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On special subalgebras of derivations of Leibniz algebras

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Abstract. Our aim in this work is to study the central derivations of Leibniz algebras and investigate the properties of Leibniz algebras by comparing the set of central derivations with the inner derivations. We prove that, the set of all central derivations of a Leibniz algebra with non-trivial center coincide with the set of all inner derivations if and only if the Leibniz algebra is metabelian. In addition, we will show, by examples, that some statements hold for arbitrary Lie algebras, but does not hold for some Leibniz algebras.

Introduction

The interest in the study of derivations of algebras increased after the paper of Jacobson [8]. There using the nilpotency property of derivations, Jacobson specified one important class of Lie algebras called characteristically nilpotent. Since then the characteristically nilpotent class of Lie algebras have been intensively and extensively treated by many authors [6, 12]. Motivated by the progress made in 1971, Ravisankar [14] extended the concept of being characteristically nilpotent to other classes of algebras. This approach has been used for the study of Malcev algebras, as well as for associative algebras and their deformation theory [13].

In 1939 Jacobson proved that the exceptional complex simple Lie algebra G_2 of dimension 14 can be represented as the algebra of derivations

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of the Cayley algebra [9]. This result increased the interest in analyzing the derivations of Lie algebras. Two years earlier, Schenkman [15] had published his derivation tower theorem for centerless Lie algebras, which described in a nice manner the derivation algebras. This theory was not applicable to the nilpotent algebras, as the adjoint representation is not faithful. This fact led to the assumption that the structure of derivations for nilpotent Lie algebras is much more difficult than for classical algebras. A Lie algebra derivation defined by a right multiplication operator is said to be inner. All other derivations are called outer. It is well known that any derivation of a finite-dimensional semi-simple Lie algebra over a field of characteristic zero is inner. The outer derivations can be interpreted as elements of the first (co)homology group considering the algebra as a module over itself. Here the derivations are 1-(co)cycles and the inner derivations play the role of 1-(co)boundaries. Note that it was proved that any nilpotent Lie algebra has an outer derivation, i.e., there exists at least one derivation which is not the adjoint operator for a vector of the algebra. We remind also the fact that any Lie algebra over a field of characteristic zero which has non degenerate derivations is nilpotent [18,19].

The problems concerning the structure of the derivation algebra and its relations with the structure of Lie algebra have been investigated by several authors in [6,7,11,12,16,17]. Since the study of the properties of derivations of a Lie algebra play an essential role in the theory of Lie algebras, the question naturally arises whether the corresponding results can be extended to the more general framework of the Leibniz algebras.

A simple, but yet productive property from Lie theory, namely the fact that the right multiplication operator on an element of the algebra is a derivation, can also be taken as a defining property for a Leibniz algebra. In the last years, Leibniz algebras have been under active research; among the numerous papers devoted to this subject, we can find some (co)homology and deformations properties, results on various types of decompositions, structure of solvable and nilpotent Leibniz algebras and classifications of some classes of graded nilpotent Leibniz algebras. The classical results on Cartan subalgebras, Levi decomposition, Killing form, Engel's theorem, properties of solvable algebras with a given nilradical and others from the theory of Lie algebras are also true for Leibniz algebras [1–5,10].

In this work we consider some properties of derivations of Leibniz algebras. We study the central derivations of Leibniz algebras and investigate the properties of Leibniz algebras by comparing the set of central derivations with the inner derivations. We prove that, the set of all central derivations of a Leibniz algebra with non-trivial center coincide with the set of all inner derivations if and only if the Leibniz algebra is metabelian. In addition, we will show, by examples, that some statements hold for arbitrary Lie algebras, but does not hold for some Leibniz algebras.

Throughout the article all Leibniz algebras under consideration are finite dimension over a field $\mathbb{F}$ of characteristic zero.

1. Preliminaries

This section is devoted to recalling some basic notions and concepts used through the work.

Definition 1. A Leibniz algebra over the field $\mathbb{F}$ is a vector space L equipped with a bilinear map, called bracket, $[-,-] : L \times L \to L$, satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

for all $x, y, z \in L$.

The set $\mathrm{Ann}_r(L) = \{x \in L : [y,x] = 0, \ \forall y \in L\}$ is called *the right annihilator of* L. It is observed that for any $x, y \in L$ the elements $[x,x]$ and $[x,y] + [y,x]$ are always in $\mathrm{Ann}_r(L)$, and that is $\mathrm{Ann}_r(L)$ is a two-sided ideal of L.

The set $Z(L) = \{z \in L : [x,z] = [z,x] = 0, \ \forall x \in L\}$ is called *the center* of L.

Definition 2. For a given Leibniz algebra $(L, [-,-])$ the sequences of two-sided ideals defined recursively as follows:

$$L^1 = L, \ L^{k+1} = [L^k, L], \ k \geqslant 1,$$

$$L^{[1]} = L, \ L^{[s+1]} = [L^{[s]}, L^{[s]}], \ s \geqslant 1,$$

are said to be *the lower central* and *the derived series* of L, respectively.

Definition 3. A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ $(m \in \mathbb{N})$ such that $L^n = \{0\}$ (respectively, $L^{[m]} = \{0\}$). The minimal number n with such property is said to be the index of nilpotency of the algebra L.

Definition 4. The maximal nilpotent (solvable) ideal of a Leibniz algebra is called the nilradical (radical, respectively) of the algebra.

Definition 5. A linear map $d\colon L \to L$ of a Leibniz algebra $(L, [\cdot,\cdot])$ is said to be a derivation if for all $x, y \in L$, the following condition holds:

$$d([x,y]) = [d(x), y] + [x, d(y)].$$

The set of all derivations of L is denoted by $Der(L)$ and it is a Lie algebra with respect to the commutator.

For a given element x of a Leibniz algebra L, the right multiplication operator $R_x\colon L \to L$, defined by $R_x(y) = [y, x]$, is a derivation. In fact, a Leibniz algebra is characterized by this property of the right multiplication operators. As in Lie case this kind of derivations are said to be *inner derivations.* The set of all inner derivations of a Leibniz algebra L is denoted by $R(L)$, i.e. $R(L) = \{R_x \mid x \in L\}$. The set $R(L)$ inherits the Lie algebra structure from $Der(L)$:

$$[R_x, R_y] = R_x \circ R_y - R_y \circ R_x = R_{[y,x]}$$

Definition 6. A derivation d of a Leibniz algebra L is called *central* if $d(L) \subseteq Z(L)$.

We denote by $\mathrm{CDer}(L)$ the set of all central derivations of L. It should be noted that $\mathrm{CDer}(L)$ is a subalgebra of $Der(L)$.

We recall an analogue of Levi's theorem for Leibniz algebras.

Theorem 1. *[4] Let L be a finite-dimensional Leibniz algebra over a field of characteristic zero and let $\mathfrak{R}$ be its solvable radical. Then there exists a semi-simple Lie subalgebra S of L such that $L = S \dot{+} \mathfrak{R}$.*

The subalgebra S of the above theorem, similarly to Lie algebras theory, is called a *Levi subalgebra* of the Leibniz algebra L.

2. Main results

We study the derivations of a Leibniz algebra which are both inner and central. We will prove the following proposition.

Proposition 1. *Let L be a Leibniz algebra, $\mathfrak{R}$ and $\mathfrak{R}_1$ be the radicals of L and L^2, respectively. Then:*

(1) $R(L) \cap \mathrm{CDer}(L) = R(Z_1)$, where $Z_1 = \{l \in L \mid [L, l] \subseteq Z(L)\}$.

(2) $R(L) \cap \mathrm{CDer}(L) \subset R(Z_2)$, where $Z_2 = \{r \in \mathfrak{R} \mid [\mathfrak{R}_1, r] = 0\}$.

Proof. (1): We consider following cases:

- If $Z(L) = \{0\}$, then $\mathrm{CDer}(L) = \{0\}$ and $Z_1 = \mathrm{Ann}_r(L)$.

- If $L = L^2$, then $\mathrm{CDer}(L) = \{0\}$ and

$$[L, Z_1] = [[L, L], Z_1] \subseteq [L, [L, Z_1]] + [[L, Z_1], L] = \{0\},$$

that is, $Z_1 = \mathrm{Ann}_r(L)$. Therefore in these cases, $R(L) \cap \mathrm{CDer}(L) = R(Z_1) = \{0\}$.
- If $Z(L) \neq \{0\}$ and $L \neq L^2$, then for an element l of L the inner derivation R_l is in $\mathrm{CDer}(L)$ if and only if $[L, l] \subset Z(L)$, that is, l is in Z_1. Therefore $R(L) \cap \mathrm{CDer}(L) = R(Z_1)$.

(2): We have $L = S + \mathfrak{R}$ and $[L, L] = S + \mathfrak{R}_1$, $\mathfrak{R}$ and $\mathfrak{R}_1$ be the radical of L and L^2, respectively.

Now let d be any element of $R(L) \cap \mathrm{CDer}(L)$. Since $d \in R(L)$ we have

$$d = R_{s+r} \quad with \quad s \in S, \quad r \in \mathfrak{R}.$$

Since d is central, $d(L^2) = 0$ and therefore

$$d(S) = 0 \quad and \quad d(\mathfrak{R}_1) = 0.$$

It follows that

$$R_{s+r}(S) = [S, s + r] = [S, s] + [S, r] = 0,$$

whence $[S, s] = 0$ and therefore $s = 0$. Thus $d = R_r$.

Since $d(\mathfrak{R}_1) = 0$, it follows that $R_r(\mathfrak{R}_1) = [\mathfrak{R}_1, r] = 0$. □

In the examples in this article we only indicate the non-zero products of the elements of the given basis.

Example 1. Let L be the Leibniz algebra described in terms of a basis $\{e_1, e_2, e_3, e_4, e_5\}$ by the following table:

$$\begin{cases} [e_1, e_1] = e_4, & [e_1, e_2] = e_3, \\ [e_2, e_1] = -e_3, & [e_2, e_2] = e_5, \\ [e_3, e_1] = e_5, & [e_1, e_3] = -e_5. \\ [e_4, e_1] = e_5, & \end{cases}$$

The matrix form of $Der(L)$ is

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & -2a_{12} & a_{25} \\ 0 & 0 & 0 & 0 & a_{12} - a_{23} \\ 0 & 0 & 0 & 0 & a_{14} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and matrices of an inner and central derivations are

$$\begin{pmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & 0 & -a_{14} & 0 & a_{13} \\ 0 & 0 & 0 & 0 & a_{14} \\ 0 & 0 & 0 & 0 & a_{14} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & a_{15} \\ 0 & 0 & 0 & 0 & a_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $Der(L)$ is 6-dimensional, $R(L)$ is 3-dimensional and $\mathrm{CDer}(L)$ is 2-dimensional and

$$R(L) \cap \mathrm{CDer}(L) = \{R_{e_3}\}.$$

It should be noted that $R(L) \cap \mathrm{CDer}(L) \neq \{0\}$ for any non-abelian nilpotent Lie algebra [18]. In the following example we show that there exists non-Lie nilpotent Leibniz algebra, such that $R(L) \cap \mathrm{CDer}(L) = (0)$.

Example 2. We consider n-dimensional null-filiform Leibniz algebra.

$$NF_n : [e_i, e_1] = e_{i+1}, \quad 1 \leqslant i \leqslant n-1.$$

where $\{e_1, e_2, ..., e_n\}$ is a basis of the algebra NF_n. The center $Z(NF_n) = \mathrm{span}\{e_n\}$.

From [5] we know that, any derivation of the algebra NF_n has the following matrix form:

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ 0 & 2a_1 & a_2 & \dots & a_{n-1} \\ 0 & 0 & 3a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & na_1 \end{pmatrix}.$$

By using the matrix it is immediate that $\dim R(NF_n) = \dim \mathrm{CDer}(NF_n) = 1$, but $R(NF_n) \cap \mathrm{CDer}(NF_n) = \{0\}$.

Now we will study the Leibniz algebras with the condition $\mathrm{CDer}(L) \subset R(L)$. We have the following result:

Theorem 2. *Let L be a Leibniz algebra satisfying $\mathrm{CDer}(L) \subset R(L)$. The following assertions are valid.*

(1) *If $\mathfrak{R}$ is abelian, then either $Z(L) = \{0\}$ or $L = L^2$.*

(2) *If $Z(L) \neq \{0\}$ and $\mathrm{CDer}(L) \neq \{0\}$, then $\mathfrak{R}$ is not abelian.*

Proof. (1): Let $\mathfrak{R}$ be the solvable radical of the algebra L. Then there exists a semi-simple Lie subalgebra S of L such that $L = S + \mathfrak{R}$.

Suppose that the center $Z(L) \neq \{0\}$ and $L \neq L^2$. Since $L^2 = S + [S,\mathfrak{R}] + [\mathfrak{R}, S]$, it follows that $\mathfrak{R} \neq [S,\mathfrak{R}] + [\mathfrak{R}, S]$. Choose a subspace U of $\mathfrak{R}$ such that

$$\mathfrak{R} = U + [S,\mathfrak{R}] + [\mathfrak{R}, S], \quad U \cap [S,\mathfrak{R}] + [\mathfrak{R}, S] = \{0\}.$$

Define a non-zero endomorphism d of L such that

$$dU \subset Z(L), \qquad d(S + [S,\mathfrak{R}] + [\mathfrak{R}, S]) = 0.$$

Then d is a central derivation of L.

According to the condition of theorem $\mathrm{CDer}(L) \subset R(L)$, thus we get

$$d = R_{s+r}, \qquad s \in S, \qquad r \in \mathfrak{R}.$$

Since $dS = (0)$, it immediately follows that $s = 0$. Therefore $d = R_r$ and we have

$$dU = [U, r] \subset [\mathfrak{R},\mathfrak{R}] = 0,$$

which contradicts our definition of d.

Therefore we see that $Z(L) = \{0\}$ or $L = L^2$.

(2): To prove the assertion of (2), assume that $Z(L) \neq \{0\}$ and $\mathrm{CDer}(L) \subset R(L)$. If $\mathfrak{R}$ is abelian, then by (1) we see that $L = L^2$. Therefore $\mathrm{CDer}(L) = \{0\}$. This is a contradiction. □

Example 3. Let L be the 5-dimensional Leibniz algebra with the basis $\{e_1, e_2, e_3, e_4, e_5\}$ such that products of the basis vectors in L are represented as follows:

$$\begin{cases} [e_1,e_2] = 2e_2, & [e_2,e_3] = e_1, & [e_4,e_2] = e_5, \\ [e_2,e_1] = -2e_2, & [e_3,e_2] = -e_1, & [e_5,e_1] = -e_5, \\ [e_1,e_3] = -2e_3, & [e_4,e_1] = e_4, & [e_5,e_3] = e_4. \\ [e_3,e_1] = 2e_3, & & \end{cases}$$

$S = \mathrm{span}\{e_1, e_2, e_3\}$ is semi-simple subalgebra and $\mathfrak{R} = \mathrm{span}\{e_4, e_5\}$ is the abelian radical. $Z(L) = \{0\}$ and $L^2 = L$ and the central derivation $\mathrm{CDer}(L) = \{0\}$.

Example 4. Let L be the direct sum of the Leibniz algebra in Example 3 and a non-zero abelian Leibniz algebra. Then $Z(L) \neq \{0\}$ and $\mathrm{CDer}(L) \neq \{0\}$. It is not difficult to see that $\mathrm{CDer}(L)$ is not contained in $R(L)$.

In particular, we consider the Leibniz algebra $L = L_1 \oplus \mathrm{span}\{e_6\}$, where L_1 is 5-dimensional algebra in Example 3 and $\mathrm{span}\{e_6\}$ is an abelian ideal. Then the matrix of $Der(L)$ is

$$\begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 0 & 0 \\ -\frac{1}{2}a_{13} & a_{22} & 0 & 0 & 0 & 0 \\ -\frac{1}{2}a_{12} & 0 & -a_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & \frac{1}{2}a_{12} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}a_{13} & a_{22}+a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{pmatrix}$$

From this matrix we see that L has a non-trivial central derivation, but it is not contained in $R(L)$.

We consider the Leibniz algebras whose all inner derivations are central. We will prove the following

Theorem 3. *Let L be a Leibniz algebra. Then:*

(1) $R(L) \subset \mathrm{CDer}(L)$ if and only if $L^3 = \{0\}$.

(2) If $Z(L) \neq \{0\}$ and $R(L) = \mathrm{CDer}(L)$, then $L^2 = Z(L)$ and $\dim Z(L) = 1$.

(3) $Der(L) = \mathrm{CDer}(L)$ if and only if L is abelian.

Proof. (1): $R(L) \subset \mathrm{CDer}(L)$ if and only if $L^2 \subset Z(L)$ if and only if $L^3 = [[L, L]L] = \{0\}$.

(2): Now suppose that $R(L) = \mathrm{CDer}(L)$, then $L^2 \subset Z(L)$. If $L^2 \neq Z(L)$, then L is the direct sum of a non-zero central ideal Z_1 and an ideal L_1 containing L^2. The identity mapping of Z_1 can be trivially extended to the derivation of L which we denote by d. Then d is central, but non inner. This contradicts our assumption. Thus we see that $L^2 = Z(L)$. By the facts that

$$\dim R(L) = \dim L/\operatorname{Ann}_r(L) \ \ \textit{and} \ \ \dim \mathrm{CDer}(L) = \dim L/L^2 \times \dim Z(L),$$

$$\dim R(L) \leqslant \dim L/Z(L) \leqslant \dim L/Z(L) \times \dim Z(L) = \dim \mathrm{CDer}(L).$$

Since $R(L) = \mathrm{CDer}(L)$, we have $\dim Z(L) = 1$, i.e. there exists a subspace U of L of codimension 1. Hence $L = Z(L) + U$, $[U, U] = Z(L)$ and $\dim \mathrm{CDer}(L) = \dim U = \dim R(L)$, i.e. $U \cap \operatorname{Ann}_r(L) = (0)$ and $\operatorname{Ann}_r(L) = Z(L)$.

(3): Suppose that $Der(L) = \mathrm{CDer}(L)$. If $Z(L) = \{0\}$, we have $\mathrm{CDer}(L) = \{0\}$ and therefore $Der(L) = \{0\}$, hence $L = (0)$. Therefore, we may assume that $Z(L) \neq \{0\}$. Then by (1) it follows that $L^3 = \{0\}$. If $L^2 \neq \{0\}$, we take a subspace $U \neq \{0\}$ such that

$$L = U + L^2, \quad U \cap L^2 = \{0\}.$$

Define an endomorphism d of L in the following way:

$$dx = \frac{1}{2}x, \ \forall x \in U, \qquad dy = y, \ \forall y \in L^2.$$

Then d is a non-central derivation of L, which contradicts our assumption. Therefore $L^2 = \{0\}$. The converse is evident. □

Example 5. We consider a Leibniz algebra satisfying the condition in the statement (1) of Theorem 3. Let L be the nilpotent Leibniz algebra described in terms of a basis $\{e_1, e_2, e_3, e_4, e_5\}$ by the following table:

$$\begin{cases} [e_1, e_1] = e_4, & [e_2, e_1] = -e_3, \\ [e_1, e_2] = e_3, & [e_2, e_2] = e_5. \end{cases}$$

For this algebra $Z(L) = \mathrm{span}\{e_3, e_4, e_5\}$ and $L^3 = \{0\}$. Moreover, the matrix of $Der(L)$ is

$$\begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{11} + a_{22} & 0 & 0 \\ 0 & 0 & 0 & 2a_{11} & 0 \\ 0 & 0 & 0 & 0 & 2a_{22} \end{pmatrix}$$

and the matrices of an inner and central derivations are obtained

$$\begin{pmatrix} 0 & 0 & a_{13} & a_{14} & 0 \\ 0 & 0 & -a_{14} & 0 & a_{13} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore $R(L) \subset \mathrm{CDer}(L)$.

Remark 1. We will here note that the converse of the statement (2) of Theorem 3 is true for Lie algebras [18], but in the next example we will show that there exists a Leibniz algebra which does not satisfy the converse of the statement (2).

Example 6. Let L be the Leibniz algebra described in terms of a basis $\{e_1, e_2, e_3, e_4, e_5\}$ by the following multiplication table:

$$\begin{cases} [e_1, e_3] = e_5, \\ [e_2, e_4] = e_5, \\ [e_3, e_1] = -e_5. \end{cases}$$

In this algebra $L^2 = Z(L) = \mathrm{span}\{e_5\}$. Moreover, see that the matrix of derivations of L :

$$\begin{pmatrix} a_{11} & 0 & a_{13} & 0 & a_{15} \\ 0 & a_{22} & 0 & 0 & a_{25} \\ a_{31} & 0 & a_{33} & 0 & a_{35} \\ 0 & 0 & 0 & a_{11} - a_{22} + a_{33} & a_{45} \\ 0 & 0 & 0 & 0 & a_{11} + a_{33} \end{pmatrix}$$

From this the matrix of an inner derivation of L is obtained

$$\begin{pmatrix} a_{11} & 0 & a_{11} & 0 & a_{15} \\ 0 & a_{11} & 0 & 0 & a_{25} \\ a_{11} & 0 & a_{11} & 0 & a_{35} \\ 0 & 0 & 0 & a_{11} & a_{11} \\ 0 & 0 & 0 & 0 & 2a_{11} \end{pmatrix}$$

and the matrix of a central derivation is obtained

$$\begin{pmatrix} a_{11} & 0 & a_{11} & 0 & a_{15} \\ 0 & a_{11} & 0 & 0 & a_{25} \\ a_{11} & 0 & a_{11} & 0 & a_{35} \\ 0 & 0 & 0 & a_{11} & a_{45} \\ 0 & 0 & 0 & 0 & 2a_{11} \end{pmatrix}$$

Thus $R(L)$ is 3-dimensional, and $\mathrm{CDer}(L)$ is 4-dimensional, it is easily shown that $R(L) \subsetneq \mathrm{CDer}(L)$.

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