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Classical groups as Frobenius complement

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Abstract. The Frobenius group G belongs to an important class of groups that more than 100 years ago was defined by F. G. Frobenius who proved that G is a semi-direct product of a normal subgroup K of G called kernel by another non-trivial subgroup H called the complement. In this case we show that a few of the classical finite groups can be Frobenius complement.

Introduction and Preliminary results

Frobenius group for the first time was introduced in [3] and up to present time there are research about different aspects of this group. Let us give two equivalent definition for this group.

Definition 1. Let G be a group and H be a non-trivial proper subgroup of G. We say G is a Frobenius group with complement H if for every $g \in G \setminus H$ the equality $H \cap H^g = 1$ holds.

Definition 2. Let G be a transitive permutation group on a set Ω. If for every $\alpha \in \Omega$, we have $1 \neq H = G_\alpha \lneqq G$, then G is called a Frobenius group with complement H if $G_{\alpha,\beta} = 1$, for all $\alpha, \beta \in \Omega$, $\alpha \neq \beta$.

Although infinite Frobenius groups exist [1], but in this note we are concerned with finite Frobenius groups. Frobenius has shown that if G is a finite Frobenius group with complement H, then

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$$K = (G \setminus \bigcup_{g\in G} H^g) \cup \{1\}$$

is a normal subgroup of G, called Frobenius kernel and $G = KH, K\cap H = 1$. But Frobenius used character theory in proving that K is a subgroup of G. A part from Frobenius proof there is no known character free proof for the fact that K is a subgroup of G.

In [6] properties of the Frobenius groups are proved in detail. In particular in page 193 the structure of Frobenius complement is studied that we quote part of it here:

Result 1. Let H be a Frobenius complement and let p, q denote distinct primes. Then

1) H contains no subgroup of type (p, p).
2) Every subgroup of H of order pq is cyclic.
3) If $|H|$ is even, then H contains a unique element of order 2 which is central.
4) Sylow p-subgroups of H are cyclic, if p is odd.
5) Sylow 2-subgroups of H are either cyclic or quaternion.

Also in [6] (page 204) a result of Zassenhaus is given as follows:

Result 2. Let H be a non-solvable Frobenius complement. Then there is a subgroup H_0 of H such that $[H : H_0] \leqslant 2$, with $H_0 \cong SL_2(5) \times M$, where M is a Z-group of order prime to 2, 3 and 5.

We remark that a finite group all of whose Sylow subgroups are cyclic is called a Z-group.

In fact $H = SL_2(5)$ is a Frobenius complement, which is shown in [6](page 202). This is done by constructing a 2-dimensional vector space V over the finite field of characteristic $p \neq 2, 3, 5$, such that $SL_2(5)$ acts on $V - \{0\}$ fixed point freely. Then the semi-direct product $G = VH$ is a Frobenius group with complement H.

Similarly we can show that the group $SL_2(3)$ is a Frobenius complement. But there are many groups that can not be Frobenius complement, for example by Result 1 (2), the group $\mathbb{S}_3$ is not a Frobenius complement. Therefore the symmetric group $\mathbb{S}_n$ is a Frobenius complement if and only if $n = 2$. But the alternating group $\mathbb{A}_4$ by Result 1 (1) can not be a Frobenius complement, hence the groups $\mathbb{A}_n$, $n \geqslant 4$ are not Frobenius complement. But $\mathbb{A}_3 \cong \mathbb{Z}_3$ is a Frobenius complement.

Motivated by this we consider the special classical finite groups, $SL_n(q)$, $SP_{2n}(q)$, $SU_n(q^2)$, $SO_{2n+1}(q)$, q odd, $SO^{\pm}_{2n}(q)$, q even, and ask which one can be a Frobenius complement. The letter S denotes the group in question consist of matrices with determinant 1. Our main result is the following:

Theorem 1. *Let H denote a special classical group over a finite field. If H is a Frobenius complement, then $H \cong SL_2(3)$, $SL_2(5)$, $\mathbb{Z}_{q-1}$ or $\mathbb{Z}_{q+1}$, where q is a prime power.*

1. Main results

We start with the special linear group.

Proposition 1. *Let $H = SL_n(q)$, $n \geqslant 2$, $q = p^m$, p prime. If H is a Frobenius complement, then $H = SL_2(3)$ or $SL_2(5)$.*

Proof. Let $A = \left\{ \begin{bmatrix} 1 & \cdots & 0 & a \\ \vdots & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \mid a \in GF(q) \right\}$, then A is a subgroup of H isomorphism to the additive group of $GF(q)$. But the additive group of $GF(q)$ is elementary abelian of order p^m, hence by Result1 (1), $m = 1$. Therefore $H = SL_n(p)$, $n \geqslant 2$. A Sylow p-subgroup S of H consists of all the upper unitriangular matrices and $|S| = p^{\binom{n}{2}}$. Again by Result 1 (4) and (5), S must be cyclic implying $n = 2$. Hence $H = SL_2(p)$, p prime.

But now if $p = 2$, then $H = SL_2(2) \cong \mathbb{S}_3$ is not a Frobenius complement and if $p = 3$, $H = SL_2(3)$ is known to be a Frobenius complement. If $p \geqslant 5$, then H is a non-solvable group and by Result 2, $SL_2(5) \trianglelefteq SL_2(p)$ implying them $p = 5$, and the proposition is proved. □

Next we consider the symplectic group $H = SP_{2n}(q)$, $n \geqslant 1$, $q = p^m$, p prime. It is well-known that $SP_2(q) \cong SL_2(q)$.

Proposition 2. *If $H = SP_{2n}(q)$ is a Frobenius complement, then $H \cong SP_2(3)$ or $SP_2(5)$.*

Proof. We use a subgroup of the symplectic group constructed in [4] in the course of investigating irreducible characters of the affine symplectic group. The stabilizer of non-zero vector in the natural action of $SP_{2n}(q)$ on the underlying vector space $V_{2n}(q)$ is called the affine subgroup of $SP_{2n}(q)$, and with a suitable choice of the symplectic form it is shown that the affine subgroup of $SP_{2n}(q)$ contains a subgroup $P(n) = \{[v, a] \mid v \in V_{2n-2}(q), a \in GF(q)\}$ of order q^{2n-1}. If q is even $P(n)$ is an elementary abelian p-group, otherwise it is a special p-group (a p-group P is called special if $Z(P) = P' = \phi(P)$ is elementary abelian).

In any case $P(n)$ has the subgroup $\{[0, a] \mid a \in GF(q)\}$ which is isomorphic to the additive group of $GF(q)$, $q = p^m$. Now by Result 1 (1), $m = 1$, hence $H = SP_{2n}(p)$. If $p = 2$, then we look at $P(n)$ which is an elementary abelian 2-group of order 2^{2n-1}, that by Result 1 (1), we must have $n = 1$. Therefore $H = SP_2(2) \cong \mathbb{S}_3$ which is not a Frobenius complement.

Hence we assume p an odd prime. A Sylow p-subgroup of $SP_{2n}(p)$ has order p^{n^2} and is not cyclic unless $n = 1$. Therefore $H = SP_2(p) \cong SL_2(p)$ and the result follows by Proposition 1. $\square$

Our next step is to consider the finite unitary group. First note that

$$SU_2(q^2) \cong SL_2(q).$$

Proposition 3. *If $H = SU_n(q^2)$, $n \geqslant 2$, $q = p^m$, p prime, is a Frobenius complement, then $H = SU_2(3^2)$ or $SU_2(5^2)$.*

Proof. By [4] a suitable choice of a Hermitian form, yields the affine subgroup of $GU_n(q^2)$ which contains a special p-group of order q^{2n-3}. In fact if f is the Hermitian form defined on $V_n(q^2)$ we have

$$P = \{[v, a] \mid v \in V_{n-2}(q^2), a \in GF(q^2), tr(a) + f(v, v) = 0\}$$

where $tr(a) = a + a^q$. In fact, P is a subgroup of $SU_n(q^2)$. If we choose $v = 0$, then P has a subgroup $Q = \{[0, a] \mid a \in GF(q^2), a + a^q = 0\}$ which is isomorphic to the additive group of $GF(q)$. Hence by Result 1 (1), we obtain $m = 1$, $q = p$, $H = SU_n(p)$. But it is known that a Sylow p-subgroup of $SU_n(p)$ is already a Sylow p-subgroup of $SL_n(p)$. New using the argument in proposition 1 the Result follows. $\square$

Finally, we turn to the special orthogonal groups. These groups are defined as the group of isometries of a non-degenerate quadratic form Q over a finite dimensional vector space V over the Galois field $GF(q)$. If the $\dim V = 2n + 1$ is odd there is a unique non-degenerate quadratic form Q and its group of isometries with determinant 1 is denoted by $SO_{2n+1}(q)$. If q is even it is known that $SO_{2n+1}(q) \cong SP_{2n}(q)$. Therefore, first we deal with the special orthogonal group in odd dimension over the Galois field of odd characteristic. Note that in this case if

$$f : V_{2n+1}(q) \times V_{2n+1}(q) \longrightarrow GF(q)$$

is the symmetric bilinear form associated with Q, then $Q(v) = \frac{1}{2}f(v,v)$, for all $v \in V_{2n+1}(q)$.

Proposition 4. *Let $H = SO_{2n+1}(q)$, $q = p^f$, p an odd prime. If H is a Frobenius complement, then $n = 1$, $q = p = 3$ or 5.*

Proof. The study of affine subgroups of the orthogonal groups is contained in [2], from which we deduce that the stabilizer of a non-zero isotropic vector contains an abelian subgroup of order q^{2n-1} which defined as follows:

$$P(n) = \{[v,a] \mid v \in V_{2n-1}(q), a \in GF(q), 2a + f(v,v) = 0\}.$$

The multiplication in $P(n)$ is $[v,a][u,b] = [v+u, a+b-f(v,v)]$.

Using this multiplication it is easy to calculate $[v,a]^k = [kv, ka - \binom{k}{2}f(v,v)]$, for all v and a. Therefore $[v,a]^p = [0,0] =$ The identity element of $P(n)$. Hence $P(n)$ is an elementary abelian p-group of order q^{2n-1}. Now by Result 1 (1), $n = 1$, $q = p$, prime. Therefore $H = SO_3(p)$ which is known to be isomorphic to $SL_2(p)$. Now by the arguments used in previous propositions the result follows. $\square$

If the $\dim V = 2n$ is even there are two non-degenerate non-equivalent quadratic forms Q^t, $t = \pm$. If there is a totally isotropic subspace of dimension n, the group of isometries of V with respect to Q^+ and determinant 1 is denoted by $SO^+_{2n}(q)$, otherwise by $SO^-_{2n}(q)$.

Proposition 5. *Let $H = SO^t_{2n}(q)$, $t = \pm$, q a prime power. If H is a Frobenius complement, then $n = 1$ and $H \cong \mathbb{Z}_{q-1}$ or $H \cong \mathbb{Z}_{q+1}$.*

Proof. By [2] in any case the affine subgroup of H contains an abelian subgroup of order q^{2n-2} denoted by:

$P(n) = \{[v, \epsilon Q^\epsilon(v)] \mid v \in V_{2n-2}(q)\}$ where the multiplication in $P(n)$ is as follows:

$[v, \epsilon Q^\epsilon(v)][u, \epsilon Q^\epsilon(u)] = [v+u, \epsilon Q^\epsilon(v+u)]$, for all $v, u \in V_{2n-2}(q)$, it is easy to verify that $P(n)$ is an elementary abelian group, hence by 1 (1), $n = 1$. In this case we have $H = SO^\epsilon_2(q)$. But it is well-known that $SO^+_2(q) \cong \mathbb{Z}_{q-1}$ and $SO^-_2(q) \cong \mathbb{Z}_{q+1}$. $\square$

In the following we show that Frobenius complement isomorphic to $\mathbb{Z}_{q-1}$, and $\mathbb{Z}_{q+1}$, q prime power exist.

Example 1. Let F denote the finite field of order q and consider the group

$$G = \{f_{a,b} : F \longrightarrow F \mid f_{a,b}(x) = ax + b, a, b \in F, a \neq 0\}.$$

In this case G is a group of order $q(q-1)$ which acts transitively on F. The stabilizer of 0 is the group $H = \{f_{a,0} \mid a \in F^\times\}$ which is isomorphic to $\mathbb{Z}_{q-1}$ and acts fixed point freely on $F - \{0\}$. Therefore G is a Frobenius group with complement isomorphic to $\mathbb{Z}_{q-1}$.

Example 2. Let F be a finite field with q^2 elements where q is a prime power. By Example 1, a Frobenius group with kernel K isomorphic to the additive group of F, and complement isomorphic to the multiplicative group of F exists. But $F^\times \cong \mathbb{Z}_{q^2-1}$, and it has a unique subgroup H_1 of order $q+1$. Obviously $H_1 \leqslant N_G(K)$, hence by a result in [5], $G_1 = KH_1$ is a Frobenius group with complement $H_1 \cong \mathbb{Z}_{q+1}$.

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