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Free abelian trioids

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Abstract. We construct a free abelian trioid and describe the least abelian congruence on a free trioid.

1. Introduction

Trioids first appeared in the paper of J.-L. Loday and M. O. Ronco [5] at the study of ternary planar trees. An algebraic system $(T, \dashv, \vdash, \perp)$ with three binary associative operations $\dashv$, $\vdash$, and $\perp$ is called *a trioid* if for all $x, y, z \in T$ the following conditions hold:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{T_1}$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{T_2}$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \tag{T_3}$$
$$(x \dashv y) \dashv z = x \dashv (y \perp z), \tag{T_4}$$
$$(x \perp y) \dashv z = x \perp (y \dashv z), \tag{T_5}$$
$$(x \dashv y) \perp z = x \perp (y \vdash z), \tag{T_6}$$
$$(x \vdash y) \perp z = x \vdash (y \perp z), \tag{T_7}$$
$$(x \perp y) \vdash z = x \vdash (y \vdash z). \tag{T_8}$$

The notion of a trioid is a basis of the notion of a trialgebra [5], besides trioids generalize dimonoids [6]. Recall that a nonempty set T equipped

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with two binary associative operations $\dashv$ and $\vdash$ satisfying axioms (T_1)–(T_3) is called a *dimonoid*. Dimonoids and related algebras have been studied in different papers (see, e.g., [7, 10–12, 16, 22, 23, 29]), and now dimonoids play a prominent role in problems from the theory of Leibniz algebras [4,8]. If operations of a trioid or a dimonoid coincide then it becomes a semigroup. Thus, trioids and dimonoids are an extension of semigroups.

The construction of a free trioid of rank 1 was defined in [5]. Later, it was shown that the free trioid of an arbitrary rank has the similar structure (see [20, 28]). More convenient isomorphic constructions of the free monogenic trioid and the free trioid of an arbitrary rank were proposed in [30] and [19], respectively. Besides, in [30] the endomorphism monoid of the free trioid of rank 1 was described (see also the case for dimonoids of rank 1 [31]). The structure of free commutative trioids and other relatively free trioids was presented in [15, 18]. Certain congruences on free trioids were found in [14]. Abelian digroups and their examples appeared in [2]. Note that the idea of the notion of a digroup was proposed by J.-L. Loday [6]. A digroup is a dimonoid which satisfies some additional conditions (see, e.g., [17]). Abelian and symmetric generalized digroups, and free abelian monogenic digroups were considered in [9, 24]. Abelian dimonoids and the construction of the free abelian dimonoid of an arbitrary rank were described in [26]. Abelian doppelsemigroups and free objects in the variety of abelian doppelsemigroups were studied in [13]. Free abelian dibands and automorphisms of their endomorphism semigroups were investigated in [21, 25]. In this paper, we study the structure of free objects in the variety of abelian trioids.

The paper is organised as follows. In section 2, we define abelian trioids and give examples of such algebras. In section 3, we construct a free abelian trioid of an arbitrary rank and, in particular, consider a free abelian monogenic trioid. In section 4, we find the least congruence on the free trioid such that the corresponding quotient-trioid and the free abelian trioid are isomorphic.

2. Examples of abelian trioids

Following [2], a digroup $(D, \dashv, \vdash)$ is called *abelian* if for all $x, y \in D$,

$$x \vdash y = y \dashv x.$$

Abelianity was also considered in such classes of algebras as dimonoids, doppelsemigroups, generalized digroups and generalized dimonoids. It is quite natural to define the variety of abelian trioids.

Fix $*, \circ \in \{\dashv, \vdash, \perp\}$, where $* \neq \circ$. A trioid $(T, \dashv, \vdash, \perp)$ will be called $(*, \circ)$-*abelian* if for all $x, y \in T$,

$$x * y = y \circ x.$$

Obviously, a trioid $(T, \dashv, \vdash, \perp)$ is $(*, \circ)$-abelian if and only if it is $(\circ, *)$-abelian. It means that three classes of trioids appear, namely, $(\dashv, \vdash)$-abelian trioids, $(\dashv, \perp)$-abelian trioids and $(\vdash, \perp)$-abelian trioids. Clearly, the class of all $(*, \circ)$-abelian trioids forms a variety which does not coincide with the variety of commutative trioids [15]. A trioid which is free in the variety of $(*, \circ)$-abelian trioids will be called a *free* $(*, \circ)$-*abelian trioid*. In the present paper, we consider $(\dashv, \vdash)$-abelian trioids only and so refer to them as simply *abelian trioids*.

Remark 1. We note that if in a trioid $(T, \dashv, \vdash, \perp)$ the condition $x * y = y \circ x$ holds for all $*, \circ \in \{\dashv, \vdash, \perp\}$ with $* \neq \circ$ and $x, y \in T$, then operations of such trioid obviously coincide. If for a trioid $(T, \dashv, \vdash, \perp)$ any two from the following identities

$$\text{(i) } x \dashv y = y \vdash x, \qquad \text{(ii) } x \vdash y = y \perp x, \qquad \text{(iii) } x \perp y = y \dashv x$$

hold, then two suitable operations of this trioid coincide. For example, $\dashv = \perp$ if conditions (i) and (ii) hold.

Remark 2. We observe that normal forms of elements of free $(\dashv, \vdash)$-abelian trioids $(T, \dashv, \vdash, \perp)$ with a commutative operation $\perp$ were presented in [3].

Let $(S, \circ)$ be an arbitrary semigroup. A semigroup $(S, *)$ is called a *dual semigroup* to $(S, \circ)$ if $x * y = y \circ x$ for all $x, y \in S$. A semigroup $(S, \circ)$ is called *left commutative* (respectively, *right commutative*) if it satisfies the identity $x \circ y \circ a = y \circ x \circ a$ (respectively, $a \circ x \circ y = a \circ y \circ x$).

Proposition 1. *Let $(S, \circ)$ be an arbitrary right commutative semigroup and $(S, *)$ be a dual semigroup to $(S, \circ)$. Then algebras $(S, \circ, *, \circ)$ and $(S, \circ, *, *)$ are abelian trioids.*

Proof. It follows from Proposition 3 of [26] and the definition of a trioid. $\square$

If $(S, *)$ is a left commutative semigroup and $(S, \circ)$ is a dual semigroup to $(S, *)$, then algebras $(S, \circ, *, \circ)$ and $(S, \circ, *, *)$ are abelian trioids, too. It follows from Proposition 4 of [26] and the definition of a trioid.

Let $(S, \perp)$ be an arbitrary semigroup. We define two binary operations $\dashv$ and $\vdash$ on S in the following way

$$a \dashv b = a, \quad a \vdash b = b.$$

Proposition 2 ([19])**.** *The algebra $(S, \dashv, \vdash, \perp)$ is an abelian trioid.*

A dimonoid $(D, \dashv, \vdash)$ is *abelian* [26] if the semigroup $(D, \vdash)$ is dual to $(D, \dashv)$. Different examples of abelian dimonoids and, in particular, abelian digroups, can be found, e.g., in [17, 26].

Proposition 3. *Let $(D, \dashv, \vdash)$ be an arbitrary abelian dimonoid. Then algebras $(D, \dashv, \vdash, \dashv)$ and $(D, \dashv, \vdash, \vdash)$ are abelian trioids.*

Proof. It is obvious. □

Moreover, trioids $(D, \dashv, \vdash, \dashv)$ and $(D, \dashv, \vdash, \vdash)$ from Proposition 3 are $(\vdash, \perp)$-abelian trioids with $\perp = \dashv$ and, respectively, $(\dashv, \perp)$-abelian trioids such that $\perp = \vdash$.

Let S be an arbitrary additive commutative semigroup, $S_1, S_2, \ldots, S_n$, $n \geqslant 2$, be subsemigroups of S, and $S_\alpha = S$ for some $\alpha \in \{1, 2, \ldots, n\}$. We denote by S^* the direct product $\prod_{i=1}^{n} S_i$ of semigroups S_j, $1 \leqslant j \leqslant n$. For all $s = (s_1, s_2, \ldots, s_n) \in S^*$ we put $s^+ = s_1 + s_2 + \cdots + s_n$.

Take arbitrary $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in S^*$ and define three binary operations $\dashv_\alpha$, $\vdash_\alpha$, and $\perp_\alpha$ on S^* by

$$\begin{aligned}
x \dashv_\alpha y &= (x_1, \ldots, x_\alpha + y^+, \ldots, x_n),\\
x \vdash_\alpha y &= (y_1, \ldots, y_\alpha + x^+, \ldots, y_n),\\
x \perp_\alpha y &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).
\end{aligned}$$

Observe that operations $\dashv_\alpha$ and $\vdash_\alpha$ first appeared in [26].

Proposition 4. *For every $\alpha \in \{1, 2, \ldots, n\}$ the algebra $(S^*, \dashv_\alpha, \vdash_\alpha, \perp_\alpha)$ is an abelian trioid.*

Proof. Similarly as in Proposition 2 of [26] one can show that the algebra $(S^*, \dashv_\alpha, \vdash_\alpha)$ is an abelian dimonoid. Obviously, $\perp_\alpha$ is associative. Now show that axioms $(T_4) - (T_8)$ hold.

Let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n), z = (z_1, z_2, \ldots, z_n) \in S^*$. Then

$$\begin{aligned}
(x \dashv_\alpha y) \dashv_\alpha z &= (x_1, \ldots, x_\alpha + y^+, \ldots, x_n) \dashv_\alpha (z_1, z_2, \ldots, z_n)\\
&= (x_1, \ldots, x_\alpha + y^+ + z^+, \ldots, x_n)\\
&= (x_1, x_2, \ldots, x_n) \dashv_\alpha (y_1 + z_1, y_2 + z_2, \ldots, y_n + z_n)\\
&= x \dashv_\alpha (y \perp_\alpha z),
\end{aligned}$$

$$\begin{aligned}(x \perp_\alpha y) \dashv_\alpha z &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \dashv_\alpha (z_1, z_2, \ldots, z_n)\\ &= (x_1 + y_1, \ldots, x_\alpha + y_\alpha + z^+, \ldots, x_n + y_n)\\ &= (x_1, x_2, \ldots, x_n) \perp_\alpha (y_1, \ldots, y_\alpha + z^+, \ldots, y_n)\\ &= x \perp_\alpha (y \dashv_\alpha z),\end{aligned}$$

$$\begin{aligned}(x \dashv_\alpha y) \perp_\alpha z &= (x_1, \ldots, x_\alpha + y^+, \ldots, x_n) \perp_\alpha (z_1, z_2, \ldots, z_n)\\ &= (x_1 + z_1, \ldots, x_\alpha + y^+ + z_\alpha, \ldots, x_n + z_n)\\ &= (x_1, x_2, \ldots, x_n) \perp_\alpha (z_1, \ldots, z_\alpha + y^+, \ldots, z_n)\\ &= x \perp_\alpha (y \vdash_\alpha z),\end{aligned}$$

$$\begin{aligned}(x \vdash_\alpha y) \perp_\alpha z &= (y_1, \ldots, y_\alpha + x^+, \ldots, y_n) \perp_\alpha (z_1, z_2, \ldots, z_n)\\ &= (y_1 + z_1, \ldots, y_\alpha + x^+ + z_\alpha, \ldots, y_n + z_n)\\ &= (x_1, x_2, \ldots, x_n) \vdash_\alpha (y_1 + z_1, y_2 + z_2, \ldots, y_n + z_n)\\ &= x \vdash_\alpha (y \perp_\alpha z),\end{aligned}$$

$$\begin{aligned}(x \perp_\alpha y) \vdash_\alpha z &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \vdash_\alpha (z_1, z_2, \ldots, z_n)\\ &= (z_1, \ldots, z_\alpha + x^+ + y^+, \ldots, z_n)\\ &= (x_1, x_2, \ldots, x_n) \vdash_\alpha (z_1, \ldots, z_\alpha + y^+, \ldots, z_n)\\ &= x \vdash_\alpha (y \vdash_\alpha z).\end{aligned}$$

Thus, $(S^*, \dashv_\alpha, \vdash_\alpha, \perp_\alpha)$ is an abelian trioid. □

Let $(S, *)$ be an arbitrary semigroup and let $(T, \cdot)$ be a commutative semigroup such that there exists a homomorphism $\xi : S \to T$. Define three binary operations $\dashv$, $\vdash$, and $\perp$ on the direct product $S \times T$ by

$$\begin{aligned}(a, b) \dashv (c, d) &= (a, b \cdot (c\xi) \cdot d),\\ (a, b) \vdash (c, d) &= (c, d \cdot (a\xi) \cdot b),\\ (a, b) \perp (c, d) &= (a * c, b \cdot d).\end{aligned}$$

Proposition 5. *The algebra $(S \times T, \dashv, \vdash, \perp)$ is an abelian trioid.*

Proof. It is clear that operations $\dashv$, $\vdash$, and $\perp$ are associative, besides the semigroup $(S \times T, \vdash)$ is dual to $(S \times T, \dashv)$.

Let (a, b), (c, d), $(e, f) \in S \times T$. Then

$$\begin{aligned}((a, b) \dashv (c, d)) \dashv (e, f) &= (a, b \cdot c\xi \cdot d) \dashv (e, f)\\ &= (a, b \cdot c\xi \cdot d \cdot e\xi \cdot f) = (a, b \cdot e\xi \cdot f \cdot c\xi \cdot d)\\ &= (a, b) \dashv (e, f \cdot c\xi \cdot d) = (a, b) \dashv ((c, d) \vdash (e, f)),\end{aligned}$$

$$\begin{aligned}((a,b)\vdash(c,d))\dashv(e,f)&=(c,d\cdot a\xi\cdot b)\dashv(e,f)\\&=(c,d\cdot a\xi\cdot b\cdot e\xi\cdot f)=(c,d\cdot e\xi\cdot f\cdot a\xi\cdot b)\\&=(a,b)\vdash(c,d\cdot e\xi\cdot f)=(a,b)\vdash((c,d)\dashv(e,f)),\end{aligned}$$

$$\begin{aligned}((a,b)\dashv(c,d))\vdash(e,f)&=(a,b\cdot c\xi\cdot d)\vdash(e,f)\\&=(e,f\cdot a\xi\cdot b\cdot c\xi\cdot d)=(e,f\cdot c\xi\cdot d\cdot a\xi\cdot b)\\&=(a,b)\vdash(e,f\cdot c\xi\cdot d)=(a,b)\vdash((c,d)\vdash(e,f)).\end{aligned}$$

Therefore, $(S\times T,\dashv,\vdash)$ is an abelian dimonoid. Since

$$\begin{aligned}((a,b)\dashv(c,d))\dashv(e,f)&=(a,b\cdot c\xi\cdot d\cdot e\xi\cdot f)\\&=(a,b\cdot(c*e)\xi\cdot d\cdot f)=(a,b)\dashv(c*e,d\cdot f)\\&=(a,b)\dashv((c,d)\perp(e,f)),\end{aligned}$$

$$\begin{aligned}((a,b)\perp(c,d))\vdash(e,f)&=(a*c,b\cdot d)\vdash(e,f)\\&=(e,f\cdot(a*c)\xi\cdot b\cdot d)=(e,f\cdot c\xi\cdot d\cdot a\xi\cdot b)\\&=(a,b)\vdash((c,d)\vdash(e,f)),\end{aligned}$$

axioms (T_4) and (T_8) hold. In addition,

$$\begin{aligned}((a,b)\perp(c,d))\dashv(e,f)&=(a*c,b\cdot d)\dashv(e,f)\\&=(a*c,b\cdot d\cdot e\xi\cdot f)=(a,b)\perp(c,d\cdot e\xi\cdot f)\\&=(a,b)\perp((c,d)\dashv(e,f)),\end{aligned}$$

$$\begin{aligned}((a,b)\dashv(c,d))\perp(e,f)&=(a*e,b\cdot c\xi\cdot d\cdot f)\\&=(a*e,b\cdot f\cdot c\xi\cdot d)=(a,b)\perp(e,f\cdot c\xi\cdot d)\\&=(a,b)\perp((c,d)\vdash(e,f)),\end{aligned}$$

$$\begin{aligned}((a,b)\vdash(c,d))\perp(e,f)&=(c,d\cdot a\xi\cdot b)\perp(e,f)=\\&=(c*e,d\cdot a\xi\cdot b\cdot f)=(a,b)\vdash(c*e,d\cdot f)\\&=(a,b)\vdash((c,d)\perp(e,f))\end{aligned}$$

which completes the verification of axioms $(T_5)-(T_7)$. □

The obtained abelian trioid $(S\times T,\dashv,\vdash,\perp)$ is denoted by $S\times T(\xi)$.

Note that for every element t of an arbitrary $(\dashv,\vdash)$-abelian trioid the degrees

$$t^n_\dashv=\underbrace{t\dashv t\dashv\cdots\dashv t}_{n}\quad\text{and}\quad t^n_\vdash=\underbrace{t\vdash t\vdash\cdots\vdash t}_{n}$$

coincide, therefore we will write t^n instead of $t^n_\dashv\ (=t^n_\vdash)$.

3. The free abelian trioid

Let X be an arbitrary nonempty set and let N be the set of all natural numbers. Denote by $\mathrm{F}(X)$ and $\mathrm{FCm}(X)$ the free semigroup on X and, respectively, the free commutative monoid on X with the identity ε. Words of $\mathrm{FCm}(X)$ we write as $w = w_1^{\alpha_1} w_2^{\alpha_2} \dots w_n^{\alpha_n}$, where $w_1, w_2, \dots, w_n \in X$ are pairwise distinct, and $\alpha_1, \alpha_2, \dots, \alpha_n \in N \cup \{0\}$. Here $w_i^0, 1 \leqslant i \leqslant n$, is the empty word ε and $w^1 = w$ for all $w \in X$.

We denote by $*$ the homomorphism $\mathrm{F}(X) \to \mathrm{FCm}(X) : w \mapsto w^*$ which induces the least commutative semigroup congruence on the free semigroup $\mathrm{F}(X)$ (see, e.g., [1]). Further we put

$$\mathrm{FAt}(X) = \mathrm{F}(X) \times \mathrm{FCm}(X)$$

and define three binary operations $\dashv$, $\vdash$, and $\perp$ on $\mathrm{FAt}(X)$ by

$$\begin{aligned}(u, v) \dashv (p, q) &= (u, vp^*q),\\ (u, v) \vdash (p, q) &= (p, qu^*v),\\ (u, v) \ \perp \ (p, q) &= (up, vq).\end{aligned}$$

Theorem 1. *The algebra* $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ *is the free abelian trioid.*

Proof. By Proposition 5, $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ is an abelian trioid. Let $(u, v) \in \mathrm{FAt}(X)$, where $u = u_1 u_2 \dots u_m$, $u_i \in X, 1 \leqslant i \leqslant m$, $v = v_1^{\alpha_1} v_2^{\alpha_2} \dots\ v_n^{\alpha_n}$, $v_j \in X, 1 \leqslant j \leqslant n$. Taking to account Theorem 1 of [26], we have the following canonical representation:

$$\begin{aligned}(u, v) = (u_1, \varepsilon) \ \perp \ (u_2, \varepsilon) \ \perp \ \dots \ \perp \ (u_m, \varepsilon)&\\ \dashv (v_1, \varepsilon)^{\alpha_1} \dashv (v_2, \varepsilon)^{\alpha_2} \dashv \dots \dashv (v_n, \varepsilon)^{\alpha_n}&\end{aligned}$$

which is unique up to an order of $(v_j, \varepsilon), 1 \leqslant j \leqslant n$. In addition, $\langle X \times \varepsilon \rangle = (\mathrm{FAt}(X), \dashv, \vdash, \perp)$.

Show that the trioid $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ is free abelian. Let $(T', \dashv', \vdash', \perp')$ be an arbitrary abelian trioid and let ξ be any mapping of $X \times \varepsilon$ into T'. We naturally extend ξ to a mapping Ξ of $\mathrm{FAt}(X)$ into T' using the canonical representation of elements of $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$, i.e.,

$$\begin{aligned}(u, v)\Xi = (u_1, \varepsilon)\xi \ \perp' \ (u_2, \varepsilon)\xi \ \perp' \ \dots \ \perp' \ (u_m, \varepsilon)\xi&\\ \dashv' ((v_1, \varepsilon)\xi)^{\alpha_1} \dashv' ((v_2, \varepsilon)\xi)^{\alpha_2} \dashv' \dots \dashv' ((v_n, \varepsilon)\xi)^{\alpha_n}&\end{aligned}$$

for any $(u, v) \in \mathrm{FAt}(X)$.

Prove that Ξ is a homomorphism of $(\mathrm{FAt}(X), \dashv)$ into $(T', \dashv')$. Take any $(u,v), (p,q) \in \mathrm{FAt}(X)$ such that $u = u_1u_2\dots u_m, v = v_1^{\alpha_1}v_2^{\alpha_2}\dots v_n^{\alpha_n}$, and $p = p_1p_2\dots p_s, q = q_1^{\beta_1}q_2^{\beta_2}\dots q_t^{\beta_t}$, and $p^* = p_{i_1}^{\gamma_1}p_{i_2}^{\gamma_2}\dots p_{i_r}^{\gamma_r}$. Then

$$\begin{aligned}
((u,v) \dashv (p,q))\Xi &= (u, vp^*q)\Xi \\
&= (u_1,\varepsilon)\xi \ \perp' \ \dots \ \perp' \ (u_m,\varepsilon)\xi \dashv' ((v_1,\varepsilon)\xi)^{\alpha_1} \dashv' \cdots \dashv' ((v_n,\varepsilon)\xi)^{\alpha_n} \\
&\quad \dashv' ((p_{i_1},\varepsilon)\xi)^{\gamma_1} \dashv' \cdots \dashv' ((p_{i_r},\varepsilon)\xi)^{\gamma_r} \dashv' ((q_1,\varepsilon)\xi)^{\beta_1} \\
&\quad \dashv' \cdots \dashv' ((q_t,\varepsilon)\xi)^{\beta_t}.
\end{aligned}$$

On the other hand, by the help of associativity of $\dashv'$ and $\perp'$, axioms (T_4) and (T_5), the induction by s, and the right commutativity of $(T', \dashv')$ which holds in any abelian trioid, we have

$$\begin{aligned}
&(u,v)\Xi \dashv' (p,q)\Xi \\
&\quad = [(u_1,\varepsilon)\xi \perp' \cdots \perp' (u_m,\varepsilon)\xi \dashv' ((v_1,\varepsilon)\xi)^{\alpha_1} \dashv' \cdots \dashv' ((v_n,\varepsilon)\xi)^{\alpha_n}] \\
&\qquad \dashv' [(p_1,\varepsilon)\xi \perp' \cdots \perp' (p_s,\varepsilon)\xi \dashv' ((q_1,\varepsilon)\xi)^{\beta_1} \dashv' \cdots \dashv' ((q_t,\varepsilon)\xi)^{\beta_t}] \\
&\quad = (u_1,\varepsilon)\xi \perp' \cdots \perp' (u_m,\varepsilon)\xi \dashv' ((v_1,\varepsilon)\xi)^{\alpha_1} \\
&\qquad \dashv' \cdots \dashv' ((v_n,\varepsilon)\xi)^{\alpha_n} \dashv' (p_1,\varepsilon)\xi \\
&\qquad \dashv' [(p_2,\varepsilon)\xi \perp' \cdots \perp' (p_s,\varepsilon)\xi \dashv' ((q_1,\varepsilon)\xi)^{\beta_1} \dashv' \cdots \dashv' ((q_t,\varepsilon)\xi)^{\beta_t}] \\
&\quad = \cdots = (u_1,\varepsilon)\xi \perp' \cdots \perp' (u_m,\varepsilon)\xi \dashv' ((v_1,\varepsilon)\xi)^{\alpha_1} \dashv' \cdots \dashv' ((v_n,\varepsilon)\xi)^{\alpha_n} \\
&\qquad \dashv' (p_1,\varepsilon)\xi \dashv' \cdots \dashv' (p_s,\varepsilon)\xi \dashv' ((q_1,\varepsilon)\xi)^{\beta_1} \dashv' \cdots \dashv' ((q_t,\varepsilon)\xi)^{\beta_t} \\
&\quad = (u_1,\varepsilon)\xi \perp' \cdots \perp' (u_m,\varepsilon)\xi \dashv' ((v_1,\varepsilon)\xi)^{\alpha_1} \dashv' \cdots \dashv' ((v_n,\varepsilon)\xi)^{\alpha_n} \\
&\qquad \dashv' ((p_{i_1},\varepsilon)\xi)^{\gamma_1} \dashv' \cdots \dashv' ((p_{i_r},\varepsilon)\xi)^{\gamma_r} \dashv' ((q_1,\varepsilon)\xi)^{\beta_1} \\
&\qquad \dashv' \cdots \dashv' ((q_t,\varepsilon)\xi)^{\beta_t} \\
&\quad = (u, vp^*q)\Xi.
\end{aligned}$$

Using the fact that trioids $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ and $(T', \dashv', \vdash', \perp')$ are abelian, we immediately obtain

$$\begin{aligned}
((u,v) \vdash (p,q))\Xi &= ((p,q) \dashv (u,v))\Xi \\
&= (p,q)\Xi \dashv' (u,v)\Xi = (u,v)\Xi \vdash' (p,q)\Xi.
\end{aligned}$$

Further, for convenience, we put

$$\begin{aligned}
a &= (u_1,\varepsilon)\xi \perp' \cdots \perp' (u_m,\varepsilon)\xi, & b &= ((v_1,\varepsilon)\xi)^{\alpha_1} \dashv' \cdots \dashv' ((v_n,\varepsilon)\xi)^{\alpha_n}, \\
c &= (p_1,\varepsilon)\xi \perp' \cdots \perp' (p_s,\varepsilon)\xi, & d &= ((q_1,\varepsilon)\xi)^{\beta_1} \dashv' \cdots \dashv' ((q_t,\varepsilon)\xi)^{\beta_t}.
\end{aligned}$$

By means associativity of $\dashv'$, abelianity of $(T', \dashv', \vdash', \perp')$ and axioms (T_5), (T_6) we have

$$
\begin{aligned}
&(u,v)\Xi \perp' (p,q)\Xi \\
&\quad = [\,(u_1,\varepsilon)\xi \perp' \cdots \perp' (u_m,\varepsilon)\xi \dashv' ((v_1,\varepsilon)\xi)^{\alpha_1} \dashv' \cdots \dashv' ((v_n,\varepsilon)\xi)^{\alpha_n}\,] \\
&\qquad\quad \perp' \; [\,(p_1,\varepsilon)\xi \perp' \cdots \perp' (p_s,\varepsilon)\xi \dashv' ((q_1,\varepsilon)\xi)^{\beta_1} \dashv' \cdots \dashv' ((q_t,\varepsilon)\xi)^{\beta_t}\,] \\
&\quad = (a \dashv' b) \perp' (c \dashv' d) = ((a \dashv' b) \perp' c) \dashv' d \\
&\quad = (a \perp' (b \vdash' c)) \dashv' d = a \perp' ((b \vdash' c) \dashv' d) \\
&\quad = a \perp' ((c \dashv' b) \dashv' d) = a \perp' (c \dashv' (b \dashv' d)) \\
&\quad = (a \perp' c) \dashv' (b \dashv' d) \\
&\quad = (u_1,\varepsilon)\xi \;\perp'\; \ldots \;\perp'\; (u_m,\varepsilon)\xi \;\perp'\; (p_1,\varepsilon)\xi \;\perp'\; \ldots \;\perp'\; (p_s,\varepsilon)\xi \\
&\qquad\quad \dashv' ((v_1,\varepsilon)\xi)^{\alpha_1} \dashv' \cdots \dashv' ((v_n,\varepsilon)\xi)^{\alpha_n} \dashv' ((q_1,\varepsilon)\xi)^{\beta_1} \\
&\qquad\quad \dashv' \cdots \dashv' ((q_t,\varepsilon)\xi)^{\beta_t} \\
&\quad = (up, vq)\Xi = ((u,v) \perp (p,q))\Xi.
\end{aligned}
$$

Thus, Ξ is a trioid homomorphism which completes the proof. □

The cardinality of a set X is a *rank* of the constructed free abelian trioid $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$.

Remark 3. From the construction of $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ it follows that the free abelian trioid is determined uniquely up to an isomorphism by rank. Hence the automorphism group of $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ is isomorphic to the symmetric group on X.

Remark 4. For convenient, we can define operations $\dashv$ and $\vdash$ of the free abelian trioid $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ without using the homomorphism $*$ (see, e.g., [27]).

Now we consider the structure of the free abelian trioid of rank 1.

Let $(N^0, +)$ be the additive semigroup of all non-negative integers. Clearly, $\eta : N \to N^0 : x \mapsto x$ is a monomorphism of the additive semigroup $(N, +)$ into $(N^0, +)$. By Proposition 5, $N \times N^0(\eta)$ is an abelian trioid. We denote operations of this trioid by $\dashv', \vdash'$, and $\perp'$, that is, $N \times N^0(\eta) = (N \times N^0, \dashv', \vdash', \perp')$.

Proposition 6. *The free abelian trioid* $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ *of rank* 1 *is isomorphic to the trioid* $N \times N^0(\eta)$.

Proof. Let $X = \{x\}$, then $\mathrm{FAt}(X) = \{(x^n, x^m) \mid n \in N, m \in N^0\}$. Define a mapping ψ of $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ into $N \times N^0(\eta)$ by

$$\psi : (x^n, x^m) \mapsto (n, m)$$

for any $(x^n, x^m) \in \mathrm{FAt}(X)$.

It is obvious that ψ is a bijection. In addition, for all (x^n, x^m), $(x^k, x^l) \in \mathrm{FAt}(X)$ we obtain

$$\begin{aligned}((x^n, x^m) \dashv (x^k, x^l))\psi &= (x^n, x^{m+k+l})\psi = (n, m+k+l)\\ &= (n, m) \dashv' (k, l) = (x^n, x^m)\psi \dashv' (x^k, x^l)\psi,\end{aligned}$$

$$\begin{aligned}((x^n, x^m) \perp (x^k, x^l))\psi &= (x^{n+k}, x^{m+l})\psi = (n+k, m+l)\\ &= (n, m) \perp' (k, l) = (x^n, x^m)\psi \perp' (x^k, x^l)\psi.\end{aligned}$$

Clearly, abelianity of $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ and $N \times N^0(\eta)$ implies that ψ is a homomorphism of $(\mathrm{FAt}(X), \vdash)$ to $(N \times N^0, \vdash')$. Thus, trioids $(\mathrm{FAt}(X), \dashv, \vdash, \perp), |X| = 1$, and $N \times N^0(\eta)$ are isomorphic. $\square$

4. The least abelian congruence on the free trioid

Let ρ be an equivalence relation on a trioid $(T, \dashv, \vdash, \perp)$ which is stable on the left and on the right with respect to each of operations $\dashv, \vdash, \perp$. In this case ρ is called a *congruence* on $(T, \dashv, \vdash, \perp)$. A congruence ρ on a trioid $(T, \dashv, \vdash, \perp)$ is called *abelian* if the quotient-trioid $(T, \dashv, \vdash, \perp)/\rho$ is abelian. If $f : T_1 \to T_2$ is a homomorphism of trioids, then the corresponding congruence on T_1 will be denoted by $\triangle_f$.

Let X be an arbitrary set, $\overline{X} = \{\overline{x} \mid x \in X\}$ and let $\mathrm{F}(X \cup \overline{X})$ be the free semigroup on $X \cup \overline{X}$. By $\mathrm{Ft}(X)$ we denote the subsemigroup of $\mathrm{F}(X \cup \overline{X})$ which consists of words containing at least one element of type $\overline{x}$. For every $w \in \mathrm{Ft}(X)$ we denote by $\widetilde{w}$ the word obtained from w by replacing each $\overline{x}, x \in X$, by x. *The length* of $\omega \in \mathrm{Ft}(X)$ is denoted by $l(\omega)$. For example, if $w = x\overline{x}x\overline{x}xy\overline{z}$ then $\widetilde{w} = xxxxxyz$ and $l(w) = 7$.

Define three binary operations on $\mathrm{Ft}(X)$ by

$$u \prec v = u\widetilde{v}, \qquad u \succ v = \widetilde{u}v, \qquad u \uparrow v = uv.$$

Proposition 7 ([28], Proposition 1)**.** *The algebra* $(\mathrm{Ft}(X), \prec, \succ, \uparrow)$ *is the free trioid of rank* $|X|$*.*

Elements of $\mathrm{Ft}(X)$ are called *words* and $\overline{X}$ is *the generating set* of the free trioid $(\mathrm{Ft}(X), \prec, \succ, \uparrow)$.

It is well-known (see, e.g., [30]) that each element $w \in \mathrm{Ft}(X)$ can be represented uniquely in the canonical form by one of the following ways:

$$w = (\overline{u_1^{(0)}} \succ \cdots \succ \overline{u_{k_0}^{(0)}}) \succ (\overline{u_1^{(1)}} \prec \cdots \prec \overline{u_{k_1}^{(1)}}) \uparrow \ldots \uparrow (\overline{u_1^{(j)}} \prec \cdots \prec \overline{u_{k_j}^{(j)}}),$$

where $u_l^{(i)} \in X, 1 \leqslant l \leqslant k_i$ for all $i \in \{0, 1, \ldots, j\}$, or

$$w = (\overline{u_1^{(1)}} \prec \cdots \prec \overline{u_{k_1}^{(1)}}) \uparrow (\overline{u_1^{(2)}} \prec \cdots \prec \overline{u_{k_2}^{(2)}}) \uparrow \ldots \uparrow (\overline{u_1^{(j)}} \prec \cdots \prec \overline{u_{k_j}^{(j)}}),$$

where $u_l^{(i)} \in X, 1 \leqslant l \leqslant k_i$ for all $i \in \{1, 2, \ldots, j\}$.

For every $w \in \mathrm{Ft}(X)$ of the canonical form above, we put $\Theta(w) = u_1^{(1)} u_1^{(2)} \ldots u_1^{(j)}$, and

$$\Omega(w) = u_1^{(0)} \ldots u_{k_0}^{(0)} u_2^{(1)} \ldots u_{k_1}^{(1)} u_2^{(2)} \ldots u_{k_2}^{(2)} \ldots u_{k_{j-1}}^{(j-1)} u_2^{(j)} \ldots u_{k_j}^{(j)}$$

or

$$\Omega(w) = u_2^{(1)} \ldots u_{k_1}^{(1)} u_2^{(2)} \ldots u_{k_2}^{(2)} \ldots u_{k_{j-1}}^{(j-1)} u_2^{(j)} \ldots u_{k_j}^{(j)}$$

if $w \neq \Theta(w)$, and $\Omega(w) = \varepsilon$ if $w = \Theta(w)$. Besides, we denote by $q_{\overline{x}}(w)$, $x \in X$, the quantity of all elements $\overline{x} \in \overline{X}$ that are included in the canonical form of w.

Now we can define a binary relation σ on $\mathrm{Ft}(X)$ as follows: u and v of $\mathrm{Ft}(X)$ are *σ-equivalent* if for all $x \in X$,

$$q_{\overline{x}}(u) = q_{\overline{x}}(v) \quad \text{and} \quad \Theta(u) = \Theta(v).$$

We note that $q_{\overline{x}}(u) = q_{\overline{x}}(v)$ for all $x \in X$ implies $l(u) = l(v)$.

For example, if $u = \overline{d}ab\overline{b}ac$ then the canonical form of u is the following representation: $u = (\overline{d} \prec \overline{a} \prec \overline{b}) \uparrow (\overline{b} \prec \overline{a} \prec \overline{c})$. In addition, $\Theta(u) = db$, $\Omega(u) = abac$, and $q_{\overline{a}}(u) = q_{\overline{b}}(u) = 2$, $q_{\overline{d}}(u) = q_{\overline{c}}(u) = 1$.

Theorem 2. *The binary relation σ is the least abelian congruence on the free trioid* $(\mathrm{Ft}(X), \prec, \succ, \uparrow)$.

Proof. It is not hard to see that σ is an equivalence relation on $\mathrm{Ft}(X)$. Take arbitrary $w \in \mathrm{Ft}(X)$ and $u, v \in \mathrm{Ft}(X)$ such that $u\sigma v$, i.e., $\Theta(u) = \Theta(v)$ and $q_{\overline{x}}(u) = q_{\overline{x}}(v)$ for all $x \in X$. Then

$$\begin{aligned} u \prec w &= u\widetilde{w}, & v \prec w &= v\widetilde{w}, \\ u \succ w &= \widetilde{u}w, & v \succ w &= \widetilde{v}w, \\ u \uparrow w &= uw, & v \uparrow w &= vw, \end{aligned}$$

Taking into account

$$\Theta(u\widetilde{w}) = \Theta(v\widetilde{w}), \ \ \Theta(\widetilde{u}w) = \Theta(\widetilde{v}w), \ \ \Theta(uw) = \Theta(vw)$$

and

$$q_{\overline{x}}(u\widetilde{w}) = q_{\overline{x}}(v\widetilde{w}), \ \ q_{\overline{x}}(\widetilde{u}w) = q_{\overline{x}}(\widetilde{v}w), \ \ q_{\overline{x}}(uw) = q_{\overline{x}}(vw)$$

for any $x \in X$, we have $(u * w)\sigma(v * w)$, where $* \in \{\prec, \succ, \uparrow\}$. Similarly, we can show that $(w * u)\sigma(w * v)$ for every $* \in \{\prec, \succ, \uparrow\}$. Therefore, σ is a congruence.

The direct check shows that $(u \prec v)\sigma(v \succ u)$ for all $u, v \in \mathrm{Ft}(X)$, consequently $(\mathrm{Ft}(X), \prec, \succ, \uparrow)/\sigma$ is an abelian trioid, i.e., σ is abelian.

Now we show that the quotient-trioid $(\mathrm{Ft}(X), \prec, \succ, \uparrow)/\sigma$ is isomorphic to the free abelian trioid $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ (see Theorem 1). Operations of $(\mathrm{Ft}(X), \prec, \succ, \uparrow)/\sigma$ are denoted by $\prec', \succ', \uparrow'$. An equivalence class of $(\mathrm{Ft}(X), \prec, \succ, \uparrow)/\sigma$ which contains w is denoted by $[w]$. Define a mapping φ of $(\mathrm{Ft}(X), \prec, \succ, \uparrow)/\sigma$ into $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ by

$$[w]\varphi = (\Theta(w), \Omega(w))$$

for any $w \in \mathrm{Ft}(X)$. In particular, $[w]\varphi = (\Theta(w), \varepsilon)$ for any $w \in \mathrm{Ft}(X)$ such that $w = \Theta(w)$. Clearly, φ is a bijection.

For all $[u], [v] \in (\mathrm{Ft}(X), \prec, \succ, \uparrow)/\sigma$, we have

$$\begin{aligned}([u] \prec' [v])\varphi = [u\widetilde{v}]\varphi &= (\Theta(u\widetilde{v}), \Omega(u\widetilde{v})) \\ &= (\Theta(u), \Omega(u)\tilde{v}) = (\Theta(u), \Omega(u)\Theta(v)\Omega(v)) \\ &= (\Theta(u), \Omega(u)) \dashv (\Theta(v), \Omega(v)) = [u]\varphi \dashv [v]\varphi.\end{aligned}$$

In addition, $(\mathrm{Ft}(X), \prec, \succ, \uparrow)/\sigma$ and $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ are abelian trioids, therefore

$$([u] \succ' [v])\varphi = ([v] \prec' [u])\varphi = [v]\varphi \dashv [u]\varphi = [u]\varphi \vdash [v]\varphi.$$

Finally, for all $[u], [v] \in (\mathrm{Ft}(X), \prec, \succ, \uparrow)/\sigma$ we obtain

$$\begin{aligned}([u] \uparrow' [v])\varphi = [uv]\varphi = (\Theta(uv), \Omega(uv)) &= (\Theta(u)\Theta(v), \Omega(u)\Omega(v)) \\ &= (\Theta(u), \Omega(u)) \perp (\Theta(v), \Omega(v)) = [u]\varphi \perp [v]\varphi.\end{aligned}$$

Thus, $(\mathrm{Ft}(X), \prec, \succ, \uparrow)/\sigma$ is free abelian and the composition $\eta^{\natural} \circ \varphi$, where $\eta^{\natural} : (\mathrm{Ft}(X), \prec, \succ, \uparrow) \to (\mathrm{Ft}(X), \prec, \succ, \uparrow)/\sigma$ is the natural homomorphism, is an epimorphism of $(\mathrm{Ft}(X), \prec, \succ, \uparrow)$ on $(\mathrm{FAt}(X), \dashv, \vdash, \perp)$ inducing the least abelian congruence on $(\mathrm{Ft}(X), \prec, \succ, \uparrow)$. From the definition of $\eta^{\natural} \circ \varphi$ it follows that $\triangle_{\eta^{\natural} \circ \varphi} = \sigma$. $\square$

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