© Algebra and Discrete Mathematics
Volume **37** (2024). Number 1, pp. 134–159
DOI:10.12958/adm1922 RESEARCH ARTICLE

On Σ-skew reflexive-nilpotents-property for rings

Héctor Suárez, Sebastián Higuera, and Armando Reyes

Communicated by V. Futorny

Abstract. In this paper, we study the reflexive-nilpotents-property (briefly, RNP) for skew PBW extensions. With this aim, we introduce the Σ-skew CN and Σ-skew reflexive (RNP) rings. Under conditions of compatibility, we investigate the transfer of the reflexive-nilpotents-property from a ring of coefficients to a skew PBW extension. We also consider this property for localizations on these families of noncommutative rings. Our results extend those corresponding presented by Bhattacharjee [9].

1. Introduction

Throughout the paper, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of natural numbers including zero and the ring of integers, respectively. The symbol $\Bbbk$ denotes a field, and $\Bbbk^* := \Bbbk \setminus \{0\}$. Every ring is associative with identity unless otherwise stated. For a ring R, $N(R)$ denotes its set of nilpotent elements, $N_*(R)$ is its prime radical, and $N^*(R)$ is its upper nilradical (i.e.

The authors were supported by the research fund of Department of Mathematics, Faculty of Science, Universidad Nacional de Colombia - Sede Bogotá, Colombia, HERMES CODE 52464. The first author was supported by Vicerrectoría de Investigación y Extensión, Code SGI 3725, Universidad Pedagógica y Tecnológica de Colombia, Tunja, Colombia.

2020 Mathematics Subject Classification: *Primary: 16S36, 16S80, 16U99. Secondary: 16W20, 16N40.*

Key words and phrases: *RNP ring, reflexive ring, skew polynomial ring, skew PBW extension.*

the sum of all nil ideals). It is well-known that $N^*(R) \subseteq N(R)$, and if the equality $N^*(R) = N(R)$ holds, then R is called *NI* [49]. Note that by definition, R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced, that is, $R/N^*(R)$ has no non-zero nilpotent elements. Hong and Kwak [29, Corollary 13], showed that a ring R is NI if and only if every minimal strongly prime ideal of R is completely prime. If the equality $N_*(R) = N(R)$ holds, then R is called 2-*primal* [10]; equivalently, $N^*(R)$ is a completely semiprime ideal of R. 2-primal rings are clearly NI. The converse of this implication need not hold, but if R is an NI ring of bounded index nilpotency, then R is 2-primal [31, Proposition 1.4].

Following Cohn [14], R is said to be *reversible* if $ab = 0$ implies $ba = 0$, for $a, b \in R$. Commutative rings and reduced rings are clearly reversible. Reversible rings were studied under the name *zero commutative* by Habeb [20]. As a matter of fact, the class of NI rings contains nil rings and reversible rings. Lambek [42] called a ring R *symmetric* provided $abc = 0$ implies $acb = 0$, for $a, b, c \in R$. Of course, commutative rings are symmetric, and symmetric rings are reversible, but the converses do not hold [2, Examples I.5 and II.5], and [50, Examples 5 and 7]. We know that every reduced ring is symmetric [65, Lemma 1.1], but the converse does not hold [2, Example II.5]. Bell [7] used the term *Insertion-of-Factors-Property* (*IFP*) for a ring R if $ab = 0$ implies $aRb = 0$, for $a, b \in R$ (Narbonne [53] and Habeb [20] used the terms *semicommutative* and *zero insertive*, respectively). Some results about IFP rings are due to Shin [65]. Note that every reversible ring is IFP, but the converse need not hold [37, Lemma 1.4 and Example 1.5]. Reversible rings are reflexive, and there exists a reflexive and IFP ring which is not symmetric [50, Examples 5 and 7]. In fact, a ring R is reflexive and IFP if and only if R is reversible [40, Proposition 2.2]. It is easy to see that IFP rings are 2-primal. Note that IFP rings are Abelian (every idempotent is central).

Mason [51] introduced the reflexive property for right ideals by defining a right ideal I of a ring R as *reflexive* if for $a, b \in R$, $aRb \subseteq I$ implies $bRa \subseteq I$, and a ring R is called *reflexive* if the zero ideal of R is reflexive (i.e. $aRb = 0$ implies $bRa = 0$, for $a, b \in R$). Equivalently, R is reflexive if and only if $IJ = 0$ implies $JI = 0$, for ideals I, J of R [40, Lemma 2.1]. By a direct computation one can check that semiprime rings and reversible rings are reflexive, and every ideal of a fully idempotent ring R (i.e. $I^2 = I$, for every ideal of R) is reflexive [15]. From [40, Example 2.3], we know that the IFP and the reflexive ring properties do not imply each other. Kwak and Lee [40] characterized the aspects of the reflexive and one-sided idempotent reflexive properties. They established a method by which a reflexive ring, which is not semiprime, can always be constructed from any semiprime ring, and showed that the reflexive property is Morita invariant. In the literature, different generalizations of reflexive rings have been formulated. Let us recall them.

Kim [34] introduced the notion of idempotent reflexive ring as a generalization of reflexive ring. For a one-sided ideal I of a ring R, I is called *right idempotent reflexive* if $aRe \subseteq I$ implies $eRa \subseteq I$, for any $a, e^2 = e \in R$, and the ring R is called *right idempotent reflexive* if the zero ideal is a right idempotent reflexive ideal. *Left idempotent reflexive ideals* and *left idempotent reflexive rings* are defined similarly. If a ring R is both left and right idempotent reflexive, then R is called an *idempotent reflexive ring* (for more details, see [35]). As one can check, reflexive rings and Abelian rings are idempotent reflexive. By [40, Example 2.3(1)], there exists an idempotent reflexive ring which is not reflexive. Similar to the case of reflexive rings, R is right idempotent reflexive if and only if $IJ = 0$ implies $JI = 0$, for all right ideals I, J of R where J is a right ideal generated by a subset of idempotents in R, if and only if $IJ = 0$ implies $JI = 0$, for all ideals I, J of R where J is an ideal generated by a subset of idempotents in R [40, Lemma 3.4].

Kheradmand et al. [32] introduced a generalization of reflexive rings. A ring R is called *RNP* (*reflexive-nilpotents-property*) if for $a, b \in N(R)$, $aRb = 0$ implies $bRa = 0$. Of course, reflexive rings are RNP but the converse need not hold [33, Example 1.2(1)]. In the same year, Kheradmand et al. [33] defined a more general class than reflexive and RNP rings by considering nil ideals, and called a ring R *nil-reflexive* if $IJ = 0$ implies $JI = 0$, for nil ideals I, J of R. Reflexive rings are RNP and RNP rings are nil-reflexive, but the converse need not hold [33, Example 1.2]. They also showed that R is a nil-reflexive ring if and only if $aRb = 0$ implies $bRa = 0$, for elements $a, b \in N^*(R)$ [33, Proposition 2.1]. Notice that the concepts of (nil)reflexive rings and NI rings are independent of each other [33, Examples 1.5 and 2.9]. Nevertheless, if R is an NI ring, then the following assertions are equivalent: R is nil-reflexive; $aRb = 0$, for $a, b \in N(R)$, implies $bRa = 0$; $IJ = 0$ implies $JI = 0$, for all nil right (or, left) ideals I, J of R [33, Proposition 1.6].

In addition to the reduced rings and its generalizations described above, all of them have been extended by ring endomorphisms. According to Krempa [38], an endomorphism σ of a ring R is called *rigid* if $a\sigma(a) = 0$ implies $a = 0$, where $a \in R$, and R is called *σ-rigid* if there exists a rigid endomorphism σ of R. Any rigid endomorphism of a ring R is a monomorphism and σ-rigid rings are reduced rings [27, Proposition 5]. Following Başer et al. [6] an endomorphism σ of a ring R is called *right skew reversible* if whenever $ab = 0$, for $a, b \in R$, $b\sigma(a) = 0$, and the ring R is called *right* σ*-skew reversible* if there exists a right skew reversible endomorphism σ of R. Similarly, left σ-skew reversible rings are defined. A ring R is said to be σ*-skew reversible* if it is both right and left σ-reversible. It is important to say that R is a σ-rigid ring if and only if R is semiprime and right σ-skew reversible for a monomorphism σ of R [6, Proposition 2.5 (iii)].

Kwak et al. [41] extended the reflexive property to the skewed refle–xive property by ring endomorphisms. An endomorphism σ of a ring R is called *right* (resp., *left*) *skew reflexive* if for $a, b \in R$, $aRb = 0$ implies $bR\sigma(a) = 0$ (resp., $\sigma(b)Ra = 0$), and R is called *right* (resp., *left*) σ*-skew reflexive* if there exists a right (resp., left) skew reflexive endomorphism σ of R. R is said to be σ*-skew reflexive* if it is both right and left σ-skew reflexive. It is clear that σ-rigid rings are right σ-skew reflexive. More precisely, a ring R is reduced and right σ-skew reflexive for a monomorphism σ of R if and only if R is σ-rigid [41, Theorem 2.6]. Bhattacharjee [9] extend the notion of RNP rings to ring endomorphisms σ and introduced the notion of σ-skew RNP rings as a generalization of σ-skew reflexive rings. An endomorphism σ of a ring R is called *right* (resp., *left*) *skew* RNP if for $a, b \in N(R)$, $aRb = 0$ implies $bR\sigma(a) = 0$ (resp., $\sigma(b)Ra = 0$). A ring R is called *right* (resp., *left*) σ*-skew* RNP if there exists a right (resp., left) skew RNP endomorphism σ of R. R is said to be σ*-skew* RNP if it is both right and left σ-skew RNP. From [9, Remark 1.2], we know that reduced rings are σ-skew RNP for any endomorphism σ, and every right (resp., left) σ-skew reflexive ring is right (resp., left) σ-skew RNP. By [9, Example 1.3], we have that the notion of σ-skew RNP ring is not left-right symmetric. However, if R is an RNP ring with an endomorphism σ, then R is right σ-skew RNP if and only if R is left σ-skew RNP.

Taking into account our interest in noncommutative rings defined by endomorphisms, in this paper we will focus our attention on the study of ring-theoretical notions above for the skew polynomial rings (also known as Ore extensions) defined by Ore [57], and the skew PBW extensions introduced by Gallego and Lezama [17]. As is well-known, skew polynomial rings are one of the most important families of noncommutative rings of polynomial type related with the study of quantum groups, differential operators, noncommutative algebraic geometry and noncommutative differential geometry (e.g. [12], [19], [52]), and a lot of papers have been published with the aim of studying different theoretical properties of these objects. Now, regarding skew PBW extensions, their importance is that these objects generalize PBW extensions defined by Bell and Goodearl [8], families of differential operator rings, Ore extensions of injective type, several algebras appearing in noncommutative algebraic geometry, examples of quantum groups, and other families of noncommutative rings having PBW bases. Since its introduction, ring-theoretical and homological properties of skew PBW extensions have been studied by some people (e.g. [5], [21], [22], [23], [44], [47], [70], [25], [56], [61], [66], [68], [69]). As a matter of fact, a book that includes several of the works carried out for these extensions has been published (Fajardo et al. [16]).

The paper is organized as follows. In Section 2, we recall some definitions and preliminaries about skew PBW extensions. In Section 3, we define the Σ-*skew reflexive rings* and study different properties for this new family of rings. In Section 4, we introduced the Σ-*skew RNP rings* as a generalization of the σ-skew RNP rings considered by Bhattacharjee [9]. We investigate some properties of these rings and their relationships with different kind of rings widely studied in the literature. Finally, we study the behavior of the Σ-skew RNP property for Ore localization by regular elements. In particular, we present a theorem that characterizes this property for the localization of skew PBW extensions.

2. Preliminaries

Definition 1 ([17, Definition 1]). Let R and A be a rings. We say that A is a *skew PBW extension over* R (the ring of coefficients), denoted $A = \sigma(R)\langle x_1, \dots, x_n\rangle$, if the following conditions hold:

(i) R is a subring of A sharing the same identity element.

(ii) There exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the set of standard monomials

$$\mathrm{Mon}(A) := \{x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For every $1 \le i \le n$ and any $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For $1 \leq i, j \leq n$, there exists $d_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n,$$

i.e. there exist elements $r_0^{(i,j)}, r_1^{(i,j)}, \ldots, r_n^{(i,j)} \in R$ with

$$x_j x_i - d_{i,j} x_i x_j = r_0^{(i,j)} + \sum_{k=1}^{n} r_k^{(i,j)} x_k.$$

Since $\text{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $d_{i,j}$ are unique. Thus, every non-zero element $f \in A$ can be uniquely expressed as $f = \sum_{i=0}^{m} a_i X_i$, with $a_i \in R$, $X_0 = 1$, and $X_i \in \text{Mon}(A)$, for $0 \leq i \leq m$ [17, Remark 2].

Proposition 1 ([17, Proposition 3])**.** *If A is a skew PBW extension, then there exist an injective endomorphism σ_i of R and a σ_i-derivation δ_i of R such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $1 \leq i \leq n$, where $r \in R$.*

We use the notation $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and $\Delta := \{\delta_1, \ldots, \delta_n\}$ for the families of injective endomorphisms and σ_i-derivations of Proposition 1, respectively. The pair (Σ, Δ) is called a *system of endomorphisms and Σ-derivations* of R with respect to A.

Definition 2. Let A be a skew PBW extension over R.

(i) ([17, Definition 4]) A is called *quasi-commutative* if the conditions (iii) and (iv) presented above are replaced by the following:

(iii') For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$.

(iv') For every $1 \leq i, j \leq n$, there exists $d_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i = d_{i,j} x_i x_j.$$

(ii) ([17, Definition 4]) A is called *bijective* if σ_i is bijective for each $1 \leq i \leq n$, and $d_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

(iii) ([45, Definition 2.3]) If σ_i is the identity homomorphism of R for each $1 \leq i \leq n$, (we write $\sigma_i = \text{id}_R$), we say that A is a skew PBW extension of *derivation type*. Similarly, if $\delta_i \in \Delta$ is zero, for every i, then A is called a skew PBW extension of *endomorphism type*.

Remark 1. Some relationships between skew polynomial rings and skew PBW extensions are the following:

(i) If A is a quasi-commutative skew PBW extension over R, then A is isomorphic to an iterated skew polynomial ring of endomorphism type [48, Theorem 2.3].

(ii) In general, skew polynomial rings of injective type are strictly contained in skew PBW extensions [48, Example 5(3)]. This fact is not possible for PBW extensions. For instance, the quantum plane $\Bbbk\{x, y\}/\langle xy - qyx \mid q \in \Bbbk^*\rangle$ is a skew polynomial ring of injective type given by $\Bbbk[y][x; \sigma]$, where $\sigma(y) = qy$, but cannot be expressed as a PBW extension.

(iii) Skew PBW extensions of endomorphism type are more general than iterated skew polynomial rings of endomorphism type [67, Remark 2.4 (ii)].

Example 1. A great variety of algebras can be expressed as skew PBW extensions. Enveloping algebras of finite dimensional Lie algebras, some families of differential operators rings, Weyl algebras, skew polynomial rings of injective type, some types of Auslander-Gorenstein rings, some skew Calabi-Yau algebras, Artin-Schelter regular algebras, examples of quantum polynomials, some quantum universal enveloping algebras, and many other algebras of great interest in noncommutative algebraic geometry and noncommutative differential geometry illustrate the generality of skew PBW extensions (see [16, 18, 66]).

Definition 3 ([17, Section 3])**.** If A is a skew PBW extension, then:

(i) Throughout the paper, for any element $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we will write $\sigma^{\alpha} := \sigma_1^{\alpha_1} \circ \cdots \circ \sigma_n^{\alpha_n}$, $\delta^{\alpha} = \delta_1^{\alpha_1} \circ \cdots \circ \delta_n^{\alpha_n}$, where $\circ$ denotes the usual composition of functions.

(ii) Let $\succeq$ be a total order on $\mathrm{Mon}(A)$. If $x^{\alpha} \succeq x^{\beta}$ but $x^{\alpha} \neq x^{\beta}$, we write $x^{\alpha} \succ x^{\beta}$. If f is a non-zero element of A, then we use expressions as $f = a_1 x^{\alpha_1} + \cdots + a_k x^{\alpha_k}$, with $a_i \in R$, and $x^{\alpha_k} \succ \cdots \succ x^{\alpha_1}$. With this notation, we define $\mathrm{lm}(f) := x^{\alpha_k}$, the *leading monomial* of f; $\mathrm{lc}(f) := a_k$, the *leading coefficient* of f; $\mathrm{lt}(f) := a_k x^{\alpha_k}$, the *leading term* of f. Note that $\deg(f) := \max\{\deg(x^{\alpha_i})\}_{i=1}^{k}$. If $f = 0$, $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$.

The next proposition is very useful when one needs to make some computations with elements of skew PBW extensions.

Proposition 2 ([17, Theorem 7]). *If A is a polynomial ring with coefficients in R with respect to the set of indeterminates $\{x_1, \dots, x_n\}$, then A is a skew PBW extension over R if and only if the following conditions hold:*

(1) *For each $x^{\alpha} \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_{\alpha} := \sigma^{\alpha}(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, such that $x^{\alpha}r = r_{\alpha}x^{\alpha} + p_{\alpha,r}$, where $p_{\alpha,r} = 0$, or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_{α}.*

(2) *For each $x^{\alpha}, x^{\beta} \in \mathrm{Mon}(A)$, there exist unique elements $d_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^{\alpha}x^{\beta} = d_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}$, where $d_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$, or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.*

We need to establish a criterion which allows us to extend the family Σ of injective endomorphisms and the family of Σ-derivations Δ of the ring R, to any skew PBW extension A over R (c.f. Artamonov [5] and Venegas [71] who presented a study of derivations and automorphisms of skew PBW extensions, respectively). With this aim, we consider (Σ, Δ) the system of endomorphisms and Σ-derivations of R with respect to A.

Proposition 3 ([59, Theorem 5.1]). *Let A be a skew PBW extension over R. Suppose that $\sigma_i\delta_j = \delta_j\sigma_i$, $\delta_i\delta_j = \delta_j\delta_i$, and $\delta_k(d_{i,j}) = \delta_k(r_l^{(i,j)}) = 0$, for $1 \leq i, j, k, l \leq n$, where $d_{i,j}$ and $r_l^{(i,j)}$ are the elements of Definition 1. Let $f = a_1x^{\alpha_1} + \cdots + a_mx^{\alpha_m} \in A$. If $\overline{\sigma_k} : A \to A$ and $\overline{\delta_k} : A \to A$ are the functions given by $\overline{\sigma_k}(f) := \sigma_k(a_1)x^{\alpha_1} + \cdots + \sigma_k(a_m)x^{\alpha_m}$ and $\overline{\delta_k}(f) := \delta_k(a_1)x^{\alpha_1} + \cdots + \delta_k(a_m)x^{\alpha_m}$, respectively, and $\overline{\sigma_k}(r) := \sigma_k(r)$, for every $1 \leq k \leq n$ and $r \in R$, then $\overline{\sigma_k}$ is an injective endomorphism of A and $\overline{\delta_k}$ is a $\overline{\sigma_k}$-derivation of A, for each k.*

According to Krempa [38], if R is a ring and Σ is a finite family of endomorphisms of R, then Σ is called a *rigid endomorphisms family* if $a\sigma^{\alpha}(a) = 0$ implies $a = 0$, where $a \in R$ and $\alpha \in \mathbb{N}^n$. If there exists a rigid endomorphisms family Σ of R, then R is called Σ-*rigid* [59, Definition 3.1]. Following Annin [3] or Hashemi and Moussavi [24], if R is a ring, σ is an endomorphism of R, and δ is a σ-derivation of R, then R is said to be σ-*compatible* if for each $a, b \in R$, $ab = 0$ if and only if $a\sigma(b) = 0$; R is called δ-*compatible* if for each $a, b \in R$, $ab = 0$ implies $a\delta(b) = 0$. If R is both σ-compatible and δ-compatible, R is called (σ, δ)-*compatible*. The corresponding notion of compatibility have been formulated for skew PBW extensions as the following definition shows.

Definition 4. Let R be a ring, Σ be a finite family of endomorphism of R, and Δ be a finite family of Σ-derivations of R.

(1) ([22, Definition 3.1]; [60, Definition 3.2]) R is called Σ-*compatible* if for each $a, b \in R$, we have $a\sigma^{\alpha}(b) = 0$ if and only if $ab = 0$, for all $\alpha \in \mathbb{N}^n$; R is said to be Δ-*compatible* if for each $a, b \in R$, $ab = 0$ implies $a\delta^{\beta}(b) = 0$, for all $\beta \in \mathbb{N}^n$; if R is both Σ-compatible and Δ-compatible, then R is called (Σ, Δ)-*compatible.*

(2) ([62, Definition 4.1]) R is said to be *weak* Σ-*compatible* if for each $a, b \in R$, we have $a\sigma^{\alpha}(b) \in N(R)$ if and only if $ab \in N(R)$, for all $\alpha \in \mathbb{N}^n$; R is said to be *weak* Δ-*compatible* if for each $a, b \in R$, $ab \in N(R)$ implies $a\delta^{\beta}(b) \in N(R)$, for all $\beta \in \mathbb{N}^n$; if R is both weak Σ-compatible and weak Δ-compatible, then R is called *weak* (Σ, Δ)-*compatible.*

Remark 2. It is straightforward to prove that if Σ is a finite family of endomorphisms of a ring R, then R is Σ-compatible if and only if R is σ_i-compatible, for every $1 \leq i \leq n$.

Lemma 1 ([60, Proposition 3.8]). *Let R be a (Σ, Δ)-compatible ring. For every $a, b \in R$, we have the following:*

(1) *If $ab = 0$, then $a\sigma^{\theta}(b) = \sigma^{\theta}(a)b = 0$, where $\theta \in \mathbb{N}^n$.*

(2) *If $\sigma^{\beta}(a)b = 0$, for some $\beta \in \mathbb{N}^n$, then $ab = 0$.*

(3) *If $ab = 0$, then $\sigma^{\theta}(a)\delta^{\beta}(b) = \delta^{\beta}(a)\sigma^{\theta}(b) = 0$, where $\theta, \beta \in \mathbb{N}^n$.*

Some examples of skew PBW extensions over (Σ, Δ)-compatible rings include PBW extensions, some operator algebras, the class of diffusion algebras, quantizations of Weyl algebras, the family of 3-dimensional skew polynomial algebras, and other families of noncommutative algebras having PBW bases (see [61,63,64]). We present a new example of a skew PBW extension over a Σ-compatible ring.

Example 2. Let $\mathbb{F}_4 = \left\{0, 1, a, a^2\right\}$ be the finite field of four elements. Consider the ring of polynomials $\mathbb{F}_4[z]$ and let $R = \frac{\mathbb{F}_4[z]}{\langle z^2 \rangle}$. For simplicity, we identify the elements of $\mathbb{F}_4[z]$ with their images in R. Let $\Sigma = \{\sigma_{i,j}\}$ be the family of endomorphisms of R defined by $\sigma_{i,j}(a) = a^i$ and $\sigma_{i,j}(z) = a^j z$ with $1 \leq i \leq 2$ and $0 \leq j \leq 2$. Let us consider the skew PBW extension defined as $A = \sigma(R)\langle x_{1,0}, x_{1,1}, x_{1,2}, x_{2,0}, x_{2,1}, x_{2,2}\rangle$, under the following commutation relations: $x_{i,j}x_{i',j'} = x_{i',j'}x_{i,j}$, for all $1 \leq i, i' \leq 2$ and

$0 \le j, j' \le 2$. On the other hand, for $a^r z \in R$, $x_{i,j}a^r z = \sigma_{i,j}(a^r z)x_{i,j} = (a^r)^i a^j z x_{i,j} = a^{ri+j} z x_{i,j}$, where $a^{ri+j} \in \mathbb{F}_4$, $1 \le i \le 2$ and $1 \le r, j \le 2$. This example can be extended to any finite field $\mathbb{F}_{p^n}$ with p a prime number. Additionally, it is not difficult to see that R is Σ-compatible, where $\Sigma := \{\sigma_{i,j}\}$ with $1 \le i \le 2$ and $0 \le j \le 2$.

If R is a ring, Σ is a finite family of endomorphims of R, and Δ is a finite family of Σ-derivations of R, then an ideal I of R is called Σ-*ideal*, if $\sigma^{\alpha}(I) \subseteq I$, for each $\alpha \in \mathbb{N}^n$; I is Δ-*ideal*, if $\delta^{\alpha}(I) \subseteq I$, for each $\alpha \in \mathbb{N}^n$; if I is both Σ and Δ-ideal, then I is a (Σ, Δ)-ideal [23, Definition 3.1]. For instance, if $N(R)$ is an ideal of R, we have $N(R)$ is Σ-ideal. Indeed, if $r \in N(R)$ then $r^m = 0$, for some $m \ge 1$. So, $\sigma^{\alpha}(r) \in \sigma^{\alpha}(N(R))$ implies that $\sigma^{\alpha}(r^n) = (\sigma^{\alpha}(r))^n = 0$, whence $\sigma^{\alpha}(r) \in N(R)$.

Lemma 2. *If R is a Σ-compatible ring, $a, b \in R$ and $\alpha, \beta \in \mathbb{N}^n$, then the following assertions are equivalent:*

(1) $ab \in N(R)$.

(2) $a\sigma^{\alpha}(b) \in N(R)$.

(3) $\sigma^{\alpha}(b)a \in N(R)$.

(4) $\sigma^{\alpha}(a)b \in N(R)$.

(5) $b\sigma^{\alpha}(a) \in N(R)$.

(6) $\sigma^{\alpha}(a)\sigma^{\beta}(b) \in N(R)$.

Proof. (1) $\Leftrightarrow$ (2) By definition, $ab \in N(R) \Leftrightarrow (ab)^k = 0$, for some positive integer k. If $k = 1$, $ab = 0$ if and only if $a\sigma^{\alpha}(b)$. If $k = 2$, $0 = abab \Leftrightarrow 0 = a\sigma^{\alpha}(bab) = a\sigma^{\alpha}(b)\sigma^{\alpha}(ab) \Leftrightarrow a\sigma^{\alpha}(b)ab = 0 \Leftrightarrow 0 = a\sigma^{\alpha}(b)a\sigma^{\alpha}(b) = (a\sigma^{\alpha}(b))^2$, where all equivalences are due to the Σ-compatibility of R. Continuing with this process, $(ab)^k = 0 \Leftrightarrow (a\sigma^{\alpha}(b))^k = 0$ Thus, $ab \in N(R)$ if and only if $a\sigma^{\alpha}(b) \in N(R)$.

(2) $\Leftrightarrow$ (3) It is clear.

(1) $\Leftrightarrow$ (4) Again, since $ab \in N(R) \Leftrightarrow (ab)^k = 0$, for some positive integer k, if $k = 1$, Lemma 1 implies that $ab = 0 \Leftrightarrow \sigma^{\alpha}(a)b = 0$. If $k = 2$, $0 = abab \Leftrightarrow 0 = \sigma^{\alpha}(a)bab \Leftrightarrow \sigma^{\alpha}(a)b\sigma^{\alpha}(ab) = \sigma^{\alpha}(a)b\sigma^{\alpha}(a)\sigma^{\alpha}(b) \Leftrightarrow \sigma^{\alpha}(a)b\sigma^{\alpha}(a)b = 0 \Leftrightarrow (\sigma^{\alpha}(a)b)^2 = 0$, where the first equivalence is due to Lemma 1, and the others equivalences are due to the Σ-compatibility of R. Continuing in this way, we can see that $(ab)^k = 0 \Leftrightarrow (\sigma^{\alpha}(a)b)^k = 0$. Hence $ab \in N(R) \Leftrightarrow \sigma^{\alpha}(a)b \in N(R)$.

(4) ⇔ (5) It is immediate.

(5) ⇔ (6) It follows from (1) ⇔ (2) by replacing a by $\sigma^{\alpha}(a)$ and $\sigma^{\alpha}(b)$ by $\sigma^{\beta}(b)$. □

3. Σ−skew reflexive rings

In this section, we introduce the Σ-*skew reflexive rings* and present some of the original results of the paper. We start with the following definition which establishes the generalizations of skew reflexive endomorphism and σ-skew reflexive ring.

Definition 5. Let R be a ring and $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ be a finite family of endomorphisms of R. Σ is said to be *right* (resp., *left*) *skew reflexive* if for $a, b \in R$, $aRb = 0$ implies $bR\sigma^{\alpha}(a) = 0$ (resp., $\sigma^{\alpha}(b)Ra = 0$), for all $\alpha \in \mathbb{N}^n$; R is called *right* (resp., *left*) Σ-*skew reflexive* if there exists a right (resp., left) skew reflexive family of endomorphisms Σ of R; if R is both right and left Σ-skew reflexive, then R is called Σ-*skew reflexive*.

Reflexive rings are Σ-skew reflexive. Reduced and reversible rings are reflexive, so both are Σ-skew reflexive. Right (resp., left) σ-skew reflexive rings are right (resp., left) Σ-skew reflexive.

We present an example of a right Σ-skew reflexive ring.

Example 3. Let R be a ring and M be any right and left R-module. *The trivial extension of R by M* is the ring $T(R, M) := R \oplus M$ with the usual addition of $R \oplus M$ and the multiplication is defined as follows: $(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2 + m_1r_2)$, for $r_1, r_2 \in R$ and $m_1, m_2 \in M$. The ring $T(R, M)$ is isomorphic to the matrix ring (with the usual matrix operations) of the form $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$. In particular, we denote $\mathrm{S}_2(\mathbb{Z})$ the ring of matrices isomorphic to $T(\mathbb{Z}, \mathbb{Z})$

$$\mathrm{S}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let σ_2, σ_3 be two endomorphisms of $\mathrm{S}_2(\mathbb{Z})$ defined by:

$$\sigma_2\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}, \qquad \sigma_3\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Consider $ARB = 0$, for all $R \in \mathrm{S}_2(\mathbb{Z})$, where

$$A = \begin{pmatrix} a & a' \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} b & b' \\ 0 & b \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} r & r' \\ 0 & r \end{pmatrix}.$$

We have $arb = 0$ and $arb' + ar'b + bra' = 0$, for every $r \in \mathbb{Z}$, whence $a = 0$ or $b = 0$. If $a = 0$, then $bra' = 0$, for every $r \in \mathbb{Z}$, hence $b = 0$ or $a' = 0$. If $b = 0$, then $arb' = 0$, for each $r \in \mathbb{Z}$, hence $a = 0$ or $b' = 0$. First, we consider the case $BR\sigma_1(A) = BRA$:

$$BR\sigma_1(A) = \begin{pmatrix} bra & bra' + br'a + b'ra \\ 0 & bra \end{pmatrix}.$$

From the previous observation, it follows that $BR\sigma_1(A) = 0$. Now, consider the case $BR\sigma_2(A)$. A calculation shows us that

$$BR\sigma_2(A) = \begin{pmatrix} bra & -bra' + br'a + b'ra \\ 0 & bra \end{pmatrix}.$$

Making use of the initial observation, we can see that $BR\sigma_2(A) = 0$. $S_2(\mathbb{Z})$ is right σ_2-skew reflexive [41, Proposition 2.11]. Finally, we present the case $BR\sigma_3(A)$:

$$BR\sigma_3(A) = \begin{pmatrix} bra & br'a + b'ra \\ 0 & bra \end{pmatrix}.$$

We obtain that $bra = 0$, since $BRA = 0$. If $a = 0$, then $br'a + b'ra = 0$, for every $r, r' \in \mathbb{Z}$. If $b = 0$, it follows that $b'ra + br'a + bra' = br'a + b'ra = 0$, for each $r, r' \in \mathbb{Z}$. This implies that $BR\sigma_3(A) = 0$. We can also observe that $\sigma_2 \circ \sigma_3 = \sigma_3 \circ \sigma_2 = \sigma_3$, which implies that $S_2(\mathbb{Z})$ is right Σ-skew reflexive with $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ where $\sigma_1 = \mathrm{id}_{S_2(\mathbb{Z})}$ is the identity endomorphism of $S_2(\mathbb{Z})$.

It is not difficult to see that $S_2(\mathbb{Z})$ is a reflexive ring. In addition, notice that $S_2(\mathbb{Z})$ is not σ_3-compatible. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Some computations show that $A\sigma_3(B) = 0$, but $AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$. According to Remark 2, $S_2(\mathbb{Z})$ is not Σ-compatible.

Under conditions of Σ-compatibility, Proposition 4 characterizes the right Σ-skew reflexive rings and shows that the composition of right skew reflexive endomorphisms is a right skew reflexive endomorphism.

Proposition 4. *Let R be a ring and $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ be a finite family of endomorphisms of R. If R is Σ-compatible, then R is right Σ-skew reflexive if and only if σ_i is right skew reflexive for each $1 \leq i \leq n$.*

Proof. Assume that R is Σ-compatible ring and σ_i is a right skew reflexive endomorphism for all $1 \leq i \leq n$. Is is enough to show that if σ_i and σ_j are right skew reflexive endomorphism of R for $1 \leq i, j \leq n$, then $\sigma_i \circ \sigma_j$ is also a right skew reflexive endomorphism of R. Let $a, b \in R$ such that $aRb = 0$, that is, $arb = 0$, for all $r \in R$. If σ_i and σ_j are right skew reflexive endomorphisms of R for all $1 \leq i, j \leq n$, then $arb = 0$ implies $br\sigma_i(a) = br\sigma_j(a) = 0$, whence $\sigma_i(br\sigma_j(a)) = \sigma_i(br)\sigma_i(\sigma_j(a)) = 0$. Since R is a Σ-compatible ring, $\sigma_i(br)\sigma_i(\sigma_i(a)) = 0$ implies $br\sigma_i(\sigma_j(a)) = 0$, and thus $\sigma_i \circ \sigma_j$ is a right skew reflexive endomorphism of R. This proves that the composition of right skew reflexive endomorphisms is again a right skew reflexive endomorphism, and so R is a right Σ-skew reflexive ring. Now assume that R is right Σ-skew reflexive. If $aRb = 0$ for all $a, b \in R$, then $bR\sigma^{\alpha}(a) = 0$ for all $\alpha \in \mathbb{N}^n$ by the right Σ-skew reflexivity of R. In particular, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $\alpha_i = 1$ and $\alpha_j = 0$ for $i \neq j$, we have $bR\sigma^{\alpha}(a) = bR\sigma_i(a) = 0$, and hence σ_i is a right skew reflexive endomorphism of R for all $1 \leq i \leq n$. □

Following Rege and Chhawchharia [58], a ring R is called *Armendariz* if for elements $f(x) = a_0 + \cdots + a_m x^m,\ g(x) = b_0 + \cdots + b_l x^l \in R[x]$ (where $R[x]$ is the commutative polynomial ring with an indeterminate x over R), $f(x)g(x) = 0$ implies $a_i b_j = 0$, for all i, j. The importance of this notion lies in its natural and useful role in understanding the relation between the annihilators of the ring R and the annihilators of the polynomial ring $R[x]$. The Armendariz rings over skew polynomial rings has been widely studied by several authors (see [1, 4, 28, 30, 36, 43, 58]). Hirano [26] generalized the Armendariz rings in the following way: a ring R is said to be *quasi-Armendariz* if for $f(x) = a_0 + \cdots + a_m x^m,\ g(x) = b_0 + \cdots + b_l x^l \in R[x]$, we have $f(x)R[x]g(x) = 0$ implies $a_i R b_j = 0$, for all i, j. It is well known that reduced rings are Armendariz, and Armendariz rings are quasi-Armendariz, but the converse are not true in general [13, Proposition 2.1].

Following Hashemi et al. [24], it is said that R satisfies the (*SQA1*) *condition* if $f(x)R[x; \sigma, \delta]g(x) = 0$ implies $a_i R b_j = 0$ for each i, j and $f(x) = a_0 + \cdots + a_m x^m, g(x) = b_0 + \cdots + b_l x^l \in R[x; \sigma, \delta]$. Reyes and Suárez [60] introduced this condition for skew PBW extensions. Briefly, if A is a skew PBW extension over R, it is said that R satisfies the (*SQA1*) *condition* if $fAg = 0$ implies $a_i R b_j = 0$, for every i, j and for every $f = a_1 x^{\alpha_1} + \cdots + a_m x^{\alpha_m}, g = b_1 x^{\beta_1} + \cdots + b_t x^{\beta_t} \in A$. Relationships between the notions of compatibility, (SQA1) condition, and Armendariz rings in the context of skew PBW extensions have been studied by Reyes and Suárez [59, 60, 63].

Proposition 5 shows when a quasi-commutative skew PBW extension over a Σ-skew reflexive ring is a reflexive ring.

Proposition 5. *Let A be a quasi-commutative skew PBW extension over a right Σ-skew reflexive ring R. If R satisfies the (SQA1) condition, then A is reflexive.*

Proof. Let $f = a_1x^{\alpha_1} + \cdots + a_mx^{\alpha_m}$ and $g = b_1x^{\beta_1} + \cdots + b_tx^{\beta_t}$ be two elements of A. If $fAg = 0$, then $a_iRb_j = 0$ for each i, j by the (SQA1) condition of R. Since R is Σ-skew reflexive, $b_jR\sigma^{\alpha}(a_i) = \sigma^{\alpha}(b_j)Ra_i = 0$, for all $\alpha \in \mathbb{N}^n$, and each i, j. If A is quasi-commutative, then for every $x^{\alpha} \in \text{Mon}(A)$ and $0 \neq r \in R$, there exists an element $\sigma^{\alpha}(r) \in R \setminus \{0\}$ such that $x^{\alpha}r = \sigma^{\alpha}(r)x^{\alpha}$, and for every $x^{\alpha}, x^{\beta} \in \text{Mon}(A)$, there exists an element $d_{\alpha,\beta} \in R$ such that $x^{\alpha}x^{\beta} = d_{\alpha,\beta}x^{\alpha+\beta}$, where $d_{\alpha,\beta}$ is left invertible by Proposition 2. Thus, applying these commutation rules to the product ghf, where $h = c_1x^{\gamma_1} + \cdots + c_lx^{\gamma_l}$ is an arbitrary element of A, and taking into account that $b_jR\sigma^{\alpha}(a_i) = \sigma^{\alpha}(b_j)Ra_i = 0$, for all $\alpha \in \mathbb{N}^n$, and every i, j, then $gAf = 0$, that is, A is reflexive. □

We recall the notion of *semiprime ideal* and *semiprime ring* which arise as generalizations of prime ideal and prime ring. An ideal I of R is called *semiprime* if I is an intersection of prime ideals of R. A ring R is called *semiprime* if $\{0\}$ is a semiprime ideal of R [19, p. 51]. It is not difficult to see that R is semiprime if $(aRa)^2 = 0$ implies $aRa = 0$, for all $a \in R$. Proposition 6 shows that every Σ-compatible semiprime ring is a Σ-skew reflexive ring.

Proposition 6. *Let R be a ring and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be a finite family of endomorphims of R. If R is Σ-compatible semiprime, then R is Σ-skew reflexive.*

Proof. Let $a, b \in R$ such that $aRb = 0$. By the Σ-compatibility of R, $aR\sigma^{\alpha}(b) = 0$ for every $\alpha \in \mathbb{N}^n$, whence $\sigma^{\alpha}(b)RaR\sigma^{\alpha}(b)Ra = 0$. If R is a semiprime ring, then $\sigma^{\alpha}(b)Ra = 0$, and thus R is left Σ-skew reflexive. By Remark 2, $aRb = 0$ implies $\sigma^{\alpha}(a)Rb = 0$, and so $bR\sigma^{\alpha}(a)RbR\sigma^{\alpha}(a) = 0$. The semiprimeness of R implies $bR\sigma^{\alpha}(a) = 0$, whence R is right Σ-skew reflexive. This proves that R is a Σ-skew reflexive ring. □

As a consequence, if A is a quasi-commutative skew PBW extension over a Σ-compatible semiprime ring that satisfies the (SQA1) condition, then A is reflexive by Propositions 5 and 6.

Following Birkenmeier et al. [11], a ring R is called *right principally quasi-Baer* (or simply, *right p.q.-Baer*) ring if the right annihilator of a principal right ideal of R is generated by an idempotent. Thinking about the reflexive property and the Baer properties studied in the setting of skew PBW extensions [59,60], it is not difficult to see that if A is a skew PBW extension and a right p.q.-Baer ring, then A is a semiprime ring if and only if A is a reflexive ring by [40, Proposition 3.15].

4. Skew PBW extension over $\Sigma-$skew RNP rings

In this section, we introduce the Σ-*skew RNP rings* and investigate some relationships of these rings with the skew PBW extensions. The following definition generalizes the skew RNP endomorphisms and the σ-skew RNP rings defined by Bhattacharjee [9].

Definition 6. Let R be a ring and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be a finite family of endomorphisms of R. Σ is called a *right* (resp., *left*) *skew RNP family* if for $a, b \in N(R)$, $aRb = 0$ implies $bR\sigma^{\alpha}(a) = 0$ (resp., $\sigma^{\alpha}(b)Ra = 0$), for every $\alpha \in \mathbb{N}^n$; R is said to be right (resp., left) Σ-*skew RNP* if there exists a right (resp., left) skew RNP family of endomorphism Σ of R. If R is both right and left Σ-skew RNP, then R is called Σ-*skew RNP*.

It is straightforward to show that right (resp., left) Σ-skew reflexive rings are right (resp., left) Σ-skew RNP, reduced rings are Σ-skew RNP, for any finite family of endomorphisms Σ of R, and RNP rings are Σ-skew RNP.

Example 4. We present some examples of right Σ-skew RNP.

(1) Consider Example 2. Let us show that R is a Σ-skew RNP ring. Note first that $\sigma_{1,0}$ is the identity homomorphism over R and Σ is closed under composition, that is, $\sigma_{i,j}^{\alpha} \in \Sigma$, for all $\alpha \in \mathbb{N}^6$. Notice that the set of nilpotent elements of R is the ideal generated by z, that is, $N(R) = \langle z \rangle$. Let $f, g \in N(R)$ such that $fRg = 0$. Notice that $f = a^r z$ and $g = a^s z$, for some $0 \leq r, s \leq 2$. Furthermore, $\sigma_{i,j}^{\alpha}(f) = a^k z$, for all $\alpha \in \mathbb{N}^6$ and some $0 \leq k \leq 2$. Hence, we have $gR\sigma_{i,j}^{\alpha}(f) = 0$, for all $\alpha \in \mathbb{N}^6$ with $1 \leq i \leq 2$ and $0 \leq j \leq 2$. In this way, R is a Σ-skew RNP ring.

(2) Consider Example 3. It is not difficult to see that $\mathrm{S}_2(\mathbb{Z})$ is a right Σ-skew RNP ring where $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$. Additionally, notice that $N(\mathrm{S}_2(\mathbb{Z})) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$.

Under conditions of Σ-compatibility, Proposition 7 characterizes the right Σ-skew RNP rings and shows that the composition of right skew RNP endomorphisms is a right skew RNP endomorphism.

Proposition 7. *Let R be a ring and $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ be a finite family of endomorphisms of R. If R is a Σ-compatible ring, then R is right Σ-skew RNP if and only if R is right σ_i-skew RNP for each $1 \leq i \leq n$.*

Proof. The proof is similar to that of Proposition 4 and Lemma 2. □

Proposition 8 characterizes the right and left Σ-skew RNP rings over Σ-compatible rings.

Proposition 8. *If R is a Σ-compatible ring, then the following assertions are equivalent:*

(1) *R is RNP.*

(2) *R is right Σ-skew RNP.*

(3) *R is left Σ-skew RNP.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is RNP and let $a, b \in N(R)$ such that $aRb = 0$, that is, $arb = 0$ for all $r \in R$. By the Σ-compatibility of R, $\sigma^{\alpha}(a)rb = 0$, and thus $\sigma^{\alpha}(a)Rb = 0$. If $a \in N(R)$, then $\sigma^{\alpha}(a) \in N(R)$ by Lemma 2, and since R is RNP, $\sigma^{\alpha}(a)Rb = 0$ implies that $bR\sigma^{\alpha}(a) = 0$, whence R is right Σ-skew RNP.

(2) $\Rightarrow$ (1) Assume that R is right Σ-skew RNP and let $a, b \in N(R)$ such that $aRb = 0$, i.e., $arb = 0$ for any $r \in R$. Since R is Σ-compatible, $ar\sigma^{\alpha}(b) = 0$, for every $\alpha \in \mathbb{N}^n$. If $b \in N(R)$, then $\sigma^{\alpha}(b) \in N(R)$ by Lemma 2, and if R is right Σ-skew RNP, then $\sigma^{\alpha}(b)R\sigma^{\alpha}(a) = 0$. So, for each $r \in R$, $\sigma^{\alpha}(bra) = \sigma^{\alpha}(b)\sigma^{\alpha}(r)\sigma^{\alpha}(a) = 0$. Finally, σ^{α} is injective for all $\alpha \in \mathbb{N}^n$ which implies that $bra = 0$ proving that R is RNP.

(1) $\Leftrightarrow$ (3) The proof is similar to (1) $\Leftrightarrow$ (2). □

Corollary 1 ([9, Proposition 1.7])**.** *If R is a σ-compatible ring, then the following are equivalent:* (1) *R is RNP,* (2) *R is right σ-skew RNP, and* (3) *R is left σ-skew RNP.*

Proposition 9 characterizes the right and left Σ-skew reflexive rings.

Proposition 9. *If R is a Σ-compatible ring, then the following assertions are equivalent:*

(1) *R is reflexive.*

(2) *R is right Σ-skew reflexive.*

(3) *R is left Σ-skew reflexive.*

Proof. Since every Σ-skew reflexive ring is a Σ-skew RNP ring, the result follows from Proposition 8. □

It is not difficult to see that nil-reversible rings are RNP. Proposition 10 relates nil-reversible and Σ-compatible rings with Σ-skew RNP rings.

Proposition 10. *Let R be a ring and $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ be a finite family of endomorphisms of R. If R is nil-reversible and Σ-compatible, then R is Σ-skew RNP.*

Proof. Since nil-reversible rings are RNP, if $aRb = 0$, for $a, b \in N(R)$, then $bRa = 0$, and thus $bR\sigma^{\alpha}(a) = 0$ for all $\alpha \in \mathbb{N}^n$, by Σ-compatibility of R. Hence, R is right Σ-skew RNP. From Lemma 1, we have $bRa = 0$ implies $\sigma^{\alpha}(b)Ra = 0$, and so R is left Σ-skew RNP. □

If A is a skew PBW extension over a domain R, then A is a domain [16, Proposition 3.2.1], and so $N(A) = 0$. If A also satisfies the conditions established in Proposition 3, then A is $\overline{\Sigma}$-skew RNP for $\overline{\Sigma} = \{\overline{\sigma_1}, \dots, \overline{\sigma_n}\}$, where $\overline{\sigma_k}$ is as in Proposition 3.

Proposition 11. *If A is skew PBW extension over a Σ-rigid ring R, then the following assertions hold:*

(1) *Both R and A are reflexive.*

(2) *R is Σ-skew RNP.*

(3) *If the conditions established in Proposition 3 hold, then A is $\overline{\Sigma}$-skew RNP.*

Proof. (1) By [59, Theorem 4.4], we have that both R and A is reduced. Since reduced rings are reflexive, then R and A are reflexive.

(2) If R is Σ-rigid, then R is reduced by [59, Theorem 4.4], whence R is Σ-skew RNP for any finite family of endomorphisms Σ.

(3) Since A is reduced by [59, Theorem 4.4], then A is a Σ-skew RNP ring for any finite family of endomorphisms. In particular, A is $\overline{\Sigma}$-skew RNP for $\overline{\Sigma} = \{\overline{\sigma_1}, \dots, \overline{\sigma_n}\}$, where $\overline{\sigma_k}$ is as in Proposition 3. □

Following Kwak and Lee [39], R is called *CN* if $N(R[x]) \subseteq N(R)[x]$. Both NI rings and Armendariz rings are CN but not conversely. The classes of quasi-Armendariz and CN rings are independent of each other [32, Examples 3.15(1) and (2)]. Bhattacharjee [9] showed that if a ring R is CN and quasi-Armendariz, then there is an equivalence of the RNP property between the ring R and its ring of polynomials with coefficients in R.

We introduce the following definition with the aim of study the RNP property on skew polynomial rings and skew PBW extensions. Let (Σ, Δ) be the system of endomorphisms and Σ-derivations of R with respect to a skew PBW extension A built on R (see Proposition 1).

Definition 7. Let A be a skew PBW extension over R. We say that R is Σ-*skew CN* if $N(A) \subseteq N(R)A$.

Hashemi et al. [23], investigated the connections of the prime radicals and the upper nil radicals of a ring R and a skew PBW extension A over R. For instance, some results establish sufficient conditions to guarantee that $N^*(A) \subseteq N^*(R)A$, where A is bijective skew PBW extension over a (Σ, Δ)-compatible ring [23, Theorem 3.15]. In this way, if A is NI, it follows that R is Σ-skew CN. Other important result states that if A is a PBW extension over a Σ-compatible ring R and $N(R)$ is a Δ-ideal, then $N(A) \subseteq N(R)A$ [23, Proposition 4.1], that is, R is a Σ-skew CN. From [23, Corollary 4.12], it follows that if R is a 2-primal Σ-compatible ring, then R is Σ-skew CN. Reyes and Suárez [62] proved that if A is a skew PBW extension over a weak (Σ, Δ)-compatible NI ring R, then $f = a_1x^{\alpha_1} + \cdots + a_tx^{\alpha_t} \in N(A)$ if and only if $a_i \in N(R)$, for $0 \leq i \leq t$ [62, Theorem 4.6]. Thus, if $f \in N(A)$, then $a_i \in N(R)$ whence $f \in N(R)A$, and so R is Σ-skew CN.

Example 5. We present some examples of Σ-skew CN rings.

(1) If R is a σ-compatible ring and $N(R)$ is a δ-ideal of R, it follows that $N(R[x; \sigma, \delta]) \subseteq N(R)[x; \sigma, \delta]$ by [54, Proposition 2.2], and thus R is Σ-skew CN. If $R[x; \sigma, \delta]$ is NI and $N(R)$ is a σ-rigid ideal (an ideal I of R is called σ-*rigid* if $r\sigma(r) \in I$ implies $r \in I$, for all $r \in R$), then $N(R[x; \sigma, \delta]) = N(R)[x; \sigma, \delta]$ by [55, Theorem 3.1], and so R is a Σ-skew CN ring.

(2) If R is commutative, then the *universal enveloping algebra* $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ over R is a PBW extension over R. The set of nilpotent elements of the enveloping algebra satisfies

$N(U(\mathfrak{g})) \cong N(R\langle x_1, \dots, x_n\rangle) \subseteq N(R)A$ by [62, Theorem 4.6], and thus R is a Σ-skew CN ring.

(3) Consider the skew PBW extension A from Example 2. Following the same ideas of the proof of [62, Theorem 4.6], it is not difficult to prove that $N(A) = N(\sigma(R))\langle x_{1,0}, x_{1,1}, x_{1,2}, x_{2,0}, x_{2,1}, x_{2,2}\rangle$, and thus R is a Σ-skew CN ring.

Under compatibility conditions and the (SQA1) condition, Theorem 1 characterizes the skew PBW extensions over Σ-skew CN rings that are $\overline{\Sigma}$-skew RNP for $\overline{\Sigma} = \{\overline{\sigma_1}, \dots, \overline{\sigma_n}\}$, where $\overline{\sigma_k}$ is as in Proposition 3.

Theorem 1. *Let A be a skew PBW extension over a (Σ, Δ)-compatible and Σ-skew CN ring R that satisfies the (SQA1) condition. If the conditions established in Proposition 3 hold, then R is right Σ-skew RNP if and only if A is right $\overline{\Sigma}$-skew RNP.*

Proof. Let R be a right Σ-skew RNP ring and $f = a_1x^{\alpha_1} + \cdots + a_mx^{\alpha_m}$, $g = b_1x^{\beta_1} + \cdots + b_tx^{\beta_t}$ be two nilpotent elements of A such that $fAg = 0$. Since R satisfies (SQA1) condition, then $a_iRb_j = 0$, for all i, j. On the other hand, R is Σ-skew CN, that is, $f, g \in N(R)A$, which implies that $a_i, b_j \in N(R)$ for all i, j. Since R is Σ-skew RNP, we have $b_jR\sigma^{\alpha}(a_i) = 0$, for all i, j and $\alpha \in \mathbb{N}^n$. Consider an element $h \in A$. We can note that each coefficient of $gh\overline{\sigma}^{\alpha}(f)$ are products of the coefficient b_j with elements of R and several evaluations of a_i in σ's and δ's depending of the coordinates of α_i, β_j. From the previous statement and $b_jR\sigma^{\alpha}(a_i) = 0$, it follows that $gh\overline{\sigma}^{\alpha}(f) = 0$ for all $\alpha \in \mathbb{N}^n$, by the (Σ, Δ)-compatibility of R. Since h is an arbitrary element of A, we have $gA\overline{\sigma}^{\alpha}(f) = 0$. Hence, A is right $\overline{\Sigma}$-skew RNP.

Conversely, suppose that A is right $\overline{\Sigma}$-skew RNP and let $a, b \in N(R)$ such that $aRb = 0$. Since R is (Σ, Δ)-compatible, we have $aAb = 0$. This means that $bA\overline{\sigma}^{\alpha}(a) = 0$, for all $\alpha \in \mathbb{N}^n$ entailing $bR\sigma^{\alpha}(a) = 0$, by RNP property of A and $\overline{\sigma}^{\alpha}(a) \in N(R)$. Hence, R is right Σ-skew RNP. □

Corollary 2. *Let R be a ring.*

(1) ([9, Corollary 2.13]) *If R is Armendariz and σ is an endomorphism of R, then, R is right σ-skew RNP if and only if $R[x]$ is right $\overline{\sigma}$-skew RNP.*

(2) ([9, Proposition 2.12]) *Let R be a quasi-Armendariz and CN ring, and σ be an endomorphism of R. Then R is right σ-skew RNP if and only if $R[x]$ is right $\overline{\sigma}$-skew RNP.*

Proposition 12 characterizes the skew PBW extensions over NI rings that are RNP rings.

Proposition 12. *If A is a skew PBW extension over a (Σ, Δ)-compatible NI ring R, then A is nil-reflexive if and only if A is RNP.*

Proof. If R is (Σ, Δ)-compatible, then R is a weak (Σ, Δ)-compatible ring. In this way, if R is NI, then A is NI by [67, Theorem 3.3], whence $N^*(A) = N(A)$. By [33, Proposition 2.1], we have A is nil-reflexive if and only if $fAg = 0$ implies $gAf = 0$, for all $f, g \in N^*(A)$. Hence, $fAg = 0$ implies $gAf = 0$, for elements $f, g \in N(A)$. Therefore, A is nil-reflexive if and only if A is RNP. □

Corollary 3. *Let A be a skew PBW extension over an NI ring R. If A is $(\overline{\Sigma}, \overline{\Delta})$-compatible and the conditions established in Proposition 3 hold, then A is nil-reflexive if and only if A is $\overline{\Sigma}$-skew RNP.*

Proof. The assertion follows from Propositions 8 and 12. □

We recall the following result which describes the Ore localization by regular elements of the ring R over a skew PBW extension.

Proposition 13 ([46, Lemma 2.6])**.** *Let A be a skew PBW extension over R and S be the set of regular elements of R such that $\sigma_i(S) = S$, for every $1 \leq i \leq n$, where σ_i is defined by Proposition 1.*

(1) *If $S^{-1}R$ exists, then $S^{-1}A$ exists and it is a bijective skew PBW extension over $S^{-1}R$ with $S^{-1}A = \sigma(S^{-1}R)\langle x_1', \dots, x_n'\rangle$, where $x_i' := \frac{x_i}{1}$ and the system of constants of $S^{-1}R$ is given by $d_{i,j}' := \frac{d_{i,j}}{1}$, $c_{i,\frac{r}{s}}' := \frac{\sigma_i(r)}{\sigma_i(s)}$, for all $1 \leq i, j \leq n$.*

(2) *If RS^{-1} exists, then AS^{-1} exists and it is a bijective skew PBW extension over RS^{-1} with $AS^{-1} = \sigma(RS^{-1})\langle x_1', \dots, x_n'\rangle$, where $x_i'' := \frac{x_i}{1}$ and the system of constants of $S^{-1}R$ is given by $d_{i,j}'' := \frac{d_{i,j}}{1}$, $c_{i,\frac{r}{s}}'' := \frac{\sigma_i(r)}{\sigma_i(s)}$, for all $1 \leq i, j \leq n$.*

Related with Proposition 13, if S is a multiplicatively closed subset of a ring R consisting of central regular elements, and σ is an automorphism of R such that $\sigma(S) \subseteq S$, then the mapping $\overline{\sigma} : S^{-1}R \to S^{-1}R$ defined by $\overline{\sigma}(u^{-1}a) = \sigma(u)^{-1}\sigma(a)$ for $u \in S$ and $a \in R$, induces an endomorphism of $S^{-1}R$. If Σ is a set of automorphisms over R, we denote Σ_S the set of automorphisms over $S^{-1}R$, induced by Σ.

Theorem 2. *Let R be a ring, Σ be a finite family of automorphism of R, and S a multiplicatively closed subset of R consisting of central regular elements such that $\sigma^{\alpha}(S) \subseteq S$, for every $\alpha \in \mathbb{N}^n$. Then R is right Σ-skew RNP if and only if $S^{-1}R$ is right Σ_S-skew RNP.*

Proof. Assume that R is right Σ-skew RNP. Let $s_1^{-1}a, s_2^{-1}b \in N(S^{-1}R)$ such that $(s_1^{-1}a)S^{-1}R(s_2^{-1}b) = 0$. Since $N(S^{-1}R) = S^{-1}N(R)$, we have $a, b \in N(R)$. In addition, $(s_1 s s_2)^{-1}(arb) = (s_1^{-1}a)s^{-1}r(s_2^{-1}b) = 0$ for all $s^{-1}r \in S^{-1}R$. By definition of $S^{-1}R$, there exists $c \in S$ such that c is a central element of R and $0 = (arb)c = a(rc)b$. Since R is right Σ-skew RNP, it follows that $b(rc)\sigma^{\alpha}(a) = 0$, for all $\alpha \in \mathbb{N}^n$. Therefore,

$$\begin{aligned}(s_2^{-1}b)(s^{-1}r)\overline{\sigma}^{\alpha}(s_1^{-1}a) &= (s_2^{-1}b)(s^{-1}r)\sigma^{\alpha}(s_1)^{-1}\sigma^{\alpha}(a)\\ &= (s_2 s\sigma^{\alpha}(s_1))^{-1}(br\sigma^{\alpha}(a))\\ &= 0,\end{aligned}$$

since $b(r)\sigma^{\alpha}(a)c = 0$, for all $\alpha \in \mathbb{N}^n$, whence $S^{-1}R$ is right Σ_S-skew RNP.

For the other implication, let $a, b \in N(R)$ such that $aRb = 0$. We have $(1^{-1})(arb) = (1^{-1}a)(1^{-1}r)(1^{-1}b) = 0$. If $S^{-1}R$ is right Σ_S-skew RNP, $(1^{-1}b)(1^{-1}r)\overline{\sigma}^{\alpha}(1^{-1}a) = (1^{-1}b)(1^{-1}r)\sigma^{\alpha}(1)^{-1}\sigma^{\alpha}(a) = (1^{-1})(br\sigma^{\alpha}(a))$ whence $br\sigma^{\alpha}(a) = 0$ since $(1^{-1}b)(1^{-1}r)\overline{\sigma}^{\alpha}(1^{-1}a) = 0$. Hence, R is right Σ-skew RNP. □

Theorem 3. *Let A be a bijective skew PBW extension over a (Σ, Δ)-compatible ring R, S be a multiplicatively closed subset of R consisting of central regular elements such that $\overline{\sigma}^{\alpha}(S) \subseteq S$ and $S^{-1}R$ is a $\overline{\Sigma}_S$-skew CN which satisfies the $(SQA1)$ condition. If the conditions established in Proposition 3 hold, then A is right $\overline{\Sigma}$-skew RNP if and only if $S^{-1}A$ is right $\overline{\Sigma}_S$-skew RNP.*

Proof. Suppose that A is right $\overline{\Sigma}$-skew RNP. Since $S^{-1}R$ is a Σ_S-skew CN ring, then R is Σ-skew CN. Similarly, the condition (SQA1) also transfers from $S^{-1}R$ to R. Thus, we have that R is right Σ-skew RNP, by using Theorem 1. By hypothesis, S is a multiplicatively closed subset of R consisting of central regular elements, which means that $S^{-1}R$ is right Σ_S-skew RNP, by Theorem 2. In addition, by Proposition 13, we have that $S^{-1}A$ is a bijective skew PBW extension over $S^{-1}R$. Since R is (Σ, Δ)-compatible, then $S^{-1}R$ is $(\overline{\Sigma}, \overline{\Delta})$-compatible [60, Theorem 4.20]. Notice that $S^{-1}A$ is a bijective skew PBW extension over $S^{-1}R$, where $S^{-1}R$ is a $(\overline{\Sigma}, \overline{\Delta})$-compatible $\overline{\Sigma}_S$-skew CN ring which satisfies the (SQA1) condition. In this way, if $S^{-1}R$ is right Σ-skew RNP, then $S^{-1}A$ is right

$\overline{\Sigma}_S$-skew RNP, by Theorem 1. For the other implication, if $S^{-1}A$ is right $\overline{\Sigma}_S$-skew RNP, then $S^{-1}R$ is a right $\overline{\Sigma}_S$-skew RNP ring, because $S^{-1}R$ is $(\overline{\Sigma}, \overline{\Delta})$-compatible $\overline{\Sigma}_S$-skew CN and satisfies the condition (SQA1). The above statement follows from Theorem 1. Since $S^{-1}R$ is $\overline{\Sigma}_S$-skew RNP, then R is Σ-skew RNP. Finally, R is a (Σ, Δ)-compatible ring and inherits the Σ-skew CN property and the condition (SQA1) from the ring $S^{-1}R$, which implies that A is right $\overline{\Sigma}$-skew RNP, by Theorem 1. □

References

[1] Anderson, D.D., Camillo, V.: Armendariz rings and Gaussian rings. Comm. Algebra. **26**(7), 2265–2275 (1998). https://doi.org/10.1080/00927879808826274

[2] Anderson, D.D., Camillo, V.: Semigroups and rings whose zero products commute. Comm. Algebra. **27**(6), 2847-2852 (1999). https://doi.org/10.1080/00927879908826596

[3] Annin, S.: Associated primes over Ore extension rings. J. Algebra Appl. **3**(2), 193–205 (2004). https://doi.org/10.1142/S0219498804000782

[4] Armendariz, E.P.: A note on extensions of Baer and p.p.-rings. J. Austral. Math. Soc. **18**(4), 470–473 (1974). https://doi.org/10.1017/S1446788700029190

[5] Artamonov, V.A.: Derivations of skew PBW extensions. Commun. Math. Stat. **3**(4), 449–457 (2015). https://doi.org/10.1007/s40304-015-0067-9

[6] Başer, M., Hong, C.Y., Kwak, T.K.: On Extended Reversible Rings. Algebra Colloq. **16**(1), 37–48 (2009).

[7] Bell, H.E.: Near-rings in which each element is a power of itself. Bull. Aust. Math. Soc. **2**(3), 363–368 (1970).

[8] Bell, A., Goodearl, K.: Uniform rank over differential operator rings and Poincaré-Birkhoff-Witt extensions. Pacific J. Math. **131**(1), 13–37 (1998).

[9] Bhattacharjee, A.: Reflexive-nilpotents-property skewed by ring endomorphisms. Arab. J. Math. **9**(1), 63–72 (2020) https://doi.org/10.1007/s40065-018-0229-1

[10] Birkenmeier, G.F., Heatherly, H.E., Lee, E.K.: Completely prime ideals and associated radicals. In: Jain, S.K., Rizvi, S.T. (eds.) Ring Theory (Granville, OH, 1992), pp. 102-129. World Scientific, Singapore and River Edge (1993).

[11] Birkenmeier, G.F., Kim, J.Y., Park, J.K.: Principally quasi-Baer rings. Comm. Algebra. **29**(2), 639–660 (2001). https://doi.org/10.1081/AGB-100001530

[12] Brown, K.A., Goodearl, K.R.: Lectures on Algebraic Quantum Groups. CRM Barcelona. Advanced Courses in Mathematics, Birkhaüser Basel (2002).

[13] Chhawchharia, S., Rege, M.B.: Armendariz rings. Proc. Japan. Acad. Ser. A Math. Sci. **73**(1), 14–17 (1997). https://dx.doi.org/10.3792/pjaa.73.14

[14] Cohn, P.M.: Reversible Rings. Bull. Lond. Math. Soc. **31**(6), 641–648 (1999). http://dx.doi.org/10.1112/S0024609399006116

[15] Courter, R.C.: Rings all of whose factor rings are semi-prime. Canad. Math. Bull. **12**(4), 417–426 (1969). https://doi.org/10.4153/CMB-1969-052-2

[16] Fajardo, W., Gallego, C., Lezama, O., Reyes, A., Suárez, H., Venegas, H.: Skew PBW Extensions. Ring and Module-theoretic Properties, Matrix and Gröbner Methods, and Applications. Springer, Cham (2020).

[17] Gallego, C., Lezama, O.: Gröbner Bases for Ideals of σ-PBW Extensions. Comm. Algebra. **39**(1), 50–75 (2011). https://doi.org/10.1080/00927870903431209

[18] Gómez, J.Y., Suárez, H.: Double Ore extensions versus graded skew PBW extensions. Comm. Algebra. **48**(1), 185–197 (2020). https://doi.org/10.1080/00927872.2019.1635610

[19] Goodearl, K.R., Warfield, R.B.Jr.: An Introduction to Noncommutative Noetherian Rings. Cambridge University Press (2004).

[20] Habeb, J.M.: A note on zero commutative and duo rings. Math. J. Okayama Univ. **32**(1), 73–76 (1990). https://doi.org/10.18926/mjou/33313

[21] Hamidizadeh, M., Hashemi, E., Reyes, A.: A classification of ring elements in skew PBW extensions over compatible rings. Int. Electron. J. Algebra. **28**(1), 75–97 (2020). https://doi.org/10.24330/ieja.768178

[22] Hashemi, E., Khalilnezhad, Kh., Alhevaz, A.: (Σ, Δ)-Compatible skew PBW extension ring. Kyungpook Math. J. **57**(3), 401–417 (2017). https://doi.org/10.5666/KMJ.2017.57.3.401

[23] Hashemi, E., Khalilnezhad, Kh., Alhevaz, A.: Extensions of rings over 2-primal rings. Matematiche. **74**(1), 141–162 (2019).

[24] Hashemi, E., Moussavi, A.: Polinomial extensions of quasi-Baer rings. Acta Math. Hungar. **107**(3), 207–224 (2005). https://doi.org/10.1007/s10474-005-0191-1

[25] Higuera, S., Reyes, A.: A survey on the fusible property for skew PBW extensions. J. Algebr. Syst. **10**(1), 1–29 (2022). https://doi.org/10.22044/JAS.2021.10351.1513

[26] Hirano, Y.: On annihilator ideals of a polynomial ring over a noncommutative ring. J. Pure Appl. Algebra. **168**(1), 45–52 (2002). https://doi.org/10.1016/S0022-4049(01)00053-6

[27] Hong, C.Y., Kim, N.K., Kwak, T.K.: Ore extensions of Baer and p.p.-rings. J. Pure Appl. Algebra. **151**(3), 215–226 (2000). https://doi.org/10.1016/S0022-4049(99)00020-1

[28] Hong, C.Y., Kim, N.K., Kwak, T.K.: On skew Armendariz rings. Comm. Algebra. **31**(1), 103–122 (2003). https://doi.org/10.1081/AGB-120016752

[29] Hong, C.Y., Kwak, T.K.: On minimal strongly prime ideals. Comm. Algebra. **28**(10), 4867–4878 (2000). https://doi.org/10.1080/00927870008827127

[30] Huh, C., Lee, Y., Smoktunowicz, A.: Armendariz rings and semicommutative rings. Comm. Algebra. **30**(2), 751–761 (2002). https://doi.org/10.1081/AGB-120013179

[31] Hwang, S.U., Jeon, Y.C., Lee, Y.: Structure and topological conditions of NI rings. J. Algebra. **302**(1), 186–199 (2006). https://doi.org/10.1016/j.jalgebra.2006.02.032

[32] Kheradmand, M., Khabazian, H., Kwak, T.K., Lee, Y.: Reflexive property restricted to nilpotents. J. Algebra Appl. **16**(3), 1750044 (2017). https://doi.org/10.1142/S021949881750044X

[33] Kheradmand, M., Kim, H.K., Kwak, T.K., Lee, Y.: Reflexive property on nil ideals. J. Algebra Appl. **17**(1), 1750022 (2017). https://doi.org/10.1142/S0219498817500220

[34] Kim, J.Y.: Certain rings whose simple singular modules are GP-injective. Proc. Japan Acad. Ser. A Math. Sci. **81**(7), 125–128 (2005).

[35] Kim, J.Y., Baik, J.U.: On idempotent reflexive rings. Kyungpook Math. J. **46**(4), 597–601 (2006).

[36] Kim, N.K., Lee, Y.: Armendariz rings and reduced rings. J. Algebra. **223**(2), 477–488 (2000). https://doi.org/10.1006/jabr.1999.8017

[37] Kim, N.K., Lee, Y.: Extensions of reversible rings. J. Pure and Appl. Algebra. **185**(1-3), 207–223 (2003). https://doi.org/10.1016/S0022-4049(03)00109-9

[38] Krempa, J.: Some examples of reduced rings. Algebra Colloq. **3**(4), 289–300 (1996).

[39] Kwak, T.K., Lee, Y.: Rings over which coefficients of nilpotent polynomials are nilpotent. Int. J. Algebra Comput. **21**(5), 745762 (2011). https://doi.org/10.1142/S0218196711006431

[40] Kwak, T.K., Lee, Y.: Reflexive property of rings. Comm. Algebra. **40**(4), 1576–1594 (2012). https://doi.org/10.1080/00927872.2011.554474

[41] Kwak, T.K., Lee, Y., Yun, S.J.: Reflexive property skewed by ring endomorphisms. Korean J. Math. **22**(2), 217–234 (2014). https://doi.org/10.11568/kjm.2014.22.2.217

[42] Lambek, J.: On the representation of modules by sheaves of factor modules. Canad. Math. Bull. **14**(3), 359–368 (1971). https://doi.org/10.4153/CMB-1971-065-1

[43] Lee, T.K., Wong, T.L.: On Armendariz rings. Houston J. Math. **29**(3), 583–593 (2003).

[44] Lezama, O.: Some open problems in the context of skew PBW extensions and semi-graded rings. Commun. Math. Stat. **9**(3), 347–378 (2021). https://doi.org/10.1007/s40304-021-00238-7

[45] Lezama, O., Acosta, J.P., Reyes, A.: Prime ideals of skew PBW extensions. Rev. Un. Mat. Argentina. **56**(2), 39–55 (2015). https://doi.org/10.48550/arXiv.1402.2321

[46] Lezama, O., Acosta, J.P., Chaparro, C., Ojeda, I., Venegas, C.: Ore and Goldie theorems for skew PBW extensions. Asian-Eur. J. Math. **6**(4), 1350061 (2013). https://doi.org/10.1142/S1793557113500617

[47] Lezama, O., Gallego, C.: d-Hermite rings and skew PBW extensions. São Paulo J. Math. Sci. **10**(1), 60–72 (2016). https://doi.org/10.1007/s40863-015-0010-8

[48] Lezama, O., Reyes, A.: Some Homological Properties of Skew PBW Extensions. Comm. Algebra. **42**(3), 1200–1230 (2014). https://doi.org/10.1080/00927872.2012.735304

[49] Marks, G.: On 2-primal Ore extensions. Comm. Algebra. **29**(5), 2113–2123 (2001). https://doi.org/10.1081/AGB-100002173

[50] Marks, G.: Reversible and symmetric rings. J. Pure and Appl. Algebra. **174**(3), 311–318 (2002). https://doi.org/10.1016/S0022-4049(02)00070-1

[51] Mason, G.: Reflexive ideals. Comm. Algebra. **9**(17), 1709–1724 (1981). https://doi.org/10.1080/00927878108822678

[52] McConnell, J., Robson, J.: Noncommutative Noetherian Rings. Graduate Studies in Mathematics, Second Edition. AMS (2001).

[53] Motais de Narbonne, L.: Anneaux semi-commutatifs et unis riels anneaux dont les id aux principaux sont idempotents. Paper presented at the 106th National Congress of Learned Societies, Perpignan, Paris: Bib. Nat. (1981).

[54] Nasr-Isfahani, A.R.: Ore extensions of 2-primal rings. J. Algebra Appl. **13**(3), 1350117 (2014). https://doi.org/10.1142/S021949881350117X

[55] Nasr-Isfahani, A.R.: On NI skew polynomial ring. Comm. Algebra. **43**(12), 5113–5120 (2015). https://doi.org/10.1080/00927872.2014.957385

[56] Niño, A., Reyes, A.: Some remarks about minimal prime ideals of skew Poincaré-Birkhoff-Witt extensions. Algebra Discrete Math. **30**(2), 207–229 (2020). https://dx.doi.org/10.12958/adm1307

[57] Ore, O.: Theory of non-commutative polynomials. Ann. of Math. (2). **34**(3), 480–508 (1933).

[58] Rege, M.B., Chhawchharia, S.: Armendariz rings. Proc. Jpn. Acad. Ser. A Math. Sci. **73**(1), 14–17 (1997). https://doi.org/10.3792/pjaa.73.14

[59] Reyes, A., Suárez, H.: σ-PBW extensions of skew Armendariz rings. Adv. Appl. Clifford Algebr. **27**(4), 3197–3224 (2017). https://doi.org/10.1007/s00006-017-0800-4

[60] Reyes, A., Suárez, H.: A notion of compatibility for Armendariz and Baer properties over skew PBW extensions. Rev. Un. Mat. Argentina. **59**(1), 157–178 (2018).

[61] Reyes, A., Suárez, H.: Skew Poincaré-Birkhoff-Witt extensions over weak zip rings. Beitr. Algebra Geom. **60**(2), 197–216 (2019). https://doi.org/10.1007/s13366-018-0412-8

[62] Reyes, A., Suárez, H.: Skew Poincaré-Birkhoff-Witt extensions over weak compatible rings, J. Algebra Appl. **19**(12), 2050225 (2020). https://doi.org/10.1142/S0219498820502254

[63] Reyes, A., Suárez, H.: Radicals and Köthe's conjecture for skew PBW extensions. Commun. Math. Stat. **9**(2), 119–138 (2021). https://doi.org/10.1007/s40304-019-00189-0

[64] Reyes, A., Suárez, H.: Skew PBW extensions over symmetric rings. Algebra Discrete Math. **31**(1), 76–102 (2021). http://dx.doi.org/10.12958/adm1767

[65] Shin, G.Y.: Prime ideals and sheaf representation of a pseudo symmetric ring. Trans. Amer. Math. Soc. **184**, 43–60 (1973).

[66] Suárez, H.: Koszulity for graded skew PBW extensions. Comm. Algebra. **45**(10), 4569–4580 (2017). https://dx.doi.org/10.1080/00927872.2016.1272694

[67] Suárez, H., Chacón, A., Reyes, A.: On NI and NJ skew PBW extensions. Comm. Algebra. **50**(8), 3261–3275 (2022). https://doi.org/10.1080/00927872.2022.2028799

[68] Suárez, H., Reyes, A.: Σ-Semicommutative rings and their skew PBW extensions. São Paulo J. Math. Sci. **17**(2), 531–554 (2023). https://doi.org/10.1007/s40863-023-00356-w

[69] Suárez, H., Reyes, A., Suárez, Y.: Homogenized skew PBW extensions. Arab. J. Math. **12**(1), 247–263 (2023). https://doi.org/10.1007/s40065-022-00410-z

[70] Tumwesigye, A.B., Richter, J., Silvestrov, S.: Centralizers in PBW Extensions. In: Silvestrov S., Malyarenko A., Rancic M. (eds.) Algebraic Structures and Applications, SPAS 2017. Springer Proceedings in Mathematics & Statistics, vol. **317**. Springer (2020).

[71] Venegas, C.: Automorphisms for skew PBW extensions and skew quantum polynomial rings. Comm. Algebra. **43**(5), 1877–1897 (2015). https://doi.org/10.1080/00927872.2013.879163

Contact information

H. Suárez

School of Mathematics and Statistics,
Grupo de Álgebra y Análisis,
Universidad Pedagógica y Tecnológica de Colombia, Tunja
E-Mail: hector.suarez@uptc.edu.co

S. Higuera,
A. Reyes

Department of Mathematics,
Faculty of Science,
Universidad Nacional de Colombia - Sede Bogotá
E-Mail: sdhiguerar@unal.edu.co
mareyesv@unal.edu.co

Received by the editors: 01.12.2021
and in final form 04.03.2024.