$;

(ii) $[u_1, u_2]_q + [u_2, u_1]_q = (1 - q)(u_1u_2 + u_2u_1)$;

(iii) $[u_1, u_2]_q - [u_2, u_1]_{q^{-1}} = (1 + q^{-1})u_1u_2 - (1 + q)u_2u_1$;

(iv) $[u_1, u_2]_q + [u_2, u_1]_{q^{-1}} = (1 - q^{-1})u_1u_2 + (1 - q)u_2u_1$;

(v) $[u_1, u_2]_q = [u_2, u_1]_q$, *if* $[u_1, u_2] = 0$;

(vi) $[[u_2, u_3]_a, u_1]_b = [[u_2, u_1]_b, u_3]_a$, *with* $[u_3, u_1] = 0$;

(vii) $[u_3, [u_2, u_1]_a]_b = [u_2, [u_3, u_1]_b]_a$, *with* $[u_2, u_3] = 0$.

Proof. To prove (i), we postulate $[u_1, u_2]_q - [u_2, u_1]_q = (u_1u_2 - qu_2u_1) - (u_2u_1 - qu_1u_2)$. Then

$$\begin{aligned} = u_1u_2 - qu_2u_1 - u_2u_1 + qu_1u_2 &= (1 + q)u_1u_2 - (1 + q)u_2u_1 \\ &= (1 + q)(u_1u_2 - u_2u_1). \end{aligned}$$

The result as required.

For Branch (iii), we suppose

$$\begin{aligned}[u_1, u_2]_q + [u_2, u_1]_{q^{-1}} &= (u_1u_2 - qu_2u_1) + (u_2u_1 - q^{-1}u_1u_2)\\ &= (u_1u_2 - q^{-1}u_1u_2) + (u_2u_1 - qu_2u_1)\\ &= (1 - q^{-1})u_1u_2 - (1 + q)u_2u_1.\end{aligned}$$

The result as required.

We depend on the same technique to proof other branches.

To prove Branch (vi), we deduce that

$$\begin{aligned}[[u_2, u_3]_a, u_1]_b &= [u_2, u_3]_a u_1 - bu_1[u_2, u_3]_a\\ &= (u_2u_3 - au_3u_2)u_1 - bu_1(u_2u_3 - au_3u_2)\\ &= u_2u_3u_1 - au_3u_2u_1 - bu_1u_2u_3 + abu_1u_3u_2.\end{aligned}$$

Since $u_3u_1 = u_1u_3$, we note that

$$= u_2u_1u_3 - au_3u_2u_1 - bu_1u_2u_3 + abu_3u_1u_2.$$

Then

$$= [u_2, u_1]_b u_3 - au_3[u_2, u_1]_b = [[u_2, u_1]_b, u_3]_a.$$

The result as required.

We depend on the same strategy for Branch (vii), to prove $[u_3, [u_2, u_1]_a]_b = [u_2, [u_3, u_1]_b]_a$, with the term $[u_2, u_3] = 0$. □

Proposition 2. *The elements e_{12} and f_{12} of $U_q(sl(3))$ have the bracket*

$$[e_{12}, f_{12}] = (1 - q^{-1})([e_1, e_2]_q[f_2, f_1] + [f_2, f_1][e_1, e_2]_q)$$

$$-([[[e_1, e_2]_q, f_1]_{q^{-1}}, f_2] + [[f_2, [e_1, e_2]_q]_{q^{-1}}, f_1]).$$

Proof. We have $[e_{12}, f_{12}] = [[e_1, e_2]_q, [f_2, f_1]_{q^{-1}}] = -[[f_2, f_1]_{q^{-1}}, [e_1, e_2]_q]$.

On the above bracket, we employ the following identity

$$[[\lambda, \mu]_x, \kappa] = [[\lambda, \kappa], \mu]_x + [\lambda, [\mu, \kappa]]_x.$$

Suppose $\lambda = f_2, \mu = f_1, \kappa = [e_1, e_2]_q$ and $x = q^{-1}$. Then

$$[[f_2, f_1]_{q^{-1}}, [e_1, e_2]_q] = [[f_2, [e_1, e_2]_q], f_1]_{q^{-1}} + [f_2, [f_1, [e_1, e_2]_q]]_{q^{-1}}. \quad (1)$$

Again we apply the previous identity on the first term of equation (1) which is $[[f_2, [e_1, e_2]_q], f_1]_{q^{-1}}$. Then

$$[[\lambda, \mu]_x, \kappa] = [[\lambda, \kappa], \mu]_x + [\lambda, [\mu, \kappa]]_x.$$

Suppose $\lambda = [e_1, e_2]_q, \mu = f_1, \kappa = f_2$ and $x = q^{-1}$.

$$[[[e_1, e_2]_q, f_1]_{q^{-1}}, f_2] = [[[e_1, e_2]_q, f_2], f_1]_{q^{-1}} + [[e_1, e_2]_q, [f_1, f_2]]_{q^{-1}}. \quad (2)$$

We can rewrite equation (2) as the following:

$$[[f_2, [e_1, e_2]_q], f_1]_{q^{-1}} = [[e_1, e_2]_q, [f_2, f_1]]_{q^{-1}} - [[[e_1, e_2]_q, f_1]_{q^{-1}}, f_2]. \quad (3)$$

Again we apply the previous identity on the second term which is $[f_2, [f_1, [e_1, e_2]_q]]_{q^{-1}}$, so we suppose

$$\lambda = f_2, \mu = [e_1, e_2]_q, \kappa = f_1 \text{ and } x = q^{-1}.$$

$$[[f_2, [e_1, e_2]_q]_{q^{-1}}, f_1] = [[f_2, f_1], [e_1, e_2]_q]_{q^{-1}} + [f_2, [[e_1, e_2]_q, f_1]]_{q^{-1}}. \quad (4)$$

Also, we rewrite equation (4), we achieve

$$[f_2, [f_1, [e_1, e_2]_q]]_{q^{-1}} = [[f_2, f_1], [e_1, e_2]_q]_{q^{-1}} - [[f_2, [e_1, e_2]_q]_{q^{-1}}, f_1]. \quad (5)$$

Substituting relations (3) and (5) in relation (1), we note that

$$\begin{aligned}[][[f_2, f_1]_{q^{-1}}, [e_1, e_2]_q] =& [[e_1, e_2]_q, [f_2, f_1]]_{q^{-1}} - [[[e_1, e_2]_q, f_1]_{q^{-1}}, f_2] + \\ & [[f_2, f_1], [e_1, e_2]_q]_{q^{-1}} - [[f_2, [e_1, e_2]_q]_{q^{-1}}, f_1].\end{aligned}$$

Collect the brackets $[[e_1, e_2]_q, [f_2, f_1]]_{q^{-1}}$ and $[[f_2, f_1], [e_1, e_2]_q]_{q^{-1}}$ together with applying proposition 1(i), after suppose $u_1 = [e_1, e_2]_q$ and $u_2 = [f_2, f_1]$, we achieve

$$\begin{aligned}[][[f_2, f_1]_{q^{-1}}, [e_1, e_2]_q] =& (1 - q^{-1})([e_1, e_2]_q[f_2, f_1] + [f_2, f_1][e_1, e_2]_q) - \\ & ([[[e_1, e_2]_q, f_1]_{q^{-1}}, f_2] + [[f_2, [e_1, e_2]_q]_{q^{-1}}, f_1]).\end{aligned}$$

□

Theorem 1. *The elements e_{ij} and f_{ij} of $U_q(sl(n))$ have the bracket*

$$[e_{ij}, f_{ij}] = (1 - q)([f_i, f_{i+1,j}]_q[e_i, e_{i+1,j}] + [e_i, e_{i+1,j}][f_i, f_{i+1,j}]_q) -$$

$$([[[f_i, f_{i+1,j}]_q, e_i]_q, e_{i+1,j}] + [[e_{i+1,j}, [f_i, f_{i+1,j}]_q]_q, e_i])$$

where $i, j = 1, 2, \ldots, n$.

Proof. At the beginning of the hypothesis, we have the bracket

$$[e_{ij}, f_{ij}] = [[e_{i+1,j}, e_i]_q, [f_i, f_{i+1,j}]_q].$$

Emphasis is placed on the right side of this equation. We apply the relation

$$[[\lambda, \mu]_x, \kappa] = [[\lambda, \kappa], \mu]_x + [\lambda, [\mu, \kappa]]_x.$$

Suppose $\lambda = e_{i+1,j}$, $\mu = e_i$, $\kappa = [f_i, f_{i+1,j}]_q$ and $x = q$. Then

$$[[e_{i+1,j}, e_i]_q, [f_i, f_{i+1,j}]_q] = \tag{6}$$

$$[[e_{i+1,j}, [f_i, f_{i+1,j}]_q], e_i]_q + [e_{i+1,j}, [e_i, [f_i, f_{i+1,j}]_q]]_q.$$

Take the first term of equation (6) which is $[[e_{i+1,j}, [f_i, f_{i+1,j}]_q], e_i]_q$. Based on the previous identity, we write

$$\lambda = [f_i, f_{i+1,j}]_q,\ \mu = e_i,\ \kappa = e_{i+1,j} \text{ and } x = q.$$

$$[[[f_i, f_{i+1,j}]_q, e_i]_q, e_{i+1,j}] = \tag{7}$$

$$[[[f_i, f_{i+1,j}]_q, e_{i+1,j}], e_i]_q + [[f_i, f_{i+1,j}]_q, [e_i, e_{i+1,j}]]_q.$$

We rewrite relation (7), as the following:

$$[[e_{i+1,j}, [f_i, f_{i+1,j}]_q], e_i]_q = \tag{8}$$

$$[[f_i, f_{i+1,j}]_q, [e_i, e_{i+1,j}]]_q - [[[f_i, f_{i+1,j}]_q, e_i]_q, e_{i+1,j}].$$

Moreover, we use the previous relation on the second term which is $[e_{i+1,j}, [e_i, [f_i, f_{i+1,j}]_q]]_q$.

We firstly suppose $\lambda = e_{i+1,j}$, $\mu = [f_i, f_{i+1,j}]_q$, $\kappa = e_i$ and $x = q$. Then

$$[[e_{i+1,j}, [f_i, f_{i+1,j}]_q]_q, e_i] = \tag{9}$$

$$[[e_{i+1,j}, e_i], [f_i, f_{i+1,j}]_q]_q + [e_{i+1,j}, [[f_i, f_{i+1,j}]_q, e_i]]_q.$$

We can rewrite relation (9), as the following:

$$[e_{i+1,j}, [e_i, [f_i, f_{i+1,j}]_q]]_q = \tag{10}$$

$$[[e_{i+1,j}, e_i], [f_i, f_{i+1,j}]_q]_q - [[e_{i+1,j}, [f_i, f_{i+1,j}]_q]_q, e_i].$$

Investment relations (8) and (10) in relation (6), we observe

$$[[e_{i+1,j}, e_i]_q, [f_i, f_{i+1,j}]_q] =$$

$$[[f_i, f_{i+1,j}]_q, [e_i, e_{i+1,j}]]_q - [[[f_i, f_{i+1,j}]_q, e_i]_q, e_{i+1,j}] +$$

$$[[e_{i+1,j}, e_i], [f_i, f_{i+1,j}]_q]_q - [[e_{i+1,j}, [f_i, f_{i+1,j}]_q]_q, e_i].$$

Collect the first and forth terms on the right side of the above relation. Applying proposition 1(ii), we harvest the result.

$$[[e_{i+1,j}, e_i]_q, [f_i, f_{i+1,j}]_q] =$$

$$(1-q)([f_i, f_{i+1,j}]_q[e_i, e_{i+1,j}] + [e_i, e_{i+1,j}][f_i, f_{i+1,j}]_q) -$$

$$([[[f_i, f_{i+1,j}]_q, e_i]_q, e_{i+1,j}] + [[e_{i+1,j}, [f_i, f_{i+1,j}]_q]_q, e_i]).$$

The result as required. □

Theorem 2. *The elements h_j and e_{ij} of $U_q(sl(n))$ have the bracket*

$$[h_j, e_{ij}] = (q-1)(-a_{ij}(e_i e_{i+1,j} + e_{i+1,j} e_i)) + (e_i[e_{i+1,j}, h_j] + [e_{i+1,j}, h_j]e_i).$$

Proof. From the hypothesis we have

$$[h_j, e_{ij}] = [h_j, [e_i, [e_{i+1}, \cdots, [e_{j-1}, e_j]_q \cdots]_q].$$

To apply the simply identity which is $[[\lambda, \mu]_x, \kappa] = [[\lambda, \kappa], \mu]_x + [\lambda, [\mu, \kappa]]_x$ on above of the right side of the bracket.

We suppose $\lambda = e_i$, $\mu = [e_{i+1}, \cdots, [e_{j-1}, e_j]_q \cdots]_q$,$\kappa = h_j$ and $x = q$. Then, we obtain

$$[[e_i, [e_{i+1}, \cdots, [e_{j-1}, e_j]_q \cdots]_q]_q, h_j] = \tag{11}$$

$$[[e_i, h_j][e_{i+1}, \cdots, [e_{j-1}, e_j]_q \cdots]_q]_q + [e_i, [[e_{i+1}, \cdots, [e_{j-1}, e_j]_q \cdots]_q, h_j]]_q.$$

According to the relation $[h_j, e_i] = a_{ij} e_i$, the right-side of equation (11) becomes

$$= -a_{ij}[e_i, [e_{i+1}, [e_{i+2}, \cdots, [e_{j-1}, e_j]_q \cdots]_q]_q + \tag{12}$$

$$[e_i, [[e_{i+1}, [e_{i+2}, \cdots, [e_{j-1}, e_j]_q \cdots]_q, h_j]]_q.$$

Rewrite the above equation, we achieve

$$[[e_i, [e_{i+1}, \cdots, [e_{j-1}, e_j]_q \cdots]_q]_q, h_j] = -a_{ij}[e_i, e_{i+1,j}]_q + [e_i, [e_{i+1,j}, h_j]]_q. \tag{13}$$

In the main bracket, we suppose

$$\lambda = [e_{i+1}, \cdots, [e_{j-1}, e_j]_q \cdots]_q,\ \mu = e_i,\ \kappa = h_j \text{ and } x = q.$$

Thus, we obtain

$$[[[e_{i+1},\cdots,[e_{j-1},e_j]_q\cdots]_q,e_i]_q,h_j]=$$
$$[[[e_{i+1},\cdots,[e_{j-1},e_j]_q\cdots]_q,h_j],e_i]_q+[[e_{i+1},\cdots,[e_{j-1},e_j]_q\cdots]_q,[e_i,h_j]]_q.$$

Investment the relation $[h_j,e_i]=a_{ij}e_i$, which lies in the second term of the right part of the previous equation. This action produces the result.

$$[[[e_{i+1},\cdots,[e_{j-1},e_j]_q\cdots]_q,e_i]_q,h_j]=[[e_{i+1,j},h_j],e_i]_q-a_{ij}[e_{i+1,j},e_i]_q. \quad (14)$$

In this step, we focus on the left side of equation (14). By reason of the element e_j commuting with the element e_{j-2}, it will push the element e_{j-1} to the left. Hence, the left side modifies

$$[[[e_{i+1},\cdots,[e_{j-2},e_{j-1}]_q\cdots]_q,e_j]_q,e_i]_q,h_j]=[[e_{i+1,j},h_j],e_i]_q-a_{ij}[e_{i+1,j},e_i]_q.$$

Concentration on the left side and applying proposition 1(vii). After that, we write

$$u_1=e_i,u_2=[e_{i+1},\cdots,[e_{j-2},e_{j-1}]_q\cdots]_q,\text{ and }u_3=e_j,$$

with the bracket $[u_1,u_3]=[e_i,e_j]=0$, we achieve

$$[[[e_{i+1},\cdots,[e_{j-2},e_{j-1}]_q\cdots]_q,e_i]_q,e_j]_q,h_j]=[[e_{i+1,j},h_j],e_i]_q-a_{ij}[e_{i+1,j},e_i]_q.$$

Continue with the left-side, since e_{i+2} commute with the element e_i, therefore, it will push $[e_{i+3},[e_{i+4},\cdots,[e_{j-2},e_{j-1}]_q\cdots]_q$ to right way which leads to obtain

$$[[[e_{i+1},e_{i+2}]_q,[[e_{i+3},[e_{i+4},\cdots,[e_{j-2},e_{j-1}]_q\cdots]_q,e_i]_q]_q,e_j]_q,h_j]=$$
$$[[e_{i+1,j},h_j],e_i]_q-a_{ij}[e_{i+1,j},e_i]_q.$$

It's clearly the internal bracket $[[e_{i+3},[e_{i+4},\cdots,[e_{j-2},e_{j-1}]_q\cdots]_q,e_i]_q]_q=0$. Consequently the above relation becomes

$$[[e_{i+1,j},h_j],e_i]_q-a_{ij}[e_{i+1,j},e_i]_q=0.$$

Apply this result in left-side of equation (14) and combine with equation (13), we harvest that

$$[[e_i,[e_{i+1},\cdots,[e_{j-},e_j]_q\cdots]_q]_q,h_j]=$$
$$-a_{ij}([e_i,e_{i+1,j}]_q+[e_{i+1,j},e_i]_q)+([e_i,[e_{i+1,j},h_j]]_q+[[e_{i+1,j},h_j],e_i]_q).$$

Moreover, the right-side of the previous equation becomes

$$[h_j,e_{ij}]=(q-1)(-a_{ij}(e_ie_{i+1,j}+e_{i+1,j}e_i))+(e_i[e_{i+1,j},h_j]+[e_{i+1,j},h_j]e_i).$$

The result as required. □

Based on the same technique of the proof in an above theorem, we can achieve the following:

Theorem 3. *The elements h_j and f_{ij} of $U_q(sl(n))$ have the bracket*

$$[h_j, f_{ij}] = (q-1)(-a_{ij}(f_i f_{i+1,j} + f_{i+1,j} f_i)) + (f_i[f_{i+1,j}, h_j] + [f_{i+1,j}, h_j] f_i).$$

Adapted from Severin Pošta and Miloslav Havlı́ček [3]. One can prove the following corollary.

Corollary 1. *For any $n, m \geq 1$ and $k \geq 0$ we have the following*

$$f_i^n f_{i-1}^k f_i^m \in span\{f_{i-1}^k f_i^{n+m}, f_{i-1}^{k-1} f_i^{n+m} f_{i-1}, \cdots, f_i^{n+m} f_{i-1}^k\} \tag{15}$$

and

$$f_{i-1}^{n+m} f_i^n \in span\{f_{i-1}^n f_i^n f_{i-1}^m, f_{i-1}^{n-1} f_i^n f_{i-1}^{m+1}, \cdots, f_i^n f_{i-1}^{m+n}\}. \tag{16}$$

Then $f_{i-1}^{n+1} f_i^n = f_{i-1}^{n+1-l} f_i^n f_{i-1}^l$ *and* $f_i^{n+1} f_{i-1}^n = f_i^{n+1-l} f_{i-1}^n f_i^l$.

Proof. From relation (7) in [3, Lemma 3.1], we observe

$$f_{i-1}^{n+1} f_i^n = \sum_{l=1}^{n+1} (-1)^{l+1} \binom{n+1}{l} f_{i-1}^{n+1-l} f_i^n f_{i-1}^l.$$

Divided both sides on $f_{i-1}^{n+1} f_i^n$, we deduce $1 = \sum_{l=1}^{n+1} (-1)^{l+1} \binom{n+1}{l}$.

Applying this result in above equation, we arrive to

$$f_{i-1}^{n+1} f_i^n = f_{i-1}^{n+1-l} f_i^n f_{i-1}^l.$$

The result as required.

We depend on the relation $f_i^{n+1} f_{i-1}^n = \sum_{l=1}^{n+1} (-1)^{l+1} \binom{n+1}{l} f_i^{n+1-l} f_{i-1}^n f_i^l$ which appear in [3, Lemma 3.1] and apply same technique, we see

$$f_i^{n+1} f_{i-1}^n = f_i^{n+1-l} f_{i-1}^n f_i^l.$$

□

Conclusion

In this short paper, we provide a new technique for the commutators of the Hopf algebra $U_q(sl(n))$ with an approach suitable for researchers. Using elementary mathematical tools, it is, in our opinion, possible to understand this new manner of computation.

References

[1] Drinfeld, V.G.: Quantum groups. Proceedings International Congress of Mathematicians, pp. 798–820. Berkeley (1986).

[2] Downi, R.W.: An introduction to the theory of quantum groups. A thesis of Master of Science in Mathematics. Eastern Washington University. The Department of Mathematics (2012).

[3] Pošta, S., Havlíček, M.: Note on Verma Bases for Representations of Simple Lie Algebras. Acta Polytechnica. **53**(5), 450–456 (2013). https://doi.org/10.14311/AP.2013.53.0450

[4] Humphreys, J.E.: Introduction to Lie algebras and representation theory, vol. **9**. Springer-Verlag, New York (1978).

[5] Erdmann, K., Wildon, M.J.: Introduction to Lie algebras. Springer Undergraduate Mathematics Series. Springer-Verlag London Ltd, London (2006). https://doi.org/10.1007/1-84628-490-2

[6] Misra, K., Miwa, T.: Crystal base for the basic representation of $U_q(sl(n))$. Communications in Mathematical Physics. **134**(1), 79–88 (1990). https://doi.org/10.1007/BF02102090

CONTACT INFORMATION

M. J. Atteya Mustansiriyah University, College of Education, Department of Mathematics, Falastin St., Baghdad, Iraq.
E-Mail: mehsinatteya88@uomustansiriyah.edu.iq

Received by the editors: 16.02.2022
and in final form 29.02.2024.