© Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **34** (2022). Number 1, pp. 132–140
DOI:10.12958/adm1935

Double-toroidal and 1-planar non-commuting graph of a group

J. C. M. Pezzott

Communicated by A. Yu. Olshanskii

Abstract. Let G be a finite non-abelian group and denote by $Z(G)$ its center. The non-commuting graph of G on a transversal of the center is the graph whose vertices are the non-central elements of a transversal of $Z(G)$ in G and two vertices x and y are adjacent whenever $xy \neq yx$. In this work, we classify the finite non-abelian groups whose non-commuting graph on a transversal of the center is double-toroidal or 1-planar.

1. Introduction

In this paper, we consider only finite groups and finite undirected graphs without loops or multiple edges. Let G be a finite non-abelian group and denote the center of G by $Z(G)$. The *non-commuting graph of* G is the graph whose vertex set is $G \setminus Z(G)$ and two vertices x and y are adjacent whenever $xy \neq yx$. The non-commuting graph of a group has been extensively studied and many papers were published on the topic: see [1–4, 9–11]. Here, we denote this graph by $\nabla(G)$.

Let T be a transversal of $Z(G)$ in G. The *non-commuting graph of* G *on a transversal of the center* is the graph denoted by $\mathbb{T}(G)$ whose vertex set is $T \setminus Z(G)$ and two vertices x and y are adjacent whenever $xy \neq yx$. So, $\mathbb{T}(G)$ has $[G : Z(G)] - 1$ vertices and it is a subgraph of $\nabla(G)$. Further, as observed in [9, p. 911], the adjacency relations in $\nabla(G)$

2020 MSC: 05C25, 05C10.
Key words and phrases: non-commuting graph, double-toroidal graph, 1-planar graph, isoclinism.

can be obtained from adjacency relations in $\mathbb{T}(G)$, because two vertices x and y of $\nabla(G)$ are adjacent if and only if there are adjacent vertices x' and y' in $\mathbb{T}(G)$ such that $x \in x'Z(G)$ and $y \in y'Z(G)$. We also note that if T' is another transversal of $Z(G)$ in G, then the non-commuting graph on a transversal obtained from T' is isomorphic to the non-commuting graph obtained from T. Hence, we study the graph $\mathbb{T}(G)$ without mentioning the choice of the transversal.

The non-commuting graph $\mathbb{T}(G)$ was investigated in [10, 12] and, in [13], we see results on the complement of graph $\mathbb{T}(G)$. Further, it is worth mentioning that the graph $\mathbb{T}(G)$ was also examined in the studies on the non-commuting graph $\nabla(G)$ in the papers [3, 4, 9, 11]. In [9, 10], the graph $\mathbb{T}(G)$ was called *the underlying graph* associated with $\nabla(G)$ and was denoted by $\nabla^u(G)$.

Basic concepts and results on graphs can be seen in [17]. Let $\mathcal{G}$ be a graph. An *embedding* of $\mathcal{G}$ into a surface is a drawing of $\mathcal{G}$ on the surface in such a way no two edges intersect except at a vertex in which both are incident. If $\mathcal{G}$ can be embedded in the plane, we say that $\mathcal{G}$ is *planar*. Given an integer $n \geqslant 0$, let $\mathbb{S}_n$ be the surface obtained from the sphere by attaching n handles. Note that $\mathbb{S}_0$ is the sphere. The smallest non-negative integer n such that a graph $\mathcal{G}$ can be embedded in $\mathbb{S}_n$ is called the *genus* of $\mathcal{G}$. A graph with genus 0 is a planar graph. A graph with genus 1 is called *toroidal* and, in this case, the graph can be embedded into a torus. A graph with genus 2 is a *double-toroidal* graph and, here, it is embedded into a double-torus. A classification of the groups whose non-commuting graph on a transversal is planar or toroidal was obtained in [12, Theorems 3.7 and 3.9]. In this paper, we determine the groups with a double-toroidal non-commuting graph on a transversal of the center: see Theorem A.

A graph is said 1-*planar* if it can be drawn in the plane in such a way that each edge is crossed at most once. We note that every planar graph is a 1-planar graph. Here, in Theorem B, we classify all finite non-abelian groups whose non-commuting graph on a transversal of the center is 1-planar.

2. Results

In this section, we prove the main results of this work (Theorems A and B). We start with some concepts and notation.

Let $\mathcal{G}$ be a graph. The vertex set and the edge set of $\mathcal{G}$ are denoted, respectively, by $V(\mathcal{G})$ and $E(\mathcal{G})$. Given a subset V' of $V(\mathcal{G})$, the *subgraph of* $\mathcal{G}$ *induced by* V' is the graph whose vertex set is V' and the edge set is $\{\{u,v\} : u,v \in V', u \neq v, \{u,v\} \in E(\mathcal{G})\}$. A graph $\mathcal{G}'$ is a *spanning subgraph* of $\mathcal{G}$ if $V(\mathcal{G}') = V(\mathcal{G})$ and $E(\mathcal{G}') \subset E(\mathcal{G})$. As usual, the complete graph on n vertices is denoted by K_n and the complete multipartite graph with m partite sets of sizes $n_1, n_2, \ldots, n_m$, with $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_m$, is denoted by $K_{n_1,n_2,\ldots,n_m}$.

Given an integer $m \geqslant 3$, the dihedral group of order $2m$ is denoted by D_{2m}. Let G be a group. Given $x, y \in G$, the *commutator* $[x,y]$ of x and y is $[x,y] = xyx^{-1}y^{-1}$ and the derived subgroup of G is denoted by G'. We see that the commutator map $\alpha_G : G/Z(G) \times G/Z(G) \to G'$ given by $\alpha_G(xZ(G), yZ(G)) = [x,y]$ is well defined. We say that the groups G and H are *isoclinic* (see [6]) if there is a pair (φ, ψ) such that φ is an isomorphism from $G/Z(G)$ to $H/Z(H)$, ψ is an isomorphism from G' to H' and $\psi(\alpha_G(xZ(G), yZ(G))) = \alpha_H(\varphi(xZ(G)), \varphi(yZ(G)))$, for all $x, y \in G$. The pair (φ, ψ) is an *isoclinism* from G to H. We know that isomorphic groups are isoclinic (by [8, Lemma 2.3]) and we observe that the dihedral group D_8 and the quaternion group of order 8 are isoclinic, but they are not isomorphic.

Given a prime number p, we say that a p-group P is *extraspecial* if $|Z(P)| = p$ and $P' = Z(P) = \Phi(P)$, where $\Phi(P)$ is the Frattini subgroup of P. An extraspecial p-group has order p^{2n+1}, for some integer $n \geqslant 1$ (see [15, 5.3.8]). Further, every non-abelian group of order p^3 is extraspecial.

The *degree of commutativity* $P(G)$ of a finite group G is the probability that two randomly chosen elements commute, that is,

$$P(G) = \frac{|\{(x,y) \in G \times G : xy = yx\}|}{|G|^2}.$$

If G and J are isoclinic groups, then $P(G) = P(J)$ (see [8, Lemma 2.4]). In this paper, we will use the classification of the groups with degree of commutativity greater than or equal to $1/2$ obtained in [8].

Theorem 1 ([8, Theorem 3.1])**.** *Let G be a finite non-abelian group. We have $P(G) \geqslant 1/2$ if and only if G is isoclinic to D_6 or it is isoclinic to an extraspecial 2-group.*

Some results on the non-commuting graph $\mathbb{T}(G)$ are given below.

Lemma 1. *Let G be a finite non-abelian group.*

(i) *If a group J is isoclinic to G, then $\mathbb{T}(G)$ and $\mathbb{T}(J)$ are isomorphic graphs.*

(ii) $2|E(\mathbb{T}(G))| = (1 - P(G))(|V(\mathbb{T}(G))| + 1)^2$.

(iii) *If* $|V(\mathbb{T}(G))| \leqslant 10$*, then* $\mathbb{T}(G)$ *is isomorphic to one of the following graphs:* K_3*,* $K_{1,1,1,2}$*,* K_7*,* $K_{1,1,1,1,3}$*,* $K_{2,2,2,2}$ *or* $K_{1,1,1,1,1,4}$*.*

(iv) *The non-commuting graph* $\mathbb{T}(G)$ *is isomorphic to* $K_{2,2,2,2}$ *if and only if* G *is isoclinic to an extraspecial* 3*-group of order* 27*.*

(v) *The non-commuting graph* $\mathbb{T}(G)$ *is planar if and only if* G *is isoclinic to* D_6 *or* D_8*.*

(vi) *The non-commuting graph* $\mathbb{T}(G)$ *is toroidal if and only if* $\mathbb{T}(G)$ *is isomorphic to* $K_{1,1,1,1,3}$*,* K_7 *or* $K_{2,2,2,2}$*.*

(vii) *If* G *is isoclinic to an extraspecial* 2*-group of order* 2^{2n+1}*, with* $n \geqslant 2$*, then* $\mathbb{T}(G)$ *is a graph with* $2^{2n}-1$ *vertices and* $(2^{2n}-1)2^{2n-2}$ *edges.*

Proof. The proofs of statements (i), (ii), (iv) and (v) can be seen in [12] (respectively, Proposition 3.1, Theorem 3.4, Proposition 3.3 and Theorem 3.7). To prove (iii), we observe that $[G : Z(G)] \neq 11$ (because G is non-abelian) and so $|V(\mathbb{T}(G))| \neq 10$; now, the proof of (iii) follows from [12, Lemma 3.12]. The statement (vi) is a consequence of parts (i) and (iv) and [12, Theorem 3.9 and Lemmas 3.10 and 3.11]. To prove (vii), consider an extraspecial 2-group E of order 2^{2n+1}, with $n \geqslant 2$. We note that the complement graph of $\mathbb{T}(E)$ is a graph with $2^{2n}-1$ vertices and $(2^{2n}-1)(2^{2n-2}-1)$ edges (see [13, Proposition 3.2]). So, the graph $\mathbb{T}(E)$ has $2^{2n}-1$ vertices and $(2^{2n}-1)2^{2n-2}$ edges. By part (i), if G is isoclinic to E, then $\mathbb{T}(G)$ has $2^{2n}-1$ vertices and $(2^{2n}-1)2^{2n-2}$ edges. □

In the next result, we describe the structure of $G/Z(G)$ in the case where $[G : Z(G)] = 12$.

Proposition 1. *If* G *is a non-abelian group such that* $[G : Z(G)] = 12$*, then* $G/Z(G)$ *is isomorphic to* D_{12} *or to the alternating group on* 4 *letters* A_4*. Further, if* $G/Z(G)$ *is isomorphic to* D_{12}*, then* $\mathbb{T}(G)$ *is isomorphic to* $K_{1,1,1,1,1,1,5}$*.*

Proof. By [16, p. 85], we know that, up to isomorphism, there are only five groups of order 12: $\mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, D_{12}, A_4 and the group

$$\langle a, b \,|\, a^6 = 1,\ b^2 = a^3 = (ab)^2 \rangle.$$

Let G be a non-abelian group such that $[G : Z(G)] = 12$. Looking at the list of the groups of order 12, we get that if $G/Z(G)$ has no cyclic subgroup of order 6, then $G/Z(G)$ is isomorphic to A_4.

Suppose that $G/Z(G)$ has a cyclic subgroup of order 6. We will show that $\mathbb{T}(G)$ is isomorphic to $K_{1,1,1,1,1,1,5}$ and $G/Z(G)$ is isomorphic to D_{12}. Consider the homomorphism $f : G \to G/Z(G)$ given by $f(x) = xZ(G)$

and let B be the subgroup of G such that $f(B)$ is the cyclic subgroup of order 6 of $G/Z(G)$. It is clear that B is abelian and $[G : B] = 2$. So, $|V(\mathbb{T}(G)) \cap B| = 5$ and the subgraph induced by $V(\mathbb{T}(G)) \cap B$ has no edges. Given $x \in V(\mathbb{T}(G)) \setminus B$, arguing as in the third paragraph of the proof of [12, Lemma 3.12], we can prove that x is adjacent to all other vertices of $\mathbb{T}(G)$. Thus, $\mathbb{T}(G)$ is isomorphic to $K_{1,1,1,1,1,1,5}$. Further, since $f(B)$ is a cyclic subgroup of order 6 of $G/Z(G)$, we get that $f(B)$ has one element of order 2, two elements of order 3 and two elements of order 6. We also observe that all elements of $(G/Z(G)) \setminus f(B)$ have order 2, because each vertex of $V(\mathbb{T}(G)) \setminus B$ is adjacent to all other vertices of $\mathbb{T}(G)$. So, using the classification of the groups of order 12, we conclude that $G/Z(G)$ is isomorphic to D_{12}. Therefore, if $[G : Z(G)] = 12$, then $G/Z(G)$ is isomorphic to D_{12} or A_4.

We note that if $G/Z(G)$ is isomorphic to D_{12}, then $G/Z(G)$ has a cyclic subgroup of order 6. In this case, as shown in the paragraph above, $\mathbb{T}(G)$ is isomorphic to $K_{1,1,1,1,1,1,5}$. This proves the second statement of this result. □

Now, we classify the finite groups whose non-commuting graph of a transversal of the center is double-toroidal.

Theorem A. Let G be a finite non-abelian group. The non-commuting graph $\mathbb{T}(G)$ is double-toroidal if and only if G is isoclinic to the dihedral group D_{10}.

Proof. First, we observe that $\mathbb{T}(D_{10})$ is isomorphic to $\nabla(D_{10})$ (because $Z(D_{10}) = \{1\}$), that is, $\mathbb{T}(D_{10})$ is isomorphic to $K_{1,1,1,1,1,4}$. Thus, by [14, Figure 1], we get that $\mathbb{T}(D_{10})$ can be embedded on a double-torus. Using Lemma 1 (parts (v) and (vi)), we concluded that $\mathbb{T}(D_{10})$ is a double-toroidal graph. So, by Lemma 1(i), if a group G is isoclinic to D_{10}, then $\mathbb{T}(G)$ is double-toroidal.

Conversely, suppose that $\mathbb{T}(G)$ is a double-toroidal graph. Hence, by [17, Lemma 6.3.24],

$$|E(\mathbb{T}(G))| \leqslant 3|V(\mathbb{T}(G))| + 6. \tag{1}$$

The graphs $\mathbb{T}(D_6)$ and $\mathbb{T}(D_8)$ are planar (see Lemma 1(v)). By (1) and Lemma 1(vii), we have that G is not isoclinic to an extraspecial 2-group of order 2^{2n+1}, with $n \geqslant 2$. Hence, by Theorem 1, we obtain that $P(G) < 1/2$. So, using (1) and Lemma 1(ii), we have

$$|E(\mathbb{T}(G))| = \frac{1}{2}(1 - P(G))(|V(\mathbb{T}(G))| + 1)^2 \leqslant 3|V(\mathbb{T}(G))| + 6,$$

and thus

$$1 - \frac{6|V(\mathbb{T}(G))| + 12}{(|V(\mathbb{T}(G))| + 1)^2} \leqslant P(G) < \frac{1}{2},$$

that is,

$$\frac{(|V(\mathbb{T}(G))| + 1)^2 - 12|V(\mathbb{T}(G))| - 24}{2(|V(\mathbb{T}(G))| + 1)^2} < 0,$$

which implies $|V(\mathbb{T}(G))| \leqslant 11$.

Let us show that $|V(\mathbb{T}(G))| \neq 11$. To this end, we suppose the contrary, that is, suppose that $|V(\mathbb{T}(G))| = 11$, that is, $[G : Z(G)] = 12$. So, $G/Z(G)$ is isomorphic to D_{12} or A_4 (see Proposition 1). If $G/Z(G)$ is isomorphic to D_{12}, then Proposition 1 tells us that $\mathbb{T}(G)$ is isomorphic to $K_{1,1,1,1,1,1,5}$, which contradicts (1). Thus, we get that $G/Z(G)$ is isomorphic to A_4. It is routine to verify that $\mathbb{T}(A_4)$ is isomorphic to $K_{2,2,2,2,3}$. So, $\mathbb{T}(G/Z(G))$ is isomorphic to $K_{2,2,2,2,3}$. We note that $V(\mathbb{T}(G/Z(G))) = \{xZ(G) : x \in V(\mathbb{T}(G))\}$, because $|Z(G/Z(G))| = 1$. Given $x, y \in V(\mathbb{T}(G))$, with $x \neq y$, it is easy to see that if $xZ(G)$ and $yZ(G)$ are adjacent vertices in $\mathbb{T}(G/Z(G))$, then x and y are adjacent vertices in $\mathbb{T}(G)$. Hence, $\mathbb{T}(G/Z(G))$ is isomorphic to a spanning subgraph of $\mathbb{T}(G)$, that is, $K_{2,2,2,2,3}$ is a subgraph of $\mathbb{T}(G)$, which contradicts (1), because $K_{2,2,2,2,3}$ has 11 vertices and 48 edges. Hence, $|V(\mathbb{T}(G))| \neq 11$ and, therefore, $|V(\mathbb{T}(G))| \leqslant 10$.

It follows from Lemma 1(iii) that $\mathbb{T}(G)$ is isomorphic to K_3, $K_{1,1,1,2}$, K_7, $K_{1,1,1,1,3}$, $K_{2,2,2,2}$ or $K_{1,1,1,1,1,4}$. It is clear that K_3 and $K_{1,1,1,2}$ are planar graphs and we have that K_7, $K_{1,1,1,1,3}$ and $K_{2,2,2,2}$ are toroidal graphs (see Lemma 1(vi)). Hence, we obtain that $\mathbb{T}(G)$ is isomorphic to $K_{1,1,1,1,1,4}$. Therefore, by [12, Lemma 3.5], G is isoclinic to D_{10}. □

The next result gives us the groups with a 1-planar non-commuting graph on a transversal of the center. In view of Lemma 1(v), we can consider only the case where the graph $\mathbb{T}(G)$ is non-planar.

Theorem B. Let G be a finite non-abelian group such that $\mathbb{T}(G)$ is non-planar. The non-commuting graph $\mathbb{T}(G)$ is 1-planar if and only if G is isoclinic to an extraspecial 3-group of order 27.

Proof. The non-commuting graph on a transversal of the center of an extraspecial 3-group of order 27 is isomorphic to the graph $K_{2,2,2,2}$ (see Lemma 1(iv)). In Figure 1, we see a 1-planar drawing of the graph $K_{2,2,2,2}$

(we consider $K_{2,2,2,2}$ with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and partition $\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_7, v_8\}\}$). Hence, if G is isoclinic to an extraspecial 3-group of order 27, then $\mathbb{T}(G)$ is 1-planar.

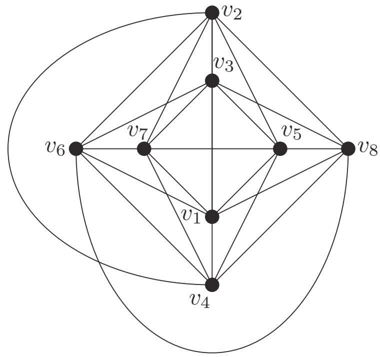

FIGURE 1. $K_{2,2,2,2}$ is 1-planar

Conversely, let G be a finite non-abelian group such that $\mathbb{T}(G)$ is non-planar and suppose that $\mathbb{T}(G)$ is 1-planar. By [5, Lemma 2.2], we have that

$$|E(\mathbb{T}(G))| \leqslant 4|V(\mathbb{T}(G))| - 8. \tag{2}$$

Using (2), parts (v) and (vii) of Lemma 1 and Theorem 1, we obtain that $P(G) < 1/2$. Hence, by (2) and Lemma 1(ii), we have

$$|E(\mathbb{T}(G))| = \frac{1}{2}(1 - P(G))(|V(\mathbb{T}(G))| + 1)^2 \leqslant 4|V(\mathbb{T}(G))| - 8$$

and so

$$1 - \frac{8|V(\mathbb{T}(G))| - 16}{(|V(\mathbb{T}(G))| + 1)^2} \leqslant P(G) < \frac{1}{2}.$$

Consequently,

$$\frac{(|V(\mathbb{T}(G))| + 1)^2 - 16|V(\mathbb{T}(G))| + 32}{2(|V(\mathbb{T}(G))| + 1)^2} < 0$$

and thus $|V(\mathbb{T}(G))| \leqslant 10$. By Lemma 1(iii), we get that $\mathbb{T}(G)$ is isomorphic to K_3, $K_{1,1,1,2}$, $K_{1,1,1,1,3}$, K_7, $K_{2,2,2,2}$ or $K_{1,1,1,1,1,4}$. We know that K_3 and $K_{1,1,1,2}$ are planar graphs. By [7, Lemma 7], $K_{1,1,1,1,3}$ is not 1-planar and, thus, we have that K_7 and $K_{1,1,1,1,1,4}$ are not 1-planar. We conclude that $\mathbb{T}(G)$ is isomorphic to $K_{2,2,2,2}$. It follows from Lemma 1(iv) that G is isoclinic to an extraspecial 3-group of order 27. □

Acknowledgements

I would like to thank the anonymous referee for careful reading of the manuscript and suggestions that improved the paper.

References

[1] A. Abdollahi, S. Akbari and H. R. Maimani. *Non-commuting graph of a group*. J. Algebra, 298(2) (2006), 468-492.

[2] M. Afkhami, D. G. M. Farrokhi and K. Khashyarmanesh. *Planar, toroidal, and projective commuting and noncommuting graphs*. Comm. Algebra 43(7) (2015), 2964–2970.

[3] M. Akbari and A. R. Moghaddamfar. *Groups for which the noncommuting graph is a split graph*. Int. J. Group Theory, 6(1) (2017), 29–35.

[4] M. Akbari and A. R. Moghaddamfar. *The existence or nonexistence of non-commuting graphs with particular properties*. J. Algebra Appl. 13(1) (2014), 1350064.

[5] I. Fabrici and T. Madaras. *The structure of 1-planar graphs*. Discrete Math., 307 (2007), 854-865.

[6] P. Hall. *The classification of prime-power groups*. J. Reine Angew. Math. 182 (1940), 130-141.

[7] V. P. Korzhik. *Minimal non-1-planar graphs*. Discrete Math., 308(7) (2008), 1319-1327.

[8] P. Lescot. *Isoclinism classes and commutativity degrees of finite groups*. J. Algebra, 177(3) (1995), 847-869.

[9] A. R. Moghaddamfar. *About noncommuting graphs*. Siberian Math. J., 47(6) (2006), 911–914.

[10] A. R. Moghaddamfar. *Some results concerning noncommuting graphs associated with finite groups*. Southeast Asian Bull. Math., 38(5) (2014), 661–676.

[11] A. R. Moghaddamfar, W. J. Shi, W. Zhou and A. R. Zokayi. *On the noncommuting graph associated with a finite group*. Siberian Math. J., 46(2) (2005), 325–332.

[12] J. C. M. Pezzott. *Groups whose non-commuting graph on a transversal is planar or toroidal*. J. Algebra Appl. (2021). https://doi.org/10.1142/S0219498822501985.

[13] J. C. M. Pezzott and I. N. Nakaoka. *On groups whose commuting graph on a transversal is strongly regular*. Discrete Math., 342(12) (2019), 111626.

[14] D. Nongsiang and P. K. Saikia. *On the non-nilpotent graphs of a group*. Int. Electron. J. Algebra, 22 (2017), 78-96.

[15] D. J. Robinson. *A Course in the Theory of Groups*. Springer-Verlag, New York, 1996.

[16] J. J. Rotman. *An Introduction to the Theory of Groups*. Fourth edition. Springer-Verlag, New York: 1995.

[17] D. B. West. *Introduction to Graph Theory*. Second Edition, Prentice Hall, 2001.

Contact information

Julio Cesar Moraes Pezzott
Departamento de Matemática,
Universidade Estadual de Maringá,
Maringá, PR, CEP: 87020-900
Brazil
E-Mail(s): juliopezzott@gmail.com

Received by the editors: 21.12.2021.