© Algebra and Discrete Mathematics
Volume **35** (2023). Number 2, pp. 200–219
DOI:10.12958/adm1886 RESEARCH ARTICLE

Orbit isomorphic skeleton groups*

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Communicated by I. Ya. Subbotin

Abstract. Recent development in the classification of p-groups often concentrate on the coclass graph $\mathcal{G}(p,r)$ associated with the finite p-groups coclass r, specially on periodicity results on these graphs. In particular, the structure of the subgraph induced by 'skeleton groups' is of notable interest. Given their importance, in this paper, we investigate periodicity results of skeleton groups. Our results concentrate on the skeleton groups in $\mathcal{G}(p,1)$. We find a family of skeleton groups in $\mathcal{G}(7,1)$ whose 6-step parent is not a periodic parent. This shows that the periodicity results available in the current literature for primes $p \equiv 5 \bmod 6$ do not hold for the primes $p \equiv 1 \bmod 6$. We also improve a known periodicity result in a special case of skeleton groups.

1. Introduction

Classification of p-groups is one of the main themes in group theory. Since a classification of p-groups by order p^n seems out of reach for large n, other invariants of groups have been used to attempt a classification; a particularly intriguing invariant is *coclass*. A finite p-group of order p^n and nilpotency class c has coclass $r = n - c$. The investigation of p-groups by coclass has been initiated by Leedham-Green & Newman [14] and recent work in coclass theory is often concerned with the study of the *coclass graph* $\mathcal{G}(p,r)$ associated with the finite p-groups of coclass r. The vertices

*This work forms part of the author's PhD thesis at Monash University, supported by an Australian Government Research Training Program (RTP) Scholarship.

2020 MSC: 20D15, 20D99.

Key words and phrases: finite groups, p-groups, coclass.

of the coclass graph $\mathcal{G}(p,r)$ are (isomorphism type representatives of) the finite p-groups of coclass r, and there is an edge $G \to H$ if and only if G is isomorphic to $H/\Gamma(H)$ where $\Gamma(H)$ is the last non-trivial term of the lower central series of H. One beauty of recent developments in coclass theory is the interplay between the intricate structure of p-groups and graphical visualisations in the form of coclass graphs $\mathcal{G}(p,r)$. The explicit computations of parts of these graphs have revealed surprising patterns, which in some cases have been proved on a group-theoretic level. For example, it is known that $\mathcal{G}(p,r)$ can be partitioned into finitely many groups and finitely many infinite trees (*coclass trees*), each having exactly one infinite path (*mainline*) starting at its root. It is a deep result that there is a one-to-one correspondence between the mainlines in $\mathcal{G}(p,r)$ and the isomorphism types of *infinite pro-p-groups of coclass* r, a brief discussion on these groups is given in Section 2.2. Since coclass trees are building blocks of a coclass graph, one aim of coclass theory is to study these trees. Many computer experiments suggest that significant parts of these trees exhibit periodic patterns; a good deal of current research, for example [1–3], is now concentrated on studying periodicity results. The first major periodicity result (known as periodicity of type I) was independently proved by du Sautoy [6] and Eick & Leedham-Green [8]. According to this, if $\mathcal{T}$ is a coclass tree with branches $\mathcal{B}_1, \mathcal{B}_2, \ldots$ then for any fixed k and all large enough n, there is a graph isomorphism $\mathcal{B}_n[k] \to \mathcal{B}_{n+d}[k]$ where d is the dimension of infinite prop-p-group associated with $\mathcal{T}$ and $\mathcal{B}_n[k]$ is the shaved subtree of $\mathcal{B}_n$ induced by the all groups at a distance at most k from the root of $\mathcal{B}_n$. However, [8, Remark 4] explains why, in addition to periodicity of type I, it is necessary to consider unbounded growth of the branches. Recently it has become apparent that a feasible approach is to first focus on so-called *skeleton groups* since almost every group in a class tree $\mathcal{T}$ has bounded (in terms of p and r) distance to a skeleton group, a detailed description can be found in [4]. Below we briefly elaborate this and explain why skeleton groups are important. The content of the next section is based on [4, Section 1.1].

1.1. Why skeleton groups?

For a fixed coclass tree, the infinitely many groups on its mainline can be recognised as the nilpotent quotients of an associated infinite pro-p group of coclass r; see Section 2.2. In a special case these pro-p-groups are split extensions, that is they have the form $S = P \ltimes T$ where $T \cong \mathbb{Z}_p^d$ is a free p-adic $\mathbb{Z}_p$-module of rank d and P is a finite p-group acting uniserially on T. The central conjecture in coclass theory is that the graph $\mathcal{G}(p,r)$ can be described by a finite subgraph and certain graph periodicities. This conjecture is a theorem for $p=2$, but it remains open for odd primes; the main reason is that the coclass trees in $\mathcal{G}(p,r)$ grow drastically in width, something the results for $p=2$ are not able to cover. It is therefore necessary to investigate this growth in more detail, and it turns out that skeleton groups provide a meaningful approach for that. For odd p, let S be an infinite pro-p group defining the mainline of a coclass tree $\mathcal{T}$ in $\mathcal{G}(p,r)$; for simplicity suppose $S = P \ltimes T$ as above. Unlike other groups in $\mathcal{T}$, skeleton groups can informally be described as twisted finite quotients of S, where the twisting is induced by a $\mathbb{Z}_p P$-module homomorphism from $T \wedge T$ into T; more precisely, every skeleton group for S has the form $P \ltimes T_{\gamma,m}$ where $\gamma\colon T \wedge T \to T$ is a $\mathbb{Z}_p P$-module homomorphism and $T_{\gamma,m}$ is a finite quotient of T whose group operation is defined via γ; we give more details in Section 2.3. Importantly, every group in $\mathcal{T}$ has bounded distance to such a skeleton group, see [4]. This shows that the broad structure of each coclass tree $\mathcal{T}$, and hence the broad structure of $\mathcal{G}(p,r)$, is determined by the subtree(s) induced by all these groups. These make skeleton groups interesting and important to study. It is therefore desirable to understand skeleton groups in more detail, and to be able to construct them up to isomorphism. Ideally the isomorphisms between skeleton groups corresponding to an infinite pro-p-group S can be expected to be induced by the automorphisms of S but unfortunately there are many exceptions [7], see Remark 2.3. Isomorphism between two skeleton groups which is induced by an automorphism of the pro-p-group associated with the coclass tree is called an orbit isomorphism in [7].

1.2. Main results

Coming back to the periodicity results, it is shown in [8] that a shaved subtree of $\mathcal{B}_n$ of depth approximately $n/6d$ can be embedded into $\mathcal{B}_{n+d}$. However, this bound is significantly improved in the case of coclass 1 by Dietrich [2]. In this paper we demonstrate a similar improvement for a special class of skeleton groups where any isomorphism between two skeleton groups is induced by an automorphism of the pro-p-group associated with the coclass tree; such an isomorphism is called orbit isomorphism in [7]. We prove the following in Section 4.1.

Theorem 1.1. Let G be an infinite pro-p-group of coclass r corresponding to the coclass tree $\mathcal{T}_G$ in $\mathcal{G}(p,r)$ such that G is split and if two skeleton groups in $\mathcal{T}_G$ are isomorphic, then they are also orbit isomorphic. Then there exists an integer $d = d(\mathcal{T}_G)$ such that for all large enough j, we have $\mathcal{S}_j \cong \mathcal{S}_{j+d}[j-d]$ as rooted trees.

In order to investigate the periodicities deep in the branches of a coclass tree, one considers *periodic parents*, see [1, 3] for details. Let k be an integer and define the k-step descendant tree $\mathcal{D}_k(K)$ of a group K in the branch $\mathcal{B}_n$ of a coclass tree $\mathcal{T}$ as the subtree of $\mathcal{B}_n$ induced by the descendants of distance at most k from K. Computer experiments [1] suggest that for $r = 1$ and $p = 5, 11$, almost always the unique ancestor H at distance $p-1$ from K satisfies $\mathcal{D}_{p-1}(K) \cong \mathcal{D}_{p-1}(H)$. In [1], such a group H is called a periodic parent of G and a result of this flavour is proved in [1, Theorem 1.2] for $p \equiv 5 \bmod 6$. However, computer experiments for $p \geqslant 7$ also suggest that there are infinitely many groups (deep within the branches) for which the $(p-1)$-step parent is not a periodic parent. We find such an infinite family of groups in $\mathcal{G}(7,1)$ which shows that the statement [1, Theorem 1.2] can not be extended to the case $p \equiv 1 \bmod 6$. In particular, we prove the following in Section 4.2.

Theorem 1.2. In $\mathcal{G}(7,1)$, there is an infinite family of pairs of skeleton groups at different depths, one being the 6-step parent of the other, where both groups have a 7-group as automorphism group, but non-isomorphic descendent trees.

In general, the common observation is that for groups deep enough in a branch, the $(p-1)$-step parent often is a period parent, [1, Theorem 1.2] is one such example. But computer experiments also show that there are cases where this is not true, and the question is whether this is a general fact or only an anomaly. Theorem 1.2 shows that this is not an anomaly by relaxing the condition $p \equiv 5 \bmod 6$. Throughout this paper, $r \geqslant 1$ is an integer, p denotes an odd prime, $\mathbb{Q}_p$ is the field of p-adic rationals, and $\mathbb{Z}_p$ is the ring of p-adic integers.

2. Preliminaries

In this section, we introduce the notation and results required for the rest of the paper.

2.1. Number theory

The results of this section are from [15, Section II, Chapters 5 & 7]. Let θ be a primitive p^s-th root of unity over $\mathbb{Q}_p$ for some $s \geqslant 1$. The p^s-th local cyclotomic field $\mathbb{Q}_p(\theta)$ has degree $d_s = (p-1)p^{s-1}$ over $\mathbb{Q}_p$, a

$\mathbb{Q}_p$-basis $\{1, \theta, \dots, \theta^{d_s-1}\}$, and a cyclic Galois group of order d_s generated by an automorphism $\sigma_k : \theta \mapsto \theta^k$ for some $1 < k < p^s$ with $p \nmid k$ and k is a primitive root modulo p^s. The ring of integers $\mathbb{Z}_p[\theta]$ is a principal ideal domain with unique maximal ideal $\mathfrak{p} = (\kappa)$, where $\kappa = \theta - 1$. For $z \in \mathbb{N}$ define $\mathfrak{p}^z = (\kappa^z)$; note that each $[\mathfrak{p}^z : \mathfrak{p}^{z+1}] = p$. The group of units $\mathcal{U}_{p^s} = (\mathbb{Z}_p[\theta]^\star, \cdot)$ can be decomposed as $\mathcal{U}_{p^s} = \langle \omega \rangle \times (1 + \mathfrak{p})$ where $\omega \in \mathbb{Z}_p$ is a primitive $(p-1)$-th root of unity. For $j \geqslant 1$ write $\mathcal{U}_{p^s,j} = 1 + \mathfrak{p}^j$.

2.2. Infinite pro-p-groups of coclass r

The results mentioned in this section are from [5, Section 10] and [11, Sections 7 & 10]. A pro-p-group G with nilpotent quotients $G_j = G/\gamma_j(G)$ has coclass $\mathrm{cc}(G) = r$ if there is an integer t such that G_j is a finite p-group of coclass r for all $j \geqslant t$. Every infinite pro-p-group G of coclass r yields an infinite path in the coclass graph $\mathcal{G}(p, r)$; in the following, the label of the branch $\mathcal{B}_n$ of the corresponding coclass tree is usually chosen such that $\mathcal{B}_n$ has root G_n. The structure of such a pro-p-group G is well-understood, and in particular if G has trivial centre, then G is a *uniserial p-adic space group* of dimension d for some $d \geqslant 1$, that is, G is an extension of a characteristic subgroup $T = (\mathbb{Z}_p^d, +)$ by a finite p-group P which acts faithfully and uniserially on T. Recall that the action is uniserial if the series $T = T_0 > T_1 > T_2 \dots$ defined by $T_{i+1} = [T_i, P]$ satisfies $[T : T_i] = p^i$ for all i.

2.3. Skeleton groups

Unless mentioned otherwise, definitions and results mentioned in this section are from [4]. In this paper we only consider skeleton groups associated with an infinite pro-p-group $G = P.T$ which is split and has dimension d. These skeleton groups are often called *split skeleton groups*. Following [4, Section 3.1], we consider $T \wedge T$ as a $\mathbb{Z}_pP$-module. Every $\mathbb{Z}_pP$-homomorphism $\gamma \colon T \wedge T \to T$ has P-invariant image, hence $\gamma(T \wedge T) = T_j$ for some $j \geqslant 0$. Thus, if m satisfies $j \leqslant m \leqslant 2j - d$, then γ induces a well-defined surjective P-homomorphism $(T/T_j) \wedge (T/T_j) \to T_j/T_m$ which maps $(a + T_j \wedge b + T_j)$ to $\gamma(a \wedge b) + T_m$. We can use γ to define a new binary operation on T/T_m via $(a + T_m) \circ_\gamma (b + T_m) = a + b + \frac{1}{2}\gamma(a \wedge b) + T_m$. The group $T_{\gamma,m} = (T/T_m, \circ_\gamma)$ is of order p^m with central derived subgroup T_j/T_m.

Definition 2.1. Let $\gamma \colon T \wedge T \to T_j$ be a surjective $\mathbb{Z}_pP$-homomorphism and choose m such that $6d < j \leqslant m \leqslant 2j - d$. The split skeleton group defined by γ and m is $G_{\gamma,m} = P \ltimes T_{\gamma,m}$.

Since skeleton groups are abstractly defined via homomorphism, it is natural to ask when two such homomorphisms yield isomorphic skeleton groups. The following is [4, Lemma 4.1].

Lemma 2.2. Let $G_{\gamma,m}$ and $G_{\gamma',m}$ be two skeleton groups, and $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(\gamma(t \wedge s)) - \gamma'(\alpha(t) \wedge \alpha(s)) \in T_m$ for all $s, t \in T$. Then $G_{\gamma,m} \cong G_{\gamma',m}$.

Remark 2.3. *Isomorphisms between skeleton groups arising from automorphisms of G are called* orbit isomorphisms *in [7]. If there exists an orbit isomorphism between any two isomorphic skeleton groups, then the subgraph induced by the skeleton groups is essentially completely determined by the structure of the associated pro-p-group G, which is a favourable situation. In [4], two instances of such G have been exhibited where this holds. Unfortunately other isomorphisms (exceptional isomorphisms) can exist, see [7, p. 1249 & 1269]. However this can not happen if G is split and P is cyclic, see [4]. We use this result later in this paper.*

For the rest of this paper, we consider a split pro-p-group $G = P \ltimes T$ of dimension d and coclass r. Let $\mathcal{T}_G$ be the coclass tree in $\mathcal{G}(p,r)$ defined by G. The shaved branch $\mathcal{B}_j[k]$ is the subgraph of $\mathcal{B}_j$ consisting of the groups of depth at most k in $\mathcal{B}_j$. For any $j \geqslant 1$, we write $H_j = \mathrm{Hom}_P(T \wedge T, T_j)$ and $L_j = \{\gamma \in H_j \mid \gamma \text{ is surjective}\}$. Whenever considering a skeleton group $G_{\gamma,m}$ for $\gamma \in L_j$, we implicitly assume that all parameters are chosen appropriately, that is, if $\gamma(T \wedge T) = T_j$ then $j \leqslant m \leqslant 2j - d$. Recall from [4] that $G_{\gamma,m}$ lies at depth $m - j$ in $\mathcal{B}_j$, thus the skeleton groups in $\mathcal{B}_j$ induce a subgraph $\mathcal{S}_j$ of depth $2j - d$. By $\mathcal{S}_j[k]$ we denote the subgraph of $\mathcal{S}_j$ consisting of all skeleton groups in depth at most k for $k \leqslant 2j - d$. For the rest of this paper, a "skeleton group" we will always denote a split skeleton group.

3. Skeleton groups with cyclic point group

Motivated by the skeleton groups of maximal class, see [3], we consider p-adic uniserial space groups with cyclic point groups. It follows from [9, Lemma 11] that every such space group is split and uniquely determined, up to isomorphism, by the size of its point group; thus, the following convention covers the general case of space groups with cyclic point groups.

Notation 1. *For Section 3, we assume that $G = P \ltimes T$ is a split space group whose point group P is cyclic of order p^s, generated by g. If θ is* *a primitive p^s-th root of unity over $\mathbb{Q}_p$, then we can assume that $T = (\mathbb{Z}_p[\theta], +)$ whose uniserial series has terms $T_i = (\theta - 1)^i T = \mathfrak{p}^i$.*

The space group associated with the coclass tree of $\mathcal{G}(p, 1)$ is obtained by taking $s = 1$. This case has been investigated in [1–3]. The content of this section is motivated by the results in [1,3] and we generalise some of these results for $s \geqslant 1$ by following the methods used in [1,3].

3.1. Homomorphisms from $T \wedge T$

To get a better understanding of skeleton groups it is important to study the set of parametrising homomorphisms. Let $K = \mathbb{Q}_p(\theta)$ and recall from Section 2.1 that $\sigma_a \in \mathrm{Aut}(K)$ is defined by $\theta \mapsto \theta^a$, see Section 2.1. The following is [11, Theorem 11.4.1].

Theorem 3.1. For $a \not\equiv 0, 1 \bmod p$ define $\nu_a : K \wedge K \to K$ by $\nu_a(x \wedge y) = \sigma_a(x)\sigma_{1-a}(y) - \sigma_a(y)\sigma_{1-a}(x)$. Then $\{\nu_a \mid 2 \leqslant a \leqslant \frac{1}{2}(p^s - 1), a \not\equiv 0, 1 \bmod p\}$ is a K-basis of $\mathrm{Hom}_{\mathbb{Q}_p P}(K \wedge K, K)$.

The image of $T \wedge T$ under ν_a lies inside T, hence we can consider the restriction $\nu_a : T \wedge T \to T$ without any ambiguity. We now concentrate on the structure of the $\mathbb{Z}_p P$-module homomorphisms from $T \wedge T$ to T. The following theorem is from [11, Proposition 8.3.5]. For any integer $s \geqslant 1$ we write $m_s = \frac{1}{2}(p^s - 2p^{s-1} - 1)$.

Theorem 3.2. The $\mathbb{Z}_p P$-module $T \wedge T$ is the direct sum of a free $\mathbb{Z}_p P$-module of rank m_s

Lemma 3.3. The element $z = \sum_{0 \leqslant i < k < (p^s - 1)} \theta^i \wedge \theta^k$ is fixed under the action of P.

Proof. Using Theorem 3.2 we can write $z^g = \sum_{0 \leqslant i < k < (p^s - 1)} \theta^{i+1} \wedge \theta^{k+1}$ where g generates P. Now the result follows from a straightforward calculation using $\theta + \ldots + \theta^{p^s - 1} = -1$. □

As both $T \wedge T$ and T are $\mathbb{Z}_p$-modules of finite rank, it follows from [5, Chapter 1] that every homomorphism $T \wedge T \to T$ and $T \wedge T \to T/T_e$ (for any e) is a $\mathbb{Z}_p$-module homomorphism. Since the only fixed point of T under the action of P is 0, we define the following using Theorem 3.2 and Lemma 3.3. This follows an analogous definition in [11, Chapter 8]. We denote by $\delta_{i,k}$ the Kronecker delta with $\delta_{i,k} = 1$ if $i = k$ and $\delta_{i,k} = 0$ otherwise.

Definition 3.4. For $i, k \in \{p^{s-1}+1, \ldots, (p^s-1)/2\}$ and any $e > 0$, the P-homomorphisms $f_k : T \wedge T \to T$ and $\tilde{f}_k : T \wedge T \to T/T_e$ are defined by $f_k(1 \wedge \theta^i) = \delta_{i,k}$ and $f_k(z) = 0$, and $\tilde{f}_k = \pi \circ f_k$, where $\pi : T \to T/T_e$ is the projection. Let $\tilde{f}_1 : T \wedge T \to T/T_e$ be the P-homomorphism defined by $\tilde{f}_1(1 \wedge \theta^i) = 0$ and $\tilde{f}_1(z) = \hat{z}_e$.

Since $\mathbb{Z}_p[\theta]$ is abelian, $\mathrm{Hom}_P(T \wedge T, T)$ is a $\mathbb{Z}_p[\theta]$-module via $(f^c)(x) = cf(x)$ for all $c \in \mathbb{Z}_p[\theta]$, $x \in T \wedge T$ and $f \in \mathrm{Hom}_P(T \wedge T, T)$. The next result is an immediate corollary to Theorem 3.2.

Corollary 3.5. As $\mathbb{Z}_p[\theta]$-module, H_0 is generated by $\{f_k \mid p^{s-1}+1 \leqslant k \leqslant (p^s-1)/2\}$.

Corollary 3.6. $\mathrm{Hom}_P(T \wedge T, T/T_e)$ is a direct sum of m_s summands isomorphic to T/T_e, generated by $\tilde{f}_k$ for $p^{s-1}+1 \leqslant k \leqslant (p^s-1)/2$, and a summand of order p generated by $\tilde{f}_1$.

Proof. The image of z under any homomorphism in $\mathrm{Hom}_P(T \wedge T, T/T_e)$ must be in the subgroup generated by $\hat{z}_e = (\theta-1)^{e-1} + T_e$. Thus using Theorem 3.5 we find that $\mathrm{Hom}_P(T \wedge T, T/T_e)$ is the direct sum of the summands generated by $\tilde{f}_k$ for $p^{s-1}+1 \leqslant k \leqslant (p^s-1)/2$ and a summand generated by $\tilde{f}_1$. Finally, each of the subgroups generated by $\tilde{f}_k$ is isomorphic to T/T_e for $k \neq 2$ and the subgroup generated by $\tilde{f}_1$ is isomorphic to $T_{e-1}/T_e \cong C_p$. □

Consider $\mathcal{J}_{p,s} = [p^{s-1}+1, (p^s-1)/2] \cap \mathbb{Z}$ and $\mathcal{I}_{p,s} = \{a \in \mathbb{Z} \mid 2 \leqslant a \leqslant (p^s-1)/2, a \not\equiv 0, 1 \bmod p\}$. From Theorem 3.4 and Definition 3.1 we can see that there are two different bases of H_0 which are indexed over different sets of same size m_s, namely these bases are explicitly $\{\nu_k \mid k \in \mathcal{I}_{p,s}\}$ and $\{f_k \mid k \in \mathcal{J}_{p,s}\}$. The presence of two bases poses some notational difficulties; in order to reduce these technicalities, we relabel the ordered bases $(\nu_k)_{k \in \mathcal{I}_{p,s}}$ and $(f_k)_{k \in \mathcal{J}_{p,s}}$ as $(\overline{\nu}_k)_{k=1}^{m_s}$ and $(\overline{f}_k)_{k=1}^{m_s}$ respectively. The next result follows from Corollary 3.5 and Theorem 3.1. This is motivated by [3, Lemma 4.1].

Lemma 3.7. If $\gamma \in \mathrm{Hom}_P(T \wedge T, T)$ then

a) there exists a unique $(c_1, \ldots, c_{m_s}) \in T^{m_s}$ such that $\gamma = \sum_{a=1}^{m_s} c_a \overline{f}_a$,

b) there exists a unique $(b_1, \ldots, b_{m_s}) \in K^{m_s}$ such that $\gamma = \sum_{a=1}^{m_s} b_a \overline{\nu}_a$.

Remark 3.8. *It is shown in Lemma 3.7 that there exists an invertible matrix* $B \in \mathrm{GL}_{m_s}(K)$ *which represents the change of bases for* $\mathrm{Hom}_P(T \wedge$

T,T) from $(\overline{f}_k)_{k=1}^{m_s}$ to $(\overline{\nu}_k)_{k=1}^{m_s}$. For example if $s=1$, then [3, Section 4.1] shows that $B=(\beta_{ij})_{m_1\times m_1}$ is given by

$$\beta_{i,j}=\left(\theta^{i-2}-\theta^{3-i}\right)\left(\left(\theta^{i-2}-1\right)\left(\theta^{3-i}-1\right)\right)^{j-1}.$$

Recall that the isomorphism problem of skeleton groups can be solved via the action of $\mathrm{Aut}(G)$ on the set of homomorphisms. Our next aim is to investigate the automorphism group of the space group G.

3.2. The automorphism group

To determine the automorphism group of G, it is useful to exploit the extension structure of G. For this a cohomological argument can be used. For any group H and an H-module N, the corresponding groups of 1-cocycles, 1-coboundaries, and the first cohomology group are denoted by $Z^1(H,N), B^1(H,N)$ and $H^1(H,N)$ respectively; these are standard and can be found in [10, 16]. The next lemma describes the structure of $Z^1(P,T)$.

Lemma 3.9. $Z^1(P,T)=\{\alpha_t \mid t\in T\}$ where $\alpha_t: P\to T$ is defined as $\alpha_t(g^i)=\frac{\theta^i-1}{\theta-1}t$ for $i\geqslant 0$.

Proof. Suppose $\alpha\in Z^1(P,T)$ maps g to $t\in T$. By definition $\alpha(uv)=\alpha(u)^v+\alpha(v)$ for $u,v\in P$. Then inductively we can show that α maps g^i to $(1+\theta+\ldots+\theta^{i-1})t=\frac{\theta^i-1}{\theta-1}t$ for all $i\geqslant 0$. Hence $\alpha=\alpha_t$. Conversely take $t\in T$ and consider α_t. Then an easy calculation, using the definition of α_t, shows $\alpha_t(g^b)^{g^a}+\alpha_t(g^a)=\alpha_t(g^{a+b})$ for all $1\leqslant a,b\leqslant p^s-1$. □

The proof of the following theorem is motivated by [7, Lemma 5.4]. Recall from Section 2.1 that the Galois group G_θ of $\mathbb{Q}_p(\theta)|\mathbb{Q}_p$ is cyclic of order d_s and is generated by σ_k for a primitive root k modulo p^s.

Theorem 3.10. The automorphisms of G are $\phi(k,u,s): G\to G$ defined by

$$(g^i,t)\mapsto(g^{ik},u\sigma_k(t)+u_{ik}s),\qquad(0\leqslant i\leqslant p^s-1,\ t\in T)\tag{3.1}$$

where $k\in\{1,\ldots,d_s\}$ with $p\nmid k$, $s\in T, u\in\mathcal{U}_{p^s}$ and $u_j=\frac{\theta^j-1}{\theta-1}$ for all $j\geqslant 0$.

Proof. Since $G/T\cong P$ and T is characteristic in G, we can define a homomorphism $\lambda:\mathrm{Aut}(G)\to\mathrm{Aut}(P)$ mapping $\phi\mapsto\phi|_{G/T}$. Now σ_k induces an automorphism η_k of G mapping $(g^i,t)\mapsto(g^{ki},\sigma_k(t))$. Hence the subgroup $\langle \eta_k \rangle$ (of $\mathrm{Aut}(G)$) maps onto $\mathrm{Aut}(P)$ under λ. We now determine the kernel of λ. Consider a restriction map $\zeta : \mathrm{Aut}(G) \to \mathrm{Aut}(T)$ mapping $\phi \mapsto \phi|_T$. If $\phi \in \ker(\lambda)$ then $\phi|_T$ is a P-module automorphism of T. Now T acts on T by natural ring multiplication. So $\phi|_T$ is a T-module automorphism of T. But the group of T-module automorphisms of T is isomorphic to $\mathcal{U}_{p^s}$. Thus ζ maps $\ker(\lambda)$ onto $\mathcal{U}_{p^s}$; the surjectivity follows from the fact that multiplication by a unit induces an automorphism of G which lies in $\ker(\lambda)$. If $\phi \in \ker(\zeta) \cap \ker(\lambda)$ then $\phi(g^i, 0) = (g^i, \phi_2(g))$ for some $\phi_2 \in Z^1(P, T)$. Thus by Lemma 3.9 there is $s \in T$ such that $\phi_2(g^i) = \frac{\theta^i - 1}{\theta - 1} s$ for all $i \geqslant 0$. Each $x \in T$ induces an automorphism of G mapping $(g^i, t) \mapsto (g^i, t + \frac{(\theta^i - 1)}{\theta - 1} x)$. Finally the result follows since P is cyclic and every automorphism of P maps $g^i \mapsto g^{ik}$ for some $1 \leqslant k \leqslant d_s$ such that $p \nmid k$. $\square$

The following lemma is immediate from the proof of Theorem 3.10. This result is analogous to [3, Lemma 3.2 (a)].

Lemma 3.11. Let $\rho : \mathrm{Aut}(G) \to \mathrm{Aut}(T)$ be the natural restriction. Then the kernel of ρ is isomorphic to $Z^1(P, T)$ and the image of ρ is isomorphic to $G_\theta \ltimes \mathcal{U}_{p^s}$; a preimage of $(\sigma_z, u) \in G_\theta \ltimes \mathcal{U}_{p^s}$ under ρ is $\phi(z, u, 0)$.

3.3. Descendants of a skeleton group

In this section we describe the descendants of a skeleton group up to isomorphism. This is needed in order to investigate periodic parents in Section 4.2. We start with the following lemma; this follows directly from Lemma 2.2.

Lemma 3.12. Every $\phi \in \mathrm{Aut}(G)$ acts on $\gamma \in H_j$ via $\gamma \mapsto \gamma^\phi$, defined by $\gamma^\phi(t \wedge s) = \phi^{-1}(\gamma(\phi(t) \wedge \phi(s)))$. If γ is surjective, then so is γ^ϕ.

The action defined in Lemma 3.12 induces an action of $\mathrm{Aut}(G)$ on L_j. For $V \leqslant \mathrm{Aut}(G)$, we write $\mathrm{Stab}_V(\gamma + H_m) = \{\alpha \in V \mid \gamma^\alpha \equiv \gamma \bmod H_m\}$. Recall that G is split and the point group of G is cyclic. In this case, it is shown in [4, Proposition 5.2] that the condition given in Lemma 2.2 is a necessary sufficient condition and thus any isomorphism between skeleton groups in $\mathcal{T}_G$ is an orbit isomorphism. Now Lemma 2.2 can be rephrased as follows.

Lemma 3.13. Let $\gamma, \gamma' \in L_j$. Then $G_{\gamma,m} \cong G_{\gamma',m}$ if and only if there exists $\beta \in \mathrm{Aut}(G)$ such that $\gamma^\beta \equiv \gamma' \bmod H_m$.

The skeleton subgraph (in the coclass tree associated with G) in this case is completely determined by the structure of G: the ingredients for constructing skeleton groups are P, T, and homomorphisms $T \wedge T \to T$, and their isomorphism problem can be solved by considering the action of $\mathrm{Aut}(G)$. Since the parent of $G_{\gamma,m+1}$ is $G_{\gamma,m}$ for $\gamma \in L_j$, the following is easy to see from Lemma 3.13.

Lemma 3.14. Given $\gamma \in L_j$ and $j \leqslant m \leqslant 2j - d_s - 1$, a skeleton group $G_{\gamma',m+1}$ is an immediate descendant of $G_{\gamma,m}$ if and only if there exists an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $\gamma^\alpha \equiv \gamma' \bmod H_m$.

Noting that if $\gamma \in L_j$ then there is $s, t \in T$ such that $\gamma(t \wedge s) \in T_j \setminus T_{j+1}$, it follows from discussion in Section 2.3 that if $\gamma \in L_j$ and $\delta \in H_k$ for $k > j$ then $\gamma + \delta \in L_j$. The next lemma describes the descendants of a skeleton group.

Lemma 3.15. Let $j \geqslant 0$ and $\gamma \in L_j$. Suppose $j < m \leqslant 2j - d_s - 1$ and $1 \leqslant k \leqslant 2j - d_s - m$. Consider the skeleton group $G_{\gamma,m}$ at depth $e = m - j$ in $\mathcal{B}_j$. Then

a) A skeleton group H is a descendant of $G_{\gamma,m}$ of distance k if and only if $H \cong G_{\gamma+\delta,m+k}$ for some $\delta \in H_{j+e}$.

b) For $\delta_1, \delta_2 \in H_{j+e}$, two skeleton groups $G_{\gamma+\delta_1,j+e+k}$ and $G_{\gamma+\delta_2,j+e+k}$ are isomorphic if and only if there exists $\alpha \in \mathrm{Stab}_{\mathrm{Aut}(G)}(\gamma + H_{j+e})$ such that $\delta_1^\alpha + \gamma^\alpha - \gamma \equiv \delta_2 \bmod H_{j+e+k}$.

Proof. a) Note that $m = e + j$. Consider $\delta \in H_m$ and then $\gamma + \delta \in L_j$. Now $(\gamma + \delta) \equiv \gamma \bmod H_m$ since $\delta \in L_m$. Thus $(\gamma + \delta)^{\mathrm{id}} \equiv \gamma \bmod H_m$. So by Lemma 3.14, we conclude that $G_{\gamma+\delta,m+k}$ is a k-step descendant of $G_{\gamma,m}$. Conversely let $G_{\eta,m+k}$ be a k-step descendant of $G_{\gamma,m}$ for some $\eta \in L_j$. Then by Lemma 3.14, there is $\alpha \in \mathrm{Aut}(G)$ such that $\eta^\alpha \equiv \gamma \bmod T_m$. Thus by Lemma 3.13 we have $G_{\eta,m+k} \cong G_{\eta^\alpha,m+k}$ and hence $G_{\eta,m+k} \cong G_{\gamma+\delta,m+k}$.

b) Consider $G_{\gamma+\delta_1,m+k} \cong G_{\gamma+\delta_2,m+k}$. Now by Lemma 3.13 there is $\alpha \in \mathrm{Aut}(G)$ such that $(\gamma + \delta_1)^\alpha \equiv \gamma + \delta_2 \bmod H_{m+k}$ and hence $\delta_1^\alpha + \gamma^\alpha - \gamma \equiv \delta_2 \bmod H_{j+e+k}$. Now $\delta_1, \delta_2 \in L_m$ and $T_{m+k} \leqslant T_m$. So $\gamma^\alpha \equiv \gamma \bmod H_m$ which is same as saying $\alpha \in \mathrm{Stab}_{\mathrm{Aut}(G)}(\gamma + H_m)$. The converse is straightforward by using Lemma 3.13. □

Remark 3.16. Each $\gamma \in L_j$ can uniquely be written as $\gamma = (\theta - 1)^j F$ where $F \in L_0$. Hence every $F \in L_0$ induces a skeleton group $G_{(\theta-1)^j F,m}$ at depth $e = m - j$ in the branch $\mathcal{B}_j$ where $j \leqslant m \leqslant 2j - d_s$. Also note that multiplication by any unit induces an automorphism of G. Hence $\mathrm{Stab}_{\mathcal{U}_{p^s}}(\gamma + H_m) = \mathrm{Stab}_{\mathcal{U}_{p^s}}(F + H_{m-j})$.

Motivated by Lemma 3.15 and [2] we define the following.

Definition 3.17. For $\alpha \in \mathrm{Aut}(G)$, $F \in H_0$, $g \in H_n$ and $e \geqslant n \geqslant 0$ we write

$$(g + H_e)_\alpha = g^\alpha + (F^\alpha - F) + H_e. \tag{3.2}$$

Note that (3.2) defines an *affine action*; it is a group action if and only if $F^\alpha \equiv F \bmod H_e$. However, we have $(g + H_e)_{(\alpha\circ\beta)} = ((g + H_e)_\alpha)_\beta$ and $(g + H_e)_{\mathrm{id}} = (g + H_e)$.

Lemma 3.18. Suppose $\gamma \in L_j$. Choose m and k such that $j < m \leqslant 2j - d_s - 1$ and $1 \leqslant k \leqslant 2j - d_s - m$. Let $\mathfrak{M}_{\gamma,m,k}$ be the set of $\mathrm{Stab}_{\mathrm{Aut}(G)}(\gamma + H_m)$-representative of $\{g + H_{m+k} \mid g \in H_m\}$ under the affine action as in the Definition 3.17. Then the k-step descendants of $G_{\gamma,m}$, up to isomorphism, are given by $\{G_{\gamma+\eta,m+k} \mid \eta \in \mathfrak{M}_{\gamma,m,k}\}$.

Proof. By Lemma 3.15, the list of k-step descendants of $G_{\gamma,m}$ is given by $\{G_{\gamma+\delta,m+k} \mid \delta \in H_m\}$ and for $\delta_1, \delta_2 \in H_m$, two skeleton groups $G_{\gamma+\delta_1,m+k}$ and $G_{\gamma+\delta_2,m+k}$ from this list are isomorphic if and only if there exists $\alpha \in \mathrm{Stab}_{\mathrm{Aut}(G)}(\gamma + H_m)$ such that $\delta_1^\alpha + \gamma^\alpha - \gamma \equiv \delta_2 \bmod H_{j+e+k}$. By assumption $\gamma = (\theta - 1)^j F$ where $F \in L_0$ and hence $G_{\gamma+\delta_1,m+k}$ and $G_{\gamma+\delta_2,m+k}$ are isomorphic if and only if there exists $\alpha \in \mathrm{Stab}_{\mathrm{Aut}(G)}(\gamma + H_m)$ such that $(\delta_2 + H_m) \equiv \delta_1^\alpha + \gamma^\alpha - \gamma \bmod H_{m+k}$ which is equivalent saying $(\delta_2 + H_m)_\alpha \equiv \delta_1$ under the action defined in (3.2) for $\gamma \in L_j$. The claim follows. $\square$

3.4. Orbit isomorphisms

In this section we introduce few results involving orbit isomorphic skeleton groups. The results of this section will be used to prove Lemma 4.8. Recall that $m_s = \frac{1}{2}(p^s - 2p^{s-1} - 1)$ and the skeleton subgraph of the branch $\mathcal{B}_j$ is denoted by $\mathcal{S}_j$. Let B be the base change matrix as given in Remark 3.8. Now Lemma 3.7 allows us to define the following. Here $K = \mathbb{Q}_p$ and recall that $\mathcal{I}_{p,s} = \{a \in \mathbb{Z} \mid 2 \leqslant a \leqslant (p^s - 1)/2, a \not\equiv 0, 1 \bmod p\}$

Definition 3.19. If $\boldsymbol{c} = (c_1, \ldots, c_{m_s}) \in K^{m_s}$ and $\gamma = \sum_{a=1}^{m_s} c_a \overline{\nu}_a \in L_j$, then the skeleton group $G_{\gamma,m}$ defined by γ and m is denoted by $S_m(\boldsymbol{c})$.

Definition 3.19 shows that one can parametrise the skeleton groups by the elements of K^{m_s}. For $j \geqslant 1$, we define $\Theta_j = \{\boldsymbol{c} \in K^{m_s} \mid S_m(\boldsymbol{c}) \in \mathcal{S}_j$ for some m with $j \leqslant m \leqslant 2j - d_s\}$ and the write $\Omega_j = \{(c_1, \ldots, c_{m_s})B^{-1} \mid c_i \in \mathfrak{p}^j$ for $1 \leqslant i \leqslant m_s\}$. Next we consider the following homomorphisms defined for all $a \in \mathcal{I}_{p,s}$

$$\rho_a : \mathcal{U}_{p^s} \to \mathcal{U}_{p^s}, \quad u \mapsto u^{-1}\sigma_a(u)\sigma_{1-a}(u). \tag{3.3}$$

In the following we rewrite this action in terms of the parameters from Θ_j. This is [3, Lemma 4.3].

An element $(\sigma_n, u) \in G_\theta \ltimes \mathcal{U}_{p^s}$ acts on $\boldsymbol{c} = (c_1, \ldots, c_{m_s}) \in \Theta_j$ via

$$\boldsymbol{c}^{(\sigma_n,u)} = (\rho_1(u^{-1})\sigma_n(c_1), \ldots, \rho_{m_s}(u^{-1})\sigma_n(c_{m_s})) \tag{3.4}$$

This induces an action on Ω_j and hence on the set of cosets Ω_j/Ω_{j+e} for all $e \geqslant 1$. Now [4, Proposition 5.2] can be rephrased in terms of Θ_j and (3.4).

Theorem 3.20. Let $j > d_s$ and choose m such that $j \leqslant m \leqslant 2j - d_s$. Suppose $\boldsymbol{c}, \boldsymbol{b} \in \Theta_j$. Then $S_m(\boldsymbol{c})$ and $S_m(\boldsymbol{b})$ are isomorphic if and only if $\boldsymbol{c} + \Omega_m$ and $\boldsymbol{b} + \Omega_m$ lie in the same orbit under the action of $G_\theta \ltimes \mathcal{U}_{p^s}$ on Ω_j/Ω_m.

Proof. Let $\boldsymbol{c} = (c_1, \ldots, c_{m_s})$ and $\boldsymbol{b} = (b_1, \ldots, b_{m_s})$. Using Lemma 3.7, consider $\gamma = \sum_{a=1}^{m_s} c_a\overline{\nu}_a$ and $\gamma' = \sum_{a=1}^{m_s} b_a\overline{\nu}_a$. Lemma 3.13 shows that $S_m(\boldsymbol{c})$ and $S_m(\boldsymbol{b})$ are isomorphic if and only if there exists some automorphism ψ of G such that $(\gamma')^\psi \equiv \gamma \bmod H_m$. Now by Lemma 3.11 we can assume $\psi = \phi(n, u, 0)$ for some n with $p \nmid n$ and $u \in \mathcal{U}_{p^s}$. The action of $\phi(n, u, 0)$ corresponds to the action of $(\sigma_n, u) \in G_\theta \ltimes \mathcal{U}_{p^s}$ on Θ_j. Hence two skeleton groups $S_m(\boldsymbol{c})$ and $S_m(\boldsymbol{b})$ are isomorphic if and only if for all $t, s \in T$ we have that $\sum_{a=1}^{m_s}(b_a - \sigma_n(c_a)\rho_a(u^{-1}))\overline{\nu}_a(t \wedge s) \in T_m$. Let $\alpha_a = b_a - \sigma_n(c_a)\rho_a(u^{-1})$ for $1 \leqslant a \leqslant m_s$ and $(\alpha_1, \ldots, \alpha_{m_s})B^{-1} = (\beta_1, \ldots, \beta_{m_s})$. Now using Remark 3.8, the skeleton groups are isomorphic if and only if for all $t, s \in T$ we have $\sum_{a=1}^{m_s} \beta_a\overline{f}_a(t \wedge s) \in T_m$. Note that $(\alpha_1, \ldots, \alpha_{m_s})B^{-1} = \boldsymbol{b}B^{-1} - \boldsymbol{c}^{(\sigma_n,u)}B^{-1}$. Thus from Definition 3.4 we conclude that the skeleton groups are isomorphic if and only if $(\boldsymbol{b}B^{-1} - \boldsymbol{c}^{(\sigma_n,u)}B^{-1}) \in T_m$ and the claim follows. □

4. Periodicities in skeleton graph

4.1. Periodicity of type I

In this section we study how Lemma 3.13 improves some known periodicity results. We continue with an infinite pro-p-group G which corresponds to a coclass tree $\mathcal{T}_G$ in $\mathcal{G}(p, r)$. *For Section 4.1, we assume that G is chosen such that if two skeleton groups (defined in $\mathcal{T}_G$) are isomorphic, then they are also orbit isomorphic.* It is proved in [4] this holds whenever G has a cyclic point group; this includes the prominent example $\mathcal{G}(p, 1)$. Recall from Section 1 that the periodicity of type I was improved by Dietrich [2] for $\mathcal{G}(p, 1)$. This motivated us to establish the following which is a re-statement of Theorem 1.1.

Theorem 4.1. Under the assumption mentioned above, the following holds. For all large enough j, we have $\mathcal{S}_j \cong \mathcal{S}_{j+d}[j-d]$ as rooted trees; here d is the dimension of the associated space group.

Proof. From Definition 2.1 we find that for $6d < j \leqslant m \leqslant 2j-d$, a complete list of skeleton groups at depth $m-j$ in $\mathcal{B}_j$ is given by $S_{j,m} = \{G_{\gamma,m} \mid \gamma \in L_j\}$. Multiplication by p defines a bijection $L_j \to L_{j+d}$ and thus $S_{j+d,m+d} = \{G_{p\gamma,m+d} \mid \gamma \in L_j\}$. Clearly $(p\gamma)^\alpha = p(\gamma^\alpha)$ for $\alpha \in \mathrm{Aut}(G)$. In view of our assumption, we have $G_{\gamma,m} \cong G_{\gamma',m}$ if and only if $G_{p\gamma,m+d} \cong G_{p\gamma',m+d}$ using Lemmas 2.2 and 3.13. This proves the existence of a bijection between the isomorphism types of the skeleton groups at depth e in $\mathcal{B}_j$ and at depth e in $\mathcal{B}_{j+d}$, respectively, for all $e \leqslant j-d$. The parent of $G_{\gamma,m}$ in $\mathcal{B}_j$ is $G_{\gamma,m-1}$ for $m > j$; this also implies that the above bijection induces a graph isomorphism from $\mathcal{S}_j$ to $\mathcal{S}_{j+d}[j-d]$; recall that $\mathcal{S}_j$ has depth $j-d$. □

We now describe why Theorem 4.1 is a significant improvement over the periodicity of type I as described in [8]: it is shown in [8] that, for large enough j, one can embed $\mathcal{B}_j[e_j]$ into $\mathcal{B}_{j+d}$ where e_j is approximately $j/6d$. In contrast, Theorem 4.1 shows one can embed the *whole* skeleton tree $\mathcal{S}_j$ (of depth $j-d$) into $\mathcal{B}_{j+d}$, such that $\mathcal{S}_j \cong \mathcal{S}_{j+d}[j-d]$.

4.2. Periodicity of type II

For this section we will continue with Notation 1, that is $G = P \ltimes T$ is a split space group whose point group P is cyclic of order p^s with $s = 1$ for $\mathcal{G}(p, 1)$. This is studied in detail in [1–3, 11–13]. We are particularly interested in the descendants of a skeleton group. The case $p \equiv 5 \bmod 6$ is discussed in [1]. We here consider other primes. In view of Notation 1, here θ is a primitive p-th root of unity. Then $T = (\mathbb{Z}_p[\theta], +)$ has $\mathbb{Z}_p$-rank $d = d_s = p-1$. The associated space group with the coclass tree of $\mathcal{G}(p, 1)$ has point group P which is cyclic of order p, see [2]. Let $\mathcal{I}_{p,1} = \{2, \ldots, d/2\}$ as in Section 3.1 and denote $\mathcal{I} = \mathcal{I}_{p,1}$. It is also known from [1, section 5.1] that for $i \geqslant 2$, there exists $\mathbb{Z}_p$-module isomorphisms between T_i and

$\mathcal{U}_{p,i}$ which are defined by the usual power series of the exponential and logarithm mapping $\exp : T_i \to \mathcal{U}_{p,i}$ and $\log : \mathcal{U}_{p,i} \to T_i$ with $\exp^{-1} = \log$. For simplicity when the prime p is clear from the context, we denote $\mathcal{U}_p = \mathcal{U}$ and $\mathcal{U}_{p,i} = \mathcal{U}^{(i)}$ for all $i \geqslant 0$. Considering the homomorphism ρ_a from (3.3), we use log on the restriction of ρ_a to $\mathcal{U}^{(2)} = 1 + T_2$ to induce the $\mathbb{Q}_p(\theta)$-linear map $\tau_a : T_2 \to T_2, \quad z \mapsto -z + \sigma_a(z) + \sigma_{1-a}(z)$. We take $\omega \in \mathbb{Z}_p$ to be a primitive $(p-1)$-th root of unity. The following can be found in [1, 11].

Lemma 4.2. There exist $v_3, \ldots, v_{p+1} \in T_1$ with $v_k \in T_{k-1} \setminus T_k$ for all k such that, for all $a \in \mathcal{I}$, the following holds. If $a \equiv \omega^i \bmod p$ and $1 - a \equiv \omega^j \bmod p$, then v_k is an eigenvector of τ_a with eigenvalue $\omega_{a,k} = \omega^{ik} + \omega^{jk} - 1$. The images of $v_3, \ldots, v_{p+1}$ under exp map generate $\mathcal{U}^{(2)}$ as a $\mathbb{Z}_p$-module. If $p \equiv 5 \bmod 6$, then $\omega_{a,k} \neq 0$ for all a and k. If $p \equiv 1 \bmod 6$, then $\omega_{a,k} = 0$ for some a and k.

So, for integers a and k, if $\omega_{a,k} \neq 0$ then there exists a largest integer $p_{a,k}$ with $\omega_{a,k} \equiv 0 \bmod p^{p_{a,k}}$. Let $e \geqslant 0$, then from [3, Section 5] we define $v_{a,k,e} = \max\{\lceil (e-k+1)/d \rceil - p_{a,k}, 0\}$. Now suppose $a \in \mathcal{I}$ and we define $N(a) = \{k \in \mathbb{Z} \mid 3 \leqslant k \leqslant p+1, \omega_{a,k} \neq 0\}$. We relax the condition $p \equiv 5 \bmod 6$ from [1, Lemma 5.3] and obtain the following.

Lemma 4.3. Let $a \in \mathcal{I}$. Suppose $u \in \mathcal{U}^{(2)}$ is such that

$$u = \prod_{k \in N(a)} \exp(v_k)^{a_k},$$

then $\rho_a(u) \in \mathcal{U}^{(e)}$ if and only if $p^{v_{a,k,e}}$ divides a_k for all $k \in N(a)$.

Proof. Recall that $\omega_{a,k}$ is the eigenvalue of τ_a corresponding to v_k. Then using log and exp, it is easy to observe that $\rho_a(u) = \prod_{k \in N(a)} \exp(v_k)^{a_k \omega_{a,k}}$. Note that $\omega_{a,k} \neq 0$ and $v_k \in T_{k-1} \setminus T_k$ for all $k \in N(a)$. Hence from the definition of $\omega_{a,k}$, we find that $\rho_a(u) \in \mathcal{U}^{(e)}$ if and only if $\exp(v_k)^{a_k \omega_{a,k}} \in \mathcal{U}^{(e)}$ for all $k \in N(a)$. Using log, this is equivalent to saying $a_k \omega_{a,k} v_k \in T_e$ for all $k \in N(a)$. As $pT_i = T_{i+d}$ we see $a_k \omega_{a,k} v_k \in T_e$ if and only if $a_k \omega_{a,k} \in T_{e-k+1}$ which is equivalent to saying $a_k \in T_{e-k+1-dp_{a,k}}$ for all $k \in N(a)$ since $p^{p_{a,k}}$ is the highest power of p which divides $\omega_{a,k}$. This is true if and only if $p^{v_{a,k,e}}$ divides a_k since $pT_i = T_{i+d}$. □

Recall the definition of ν_a from Theorem 3.1. The following result, from Lemma 3.7, describes the structure of $\mathrm{Hom}_P(T \wedge T)$. See also [1, Lemma 4.4].

Lemma 4.4. Every P-homomorphism $f : T \wedge T \to T$ can be written uniquely as $f = c_2\nu_2 + \ldots + c_{d/2}\nu_{d/2}$ with $c_2, \ldots, c_{d/2} \in T_{-(p-3)^2/4}$. If f is surjective, then $c_a \notin T$ for at least one $a \in \mathcal{I}$.

Periodic parents of skeleton groups in $\mathcal{G}(7, 1)$ Consider the coclass graph $\mathcal{G}(p, 1)$. We recall from Section 1, that for any integer k, the k-step descendant tree of any group H in $\mathcal{G}(p, 1)$ is denoted by $\mathcal{D}_k(H)$. The following conjecture states one of the possibly ways describe these trees, see [1]. For any $n \geqslant 0$, let $e_p(n) = n-2p+8$ if $p \geqslant 7$ and $e_5(n) = n-4$. For any group K in $\mathcal{G}(p, 1)$ at depth $e_p(n)$ in $\mathcal{B}_{n+p-1}$ if the unique ancestor H at distance $p-1$ from K satisfies $\mathcal{D}_{p-1}(K) \cong \mathcal{D}_{p-1}(H)$ then H is called a periodic parent of K. In the interest of finding such periodic parents, the following is proved in [1, Theorem 1.2].

Theorem 4.5. Let $p \equiv 5 \bmod 6$. There is an integer $n_0 = n_0(p)$ such that, for all $n \geqslant n_0$, the following holds. Let K be a group at depth $e_p(n)$ in $\mathcal{B}_{n+p-1}$ having immediate descendants and let H be the $(p-1)$-step parent of K. If the automorphism group of H is a p-group, then H is a periodic parent of K.

Our results in the this section show that Theorem 4.5 can not be extended for the case $p \equiv 1 \bmod 6$.

In the remainder of this section, let $p = 7$, that is, θ is a primitive 7-th root of unity and T is a $\mathbb{Z}_7$-module of dimension $d = 6$. The point group P is cyclic of order 7. Retaining the notation of Section 4.2, we take $\mathcal{I} = \{2, 3\}$ and ω is a 6-th root of unity; we choose $\omega \equiv 5 \bmod 7$. Then for $a = 2$ we have $a \equiv \omega^4 \bmod 7$ and $1 - a \equiv \omega^3 \bmod 7$. Similarly for $a = 3$ we get $a \equiv \omega^5 \bmod 7$ and $1 - a \equiv \omega \bmod 7$. Recall from Lemma 4.2 that if $a \equiv \omega^i \bmod p$ and $1 - a \equiv \omega^j \bmod p$, then v_k is an eigenvector of τ_a with eigenvalue $\omega_{a,k} = \omega^{ik} + \omega^{jk} - 1$ for $k \in \{3, \ldots, 8\}$. A straightforward computation shows that τ_2 has no zero eigenvalue whereas τ_3 has two zero eigenvalues for $k = 5, 7$. It is also easy to see that $p_{2,k} = 0$ if $k \neq 7$ and $p_{2,7} = 1$. Also $p_{3,k} = 0$ for $k \neq 5, 7$. We exploit the above facts in the following results. Recall from Section 2.1 that the group of units $\mathcal{U}$ can be decomposed as $\mathcal{U} = \langle\omega\rangle \times \langle\theta\rangle \times \mathcal{U}^{(2)}$ and each $u \in \mathcal{U}$ acts on $f = \sum_{a\in\mathcal{I}} c_a\nu_a \in H_0$ via $f^u = \sum_{a\in\mathcal{I}} \rho_a(u^{-1})c_a\nu_a$. Note that if $a \in \mathcal{I}$ then $\rho_a(\theta) = 1$ and $\rho_a(\omega) = \omega$.

Notation 2. Let $n \geqslant 1$ and $m \geqslant 0$. For the rest of this section we write $h_n = (\theta - 1)^n\nu_2 + (\theta - 1)^{-1}\nu_3$ and $S(n, m) = \mathrm{Stab}_{\mathcal{U}^{(2)}}(h_n + H_m)$.

We show that certain skeleton groups (parametrised by h_n) and their 6-step parents have non-isomorphic descendant trees. We first investigate the action of $\mathcal{U}^{(2)}$ on the homomorphisms from H_0 and prove the following which shows that [1, Theorem 5.1] is not true for $p = 7$. For any group A, we denote the group of p-th power by $A^{[p]}$.

Lemma 4.6. If $n \geqslant 1$ then $h_n \in L_0$ and $S(n, n+2)^{[p]} \neq S(n, n+2+d)$.

Proof. By Lemma 4.4 we see that $h_n \in L_0$. It is thus enough to find $u \in S(n, n+2+d)$ such that there does not exist any $v \in \mathcal{U}^{(2)}$ with $u = v^p$. We find integers x and y such that $u = \exp(v_7)^x \exp(v_5)^y$ is such an element of $S(n, n+2+d)$. Now $h_n^u - h = (\rho_2(u) - 1)(\theta - 1)^n \nu_2$ as $\rho_3(u) = 1$. By Lemma 4.4, if $(\rho_2(u) - 1) \in T_4$ then $u \in S(n, n+2+d)$. Lemma 4.3 gives $\rho_2(u) \in \mathcal{U}^{(4)}$ if and only if $p^{v_{2,7,4}}$ divides x and $p^{v_{2,5,4}}$ divides y. Recall that $v_{2,7,4} = v_{2,5,4} = 0$. So we choose x, y such that $7 \nmid x, y$. The images of $v_3, \ldots, v_8$ under exp map generate $\mathcal{U}^{(2)}$ as a $\mathbb{Z}_7$-module. So if there is $v \in \mathcal{U}^{(2)}$ such that $v^p = \exp(v_7)^x \exp(v_5)^y$ then p must divide both x and y. Hence by our choice of x and y, there does not exist any $v \in \mathcal{U}^{(2)}$ with $u = v^p$. This completes the proof. □

Corollary 4.7. Let $n \geqslant 1$ and $e = n + 2$. Then there exist $g \in H_{e+d}$ and $v \in S(n, e+d) \setminus S(n, e)^{[p]}$ such that $(g + H_{e+d+1})_v \neq (g + H_{e+d+1})_{u^p}$ for all $u \in S(n, e+d)$.

Proof. Take $g = (\theta - 1)^{e+d} h_n$. By the proof of Lemma 4.6, we can choose the element $v = \exp(v_7)^y \exp(v_5)^x$ with integers $x, y \geqslant 1$ not divisible by 7. Then $(g + H_{e+d+1})_v = (\theta - 1)^n (\alpha\rho_2(v^{-1}) - 1) + (\theta - 1)^{-1}(\alpha\rho_3(v^{-1}) - 1)$ by (3.2) where $\alpha = ((\theta - 1)^{e+d} + 1)$. Suppose, for a contradiction that $(g + H_{e+d+1})_v$ is equal to $(g + H_{e+d+1})_{u^p}$ for some $u \in S(n, e+d)$. Then we find that $\alpha((\theta - 1)^n(\rho_2(v^{-1}) - \rho_2(u^{-p})) + (\theta - 1)^{-1}(\rho_3(v^{-1}) - \rho_3(u^{-p}))) \in H_{e+d+1}$. Now Lemma 4.4 shows that $(\theta - 1)^{-9}(\rho_2(v^{-1}) - \rho_2(u^{-p})) \in T_{-4}$ as α is a unit. Hence $\rho_2(vu^{-p}) \in \mathcal{U}^{(5)}$. Write $u = \prod_{k=3}^{8} \exp(v_k)^{a_k}$. Now $v_{2,5,5} = 1$ and thus Lemma 4.3 shows that 7 divides $x - 7a_5$ which is a contradiction. □

We now find a family of skeleton groups in $\mathcal{G}(7, 1)$ whose automorphism groups are 7-groups.

Lemma 4.8. Let $n = 3 + 6z$ with $z \geqslant 1$.Then the automorphism group of $G_{(\theta-1)^j h_n, j+n+2}$ is a 7-group for $j \in (18 + 6\mathbb{Z}) \setminus (15 + 42\mathbb{Z})$.

Proof. Recall the base change matrix $B = (\beta_{ij})_{2\times 2}$ from Remark 3.8. In particular $\beta_{1,2} = (\theta^3 - 1)u_1$ and $\beta_{2,2} = (\theta^3 - 1)u_2$ where $u_1 = \theta + 2\theta^2 + 2\theta^3 + 2\theta^4 + \theta^5$ and $u_2 = \theta + 3\theta^2 + 4\theta^3 + 3\theta^4 + \theta^5$ are both units. We now take $c_1 = (\theta-1)^n$, $c_2 = (\theta-1)^{-1}$, $q_1 = (\theta-1)^j c_1$, $q_2 = (\theta-1)^j c_2$. So if $f = q_1\nu_2 + q_2\nu_3$ then $H = G_{f,j+e}$ is a skeleton group at depth e in branch $\mathcal{B}_j$. Note that the Galois group of $\mathbb{Q}_7(\theta)$ is generated by σ_3 which has order 6. Also note that $\sigma_3^2 = \sigma_2$ has order 3 and $\sigma_3^3 = \sigma_6$ has order 2. Using [3, Section 5] we conclude that $\mathrm{Aut}(H)$ is a p-group if and only if there does not exist $w_i \in \mathcal{U}$ such that $(q_1,q_2)^{(\sigma_i,w_i)} \equiv (q_1,q_2) \bmod \Omega_{j+e}$ for all $i = 2,3,6$ where the action is as defined in (3.4). This is true if and only if $((q_1,q_2)^{(\sigma_i,w)} - (q_1,q_2))B \notin T_{j+e} \times T_{j+e}$ for all $w \in \mathcal{U}$ and $i = 2,3,6$; this can be observed using (3.4). Now we take $e = n+2$ and note that $(\theta^i - 1) = (\theta-1)z_i$ for $i = 2,3,6$ where $z_2 = 1+\theta$, $z_3 = 1+\theta+\theta^2$ and $z_6 = -\theta^6$. Following the definition of B, a straightforward computation shows that if $((q_1,q_2)^{(\sigma_i,w_i)} - (q_1,q_2))B \in T_{j+e} \times T_{j+e}$ for some $w_i \in \mathcal{U}$ then

$$
\begin{aligned}
&(\theta-1)^n\theta^2(1-\theta^4)(z_i^{j+n}\rho_2(w_i^{-1}) - 1) \\
&\qquad + (\theta-1)^{-1}\theta^3(1-\theta^2)(z_i^{j-1}\rho_3(w_i^{-1}) - 1) \in T_{n+2}
\end{aligned}
$$

for $i = 2,3,6$. This shows $(z_i^{j-1}\rho_3(w_i^{-1}) - 1) \in T_{n+1}$. If $\rho_3(w_i^{-1}) \in \mathcal{U}_1$ then $z_i^{j-1} - 1 = t' - tz^{j-1} \in T_1$ for some $t' \in T_{n+1}$. Note that $z_i^{18+6z-1} - 1 \notin T_1$ for all $z \geqslant 1$ and $i = 2,3$; this can be seen via binomial theorem. Further $z_6^{j-1} - 1 \notin T_1$ unless $j = 15 + 42k$ for some $k \geqslant 1$ as in such cases $z_6^{j-1} - 1 = 0$. Thus $w_i \notin \mathcal{U}_1$ for all $i = 2,3,6$ for $j \in (18+6\mathbb{Z}) \setminus (15+42\mathbb{Z})$. Finally if $\rho_3(w_i^{-1}) = 1 + s$ for $s \in T \setminus T_1$ then $z_i^{18+6z-1}(1+s) - 1 \notin T_1$ which shows there is no $w_i \in \mathcal{U}$ such that $(z_i^{j-1}\rho_3(w_i^{-1}) - 1) \in T_{n+1}$ and hence $\mathrm{Aut}(G_{f,j+e})$ is a p-group for $j \in (18+6\mathbb{Z}) \setminus (15+42\mathbb{Z})$. □

We finally find a family of skeleton groups in $\mathcal{G}(7,1)$ whose 6-step parents are *not* periodic parents. Let $F_{18+6z} = (\theta-1)^{18+6z}h_{3+6z}$ and $e(18+6z) = 5+6z$ for $z \geqslant 0$.

Theorem 4.9. Using the notation of the above paragraph, for any $j = 18+6z$ with $z \geqslant 0$, the skeleton groups $G_{F_j,j+e(j)}$ and $G_{F_j,j+e(j)+6}$ have different number of immediate descendants if $j \in (18+6\mathbb{Z}) \setminus (15+42\mathbb{Z})$.

Proof. Let $j = 18+6z$ and take $e = n+2$ where $n = 3+6z$. Take $\gamma = F_{18+6z}$ and $m = j+e$. Suppose $\mathfrak{M}_{h_n,m,1}$ be the set of $\mathrm{Stab}_{\mathrm{Aut}(G)}((\theta-1)^j h_n + H_m)$-orbit representatives of $\{g + H_{m+1} \mid g \in H_m\}$ under the action as in Definition 3.17. Then by Lemma 3.18, the immediate descendants, up to isomorphism, of the skeleton group $G_{\gamma,j+e}$ are described by the set $\{G_{\gamma+\eta,j+e+1} \mid \eta \in \mathfrak{M}_{h_n,m,1}\}$. By Lemma 4.8 we see that $\mathrm{Aut}(G_{\gamma,j+e})$ is a p-group unless $j \in (15+42\mathbb{Z})$. Hence by Remark 3.16, the immediate descendants, up to isomorphism, of the skeleton groups $G_{\gamma,j+e}$ and $G_{\gamma,j+e+6}$ are in one-one correspondence with $\mathfrak{M}'_{h_n,e,1}$ and $\mathfrak{M}'_{h_n,e+6,1}$ respectively where $\mathfrak{M}'_{h_n,e,1}$ is the set of $S(n,e)$-orbit representative of $\{g + H_{e+1} \mid g \in H_e\}$. Now take $v = \exp(v_7)^y \exp(v_5)^x$ for some integers $x, y \geqslant 1$ such that $7 \nmid x, y$. Let $g = (\theta - 1)^{e+d} h_n$. Suppose for a contradiction, $(p^{-1}g + H_{e+1})$ and $(p^{-1}g + H_{e+1})_v$ are in same orbit under the action of $S(n,e)$. Then using (3.2) we have $(p^{-1}g + H_{e+1})_u = (p^{-1}g + H_{e+1})_v$ for some $u \in S(n,e)$. A straightforward computation shows that $g^u - g^v \in H_{e+d+1}$ which yields $(\theta-1)^{n-1}(\rho_2(u^{-1}) - \rho_2(v^{-1}))\nu_2 + (\theta-1)^{-2}(\rho_3(u^{-1}) - \rho_3(v^{-1}))\nu_3 \in H_0$. Now Lemma 4.4 shows that $(\rho_2(u^{-1}) - \rho_2(v^{-1})) \in T_{-3+n}$ and $(\rho_3(u^{-1}) - \rho_3(v^{-1})) \in T_{-2}$. By choosing large p-power of u^{-1}, we can have (with abuse of notation) $(\rho_2(u^{-p}) - \rho_2(v^{-1})) \in T_{-3+n}$ and $(\rho_3(u^{-p}) - \rho_3(v^{-1})) \in T_{-2}$. Hence we have $(\theta - 1)^{n-1}(\rho_2(u^{-p}) - \rho_2(v^{-1})) \in T_{-4}$ and also we have $(\theta - 1)^{-2}(\rho_3(u^{-p}) - \rho_3(v^{-1})) \in T_{-4}$. So by Lemma 4.4 we find that, $(\theta-1)^{n-1}(\rho_2(u^{-p}) - \rho_2(v^{-1}))\nu_2 + (\theta-1)^{-2}(\rho_3(u^{-p}) - \rho_3(v^{-1}))\nu_3 \in H_0$. This means $(g + H_{e+d+1})_{u^p} = (g + H_{e+d+1})_v$ which is not possible by Corollary 4.7. This shows that $(p^{-1}g + H_{e+1})$ and $(p^{-1}g + H_{e+1})^v$ are never in the same orbit under the action of $S(n,e)$ whereas $(g + H_{e+d+1})$ and $(g + H_{e+d+1})^v$ are in same orbit under the action of $S(n, e+d)$ as $v \in S(n, e+d)$. Hence $|\mathfrak{M}'_{h_n,e,1}| \neq |\mathfrak{M}'_{h_n,e+d,1}|$. The result follows. $\square$

The proof of Theorem 1.2 now follows from Lemma 4.8 and Theorem 4.9. Theorem 1.2 shows that, in general, one cannot expect that the d-step parent of a group in $\mathcal{G}(p,1)$ always has an isomorphic descendant tree. Note from [1, Theorem 1.1], if $n = 18 + 6z$ for some $z \geqslant 1$ then the depth of $\mathcal{S}_n$ is $12 + 6z$. The examples given in Theorem 4.9 are the skeleton groups at depth $11 + 6z$ in the branch $\mathcal{B}_{18+6z}$ for $z \geqslant 1$. This shows that these examples are occurring deep in the branches.

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Received by the editors: 11.09.2021
and in final form 19.06.2023.