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Kazhdan constants and isomorphic graph pairs*

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Abstract. Let G be a finite group, and let Γ be a subset of G. The Kazhdan constant of the pair (G, Γ) is defined to be the maximum distance we can guarantee that an arbitrary unit vector in an arbitrary nontrivial irreducible unitary representation space of G can be moved by some element of Γ. The Kazhdan constant relates to the expansion properties of the Cayley graph generated by G and Γ, and has been much studied in this context. Different pairs (G_1, Γ_1) and (G_2, Γ_2) may give rise to isomorphic Cayley graphs. In this paper, we investigate the question: To what extent is the Kazhdan constant a graph invariant? In other words, if the pairs yield isomorphic Cayley graphs, must the corresponding Kazhdan constants be equal? In our main theorem, we construct an infinite family of such pairs where the Kazhdan constants are unequal. Other relevant results are presented as well.

1. Introduction

Let G be a finite group, and let Γ be a subset of G. Let ρ be a unitary representation of G, that is, a homomorphism from G to $GL(V)$, where V is a complex vector space with a Hermitian inner product $\langle \cdot, \cdot \rangle$, such that $\langle v, w \rangle = \langle \rho(g)v, \rho(g)w \rangle$ for all $g \in G, v, w \in V$. We define $\kappa(G, \Gamma, \rho)$ by

$$\kappa(G, \Gamma, \rho) = \min_{v \in S^1(V)} \max_{\gamma \in \Gamma} ||\rho(\gamma)v - v||.$$

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Here $S^1(V)$ denotes the set of all unit vectors in V. We then define the *Kazhdan constant* of the pair (G, Γ) by

$$\kappa(G,\Gamma) = \min_{\rho \in NI(G)} \kappa(G,\Gamma,\rho),$$

where $NI(G)$ denotes the set of nontrivial irreducible unitary representations of G. We can think of the Kazhdan constant intuitively as follows: it is the maximum distance we can guarantee that an arbitrary unit vector in an arbitrary nontrivial irreducible unitary representation space of G can be moved by some element of Γ. As discussed in [7], for finite groups this version of the Kazhdan constant is well-defined, e.g., the various minima and maxima are achieved.

We say that a subset Γ of G is *symmetric* if $\gamma^{-1} \in \Gamma$ whenever $\gamma \in \Gamma$. For a symmetric subset Γ of G, we define the Cayley graph $\mathrm{Cay}(G, \Gamma)$ to be the graph whose vertex set is G, where there is an edge from x to y iff $y = x\gamma$ for some $\gamma \in \Gamma$.

The Kazhdan constant relates to the expansion properties of the Cayley graph generated by G and Γ and has been much studied in this context—see, for example, [6] for more on this. Explicitly computing the Kazhdan constant can be quite difficult; generally, lower bounds suffice. See [3] and [4] for some of the few cases in which exact values are known.

It may happen that for two different pairs (G_1, Γ_1) and (G_2, Γ_2), the corresponding Cayley graphs $\mathrm{Cay}(G_1, \Gamma_1)$ and $\mathrm{Cay}(G_2, \Gamma_2)$ are isomorphic, which we denote by $\mathrm{Cay}(G_1, \Gamma_1) \cong \mathrm{Cay}(G_2, \Gamma_2)$. We may well ask, is the following statement always true?

$$\text{If } \mathrm{Cay}(G_1,\Gamma_1) \cong \mathrm{Cay}(G_2,\Gamma_2), \quad \text{then } \kappa(G_1,\Gamma_1) = \kappa(G_2,\Gamma_2). \tag{1}$$

A simple example shows that in full generality, (1) fails. Namely, let $G_1 = \mathbb{Z}_4$, the group of integers modulo 4, and let $\Gamma_1 = \{1, -1\}$. Let G_2 be the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$, and let $\Gamma_2 = \{(1,0),(0,1)\}$. Then $\mathrm{Cay}(G_1, \Gamma_1)$ and $\mathrm{Cay}(G_2, \Gamma_2)$ are both 4-cycles. However, a straightforward computation shows that $\kappa(G_1, \Gamma_1) = \sqrt{2}$ but $\kappa(G_2, \Gamma_2) = 2$.

Finding other examples where (1) fails requires some effort, however. There is a trivial way to produce two pairs (G_1, Γ_1) and (G_2, Γ_2) such that $\mathrm{Cay}(G_1, \Gamma_1)$ and $\mathrm{Cay}(G_2, \Gamma_2)$ are isomorphic: Take G_1 and G_2 to be isomorphic groups with an isomorphism $\phi\colon G_1 \to G_2$, and let $\Gamma_2 = \phi(\Gamma_1)$. In this case we say that (G_1, Γ_1) and (G_2, Γ_2) are *GS isomorphic.* (Here "GS" stands for *group-subset.*) Such pairs produce isomorphic Cayley graphs; moreover, using standard facts from representation theory, one can show that (1) holds whenever (G_1, Γ_1) and (G_2, Γ_2) are GS isomorphic. But (1) holds in many non-trivial instances as well. For example, one can produce a cycle graph of even length $\geqslant 6$ as a Cayley graph either on a cyclic or on a dihedral group, but the corresponding Kazhdan constant is the same either way, as shown by Bacher and de la Harpe. (In Section 3, we provide an alternative proof of this fact.) A construction due to Elspas and Turner (disproving a conjecture of Àdàm) demonstrates the existence of isomorphic Cayley graphs on cyclic groups arising from Cayley non-isomorphic pairs. In Section 3, we show that for a natural generalization of this construction, again the Kazhdan constants are equal.

In Section 2, we state and prove our main theorem, in which we demonstrate that infinitely many pairs with isomorphic Cayley graphs but unequal Kazhdan constants exist. The main idea behind this construction is to suitably modify the Elspas-Turner graphs.

In Section 3, in addition to the other items mentioned above, we also remark that in cases where the two Cayley graphs are isomorphic, the corresponding Kazhdan constants cannot be too far apart from one another. Explicit upper and lower bounds are given.

We note that one can easily obtain isomorphic Cayley graphs by considering two finite groups G_1 and G_2 of the same order n, then taking $\mathrm{Cay}(G_1, G_1 \setminus e_1)$ and $\mathrm{Cay}(G_2, G_2 \setminus e_2)$, where e_1 and e_2 are the identity elements of G_1 and G_2, respectively; in both cases we get the complete graph K_n. More generally, for subgroups H_1 and H_2 of G_1 and G_2, respectively, each of order m, we have that $\mathrm{Cay}(G_1, G_1 \setminus H_1)$ and $\mathrm{Cay}(G_2, G_2 \setminus H_2)$ are each isomorphic to the complete n/m-partite graph $K_{m,\ldots,m}$. Such pairs may well be worthy of future study.

We say a group G has the *Cayley isomorphism property* to mean that for all symmetric subsets Γ_1, Γ_2 of G, if $\mathrm{Cay}(G_1, \Gamma_1) \cong \mathrm{Cay}(G_2, \Gamma_2)$, then (G_1, Γ_1) is GS isomorphic to (G_2, Γ_2). This property has been studied, for example, in [1], [2], [8]. Groups which lack the Cayley isomorphism property provide further examples that may warrant future investigation.

2. Examples where the Kazhdan constants are not equal

In this section, we present our main result: an exhibition of an infinite family of pairs with isomorphic Cayley graphs but unequal Kazhdan constants.

Theorem 2.1. Let t be an even integer with $t \geqslant 6$. Let

$$\Gamma_1 = \{\pm 1, \pm(2t-1)\} \cup \{\pm 2, \pm 4, \pm 6, \ldots, \pm t\}, \text{ and}$$

$$\Gamma_2 = \{\pm(t-1), \pm(t+1)\} \cup \{\pm 2, \pm 4, \pm 6, \ldots, \pm t\}.$$

Let $n = 4t$. Then $\mathrm{Cay}(\mathbb{Z}_n, \Gamma_1) \cong \mathrm{Cay}(\mathbb{Z}_n, \Gamma_2)$ but $\kappa(\mathbb{Z}_n, \Gamma_1) \neq \kappa(\mathbb{Z}_n, \Gamma_2)$.

Proof. The function $f\colon \mathbb{Z}_n \to \mathbb{Z}_n$ given by

$$f(a) = \begin{cases} a & \text{if } a \text{ is even} \\ a - \frac{n}{4} & \text{if } a \text{ is odd} \end{cases}$$

defines a graph isomorphism from $\mathrm{Cay}(\mathbb{Z}_n, \Gamma_1)$ to $\mathrm{Cay}(\mathbb{Z}_n, \Gamma_2)$. The key point is that f maps odd elements of Γ_1 to odd elements of Γ_2, while it fixes the even elements.

For any $k \in \mathbb{Z}$, define $|k|_n$ to be the unique element of $\{0, 1, 2, \ldots, n/2\}$ congruent to either k or $-k$ modulo n. For any symmetric subset Γ of $\mathbb{Z}_n$, we define

$$\alpha(\mathbb{Z}_n, \Gamma) := \min_{1 \leqslant j \leqslant n/2} \max_{\gamma \in \Gamma} |\gamma j|_n.$$

It follows that

$$\kappa(\mathbb{Z}_n, \Gamma_2) = 2 \sin\left(\frac{\pi \alpha(\mathbb{Z}_n, \Gamma_2)}{n}\right).$$

To establish that the Kazhdan constants are not equal, we first show that $\alpha(\mathbb{Z}_n, \Gamma_2) = t + 1$.

When $j = 1$, we have that $\max_{\gamma \in \Gamma_2} |\gamma j|_n = t+1$. The maximum is achieved when $\gamma = t + 1 = -(3t - 1)$.

We now show that for all j with $1 \leqslant j \leqslant 2t$, we have $\max_{\gamma \in \Gamma_2} |\gamma j|_n \geqslant t+1$ by producing, for each j, an element $\gamma \in \Gamma_2$ such that $|j\gamma|_n \geqslant t + 1$.

For $j \equiv 1 \pmod 4$ and $j \leqslant t$, we have $|j(t+1)|_n = t + j \geqslant t + 1$.

For $j \equiv 1 \pmod 4$ with $t \leqslant j < 2t$, we have $|j(-t-1)|_n = 3t - j \geqslant t+1$.

For $j \equiv 2 \pmod 4$, we have $|j(2t)|_n = 2t \geqslant t + 1$.

For $j \equiv 3 \pmod 4$ and $j \leqslant t$, we have that $|j(-t+1)|_n = t + j \geqslant t + 1$.

For $j \equiv 3 \pmod 4$ with $t \leqslant j < 2t$, we have $|j(t-1)|_n = 3t - j \geqslant t + 1$.

For $j \equiv 0 \pmod 4$ with $2 \leqslant j \leqslant t/2$, let $\gamma = 2\lceil \frac{t+1}{2j} \rceil$. Here $\lceil x \rceil$ denotes the ceiling function of x, that is, the smallest integer greater than or equal to x. Because $j \geqslant 2$ and $t \geqslant 6$, it follows that $\gamma \leqslant t$, so $\gamma \in \Gamma_2$. Moreover, $j\gamma \leqslant 2t$, so $|j\gamma|_n = j\gamma \geqslant t + 1$.

For $j \equiv 0 \pmod 4$ with $t/2 < j < 3t/2$, we have that $|j(2)|_n = 2j$ if $j \leqslant t$ and $|j(2)|_n = 4t - 2j$ if $j \geqslant 2t$. In either case, we have $|j(2)|_n \geqslant t+1$.

For $j \equiv 0 \pmod 4$ with $3t/2 < j \leqslant 2t$, we have that $|j(t+1)|_n = j \geqslant t + 1$.

We now show that $\alpha(\mathbb{Z}_n, \Gamma_1) > t+1$. To do so, for all j with $1 \leqslant j \leqslant 2t$, we produce, for each j, an element $\gamma \in \Gamma_1$ such that $|j\gamma|_n > t+1$.

When $j = 1$, take $\gamma = 2t - 1$.

When $2 \leqslant j < t/2$, take $\gamma = 2\lfloor \frac{t}{j} \rfloor$, where $\lfloor x \rfloor$ denotes the floor function of x, that is, the largest integer less than or equal to x.

When $j = t/2$, take $\gamma = 4$.

When $t/2 < j \leqslant t$, take $\gamma = 2$.

When $j = t + 1$, take $\gamma = 2$.

When $j > t + 2$, take $\gamma = 1$. □

As a consequence of Theorem 2.1, it follows that $(\mathbb{Z}_n, \Gamma_1)$ and $(\mathbb{Z}_n, \Gamma_2)$ are not GS isomorphic; for if they were, then their Kazhdan constants would be equal.

It would be interesting to use examples such as those in Theorem 2.1 to investigate how sharp the inequalities in Theorem 3.3 are.

3. Miscellaneous results

When two Cayley graphs are isomorphic, the two corresponding Kazhdan constants cannot be that far apart from one another. In Subsection 3.1, we make this precise by proving an inequality relating one of the Kazhdan constants to the other. This inequality follows more or less immediately from known bounds for the Kazhdan constant in terms of the isoperimetric constant and the second-largest eigenvalue of the adjacency matrix, both of which are graph invariants.

In Subsection 3.2, we consider instances in which (1) holds nontrivially. First, we take the case of even cycle graphs of length $\geqslant 6$. Up to GS isomorphism, these can be realized in exactly two ways: as $\mathrm{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$ or as $\mathrm{Cay}(D_n, \{s, sr\})$. Here D_n denotes the dihedral group of order $2n$ with presentation $\langle r, s \mid r^n = s^2 = 1, sr = r^{-1}s \rangle$. In Theorem 3.4, we show that $\kappa(\mathbb{Z}_{2n}, \{\pm 1\}) = \kappa(D_n, \{s, sr\})$, thereby establishing (1) for these pairs. The computation of $\kappa(D_n, \{s, sr\})$ is due to Bacher and de la Harpe [3, Proposition 4]; we recover their result with a new proof.

Later in Subsection 3.2, we give an example where G_1 and G_2 are both cyclic groups of the same order; the pairs are not GS isomorphic; but (1) holds anyway. It is far from obvious that one can find symmetric subsets $\Gamma_1, \Gamma_2 \subset \mathbb{Z}_n$ such that $\mathrm{Cay}(\mathbb{Z}_n, \Gamma_1)$ and $\mathrm{Cay}(\mathbb{Z}_n, \Gamma_2)$ are isomorphic but $(\mathbb{Z}_n, \Gamma_1)$ and $(\mathbb{Z}_n, \Gamma_2)$ are not GS isomorphic. Indeed, Àdàm had conjectured that no such examples exist. Elspas and Turner [5] disproved Àdàm's conjecture by finding just such an example. We generalize the construction of Elspas and Turner to produce an infinite family of such examples. We then compute the corresponding Kazhdan constants and find that (1) holds for all of them.

3.1. Bounds for Kazhdan constants when the corresponding Cayley graphs are isomorphic

The Kazhdan constant of (G_1, Γ_1) relates to various invariants of the corresponding Cayley graph. For that reason, if two pairs (G_1, Γ_1) and (G_2, Γ_2) produce isomorphic Cayley graphs, we can bound one Kazhdan constant in terms of the other, by relating both to graph invariants. We now make this precise.

Let $|B|$ denote the cardinality of a set B. Let X be a graph with vertex set V. For any subset $F \subset V$, we define the *boundary* of F, denoted ∂F, to be the set of all edges with one endpoint in F and one endpoint in $V \setminus F$. For a finite graph X, we define the *isoperimetric constant* of X, denoted $h(X)$, to be the minimum, over all subsets F of V such that $|F| \leqslant |V|/2$, of $|\partial F|/|F|$.

Let A be the adjacency matrix of X. Then A is a symmetric matrix with real entries, hence all of its eigenvalues are real. Let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of A, arranged so that

$$\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{n-1}.$$

The second-largest eigenvalue, which we denote $\lambda_1(X)$, plays a significant role in the theory.

We refer to [7] for more details about these graph invariants. The following theorem was proved in [10]. It provides an upper bound for $h(X)$ in terms of $\lambda_1(X)$. This bound is stronger than the one stated in [7].

Theorem 3.1. Let X be a finite graph. Let Δ be the maximum degree of any vertex in X. Then

$$h(X) \leqslant \sqrt{\Delta^2 - \lambda_1(X)^2}.$$

We also have inequalities relating the Kazhdan constant to $h(X)$ and to $\lambda_1(X)$, respectively. Proofs can be found in [7, Props. 8.17 and 8.18]; see also [9].

Theorem 3.2. Let G be a finite group, and Γ a symmetric subset of G. Let $d = |\Gamma|$ and $X = \mathrm{Cay}(G, \Gamma)$. Let $h = h(X), \lambda_1 = \lambda_1(X)$, and $\kappa = \kappa(G, \Gamma)$. Then:

$$h(X) \geqslant \frac{\kappa^2}{4d} \text{ and } \kappa \geqslant \sqrt{\frac{2(d - \lambda_1)}{d}}.$$

Stringing together the inequalities in Theorems 3.1 and 3.2, we immediately obtain the following theorem.

Theorem 3.3. Let G_1, G_2 be finite groups, and let Γ_1, Γ_2 be symmetric subsets of G_1, G_2, respectively, such that $\mathrm{Cay}(G_1, \Gamma_1)$ and $\mathrm{Cay}(G_2, \Gamma_2)$ are isomorphic graphs. Let $\kappa_1 = \kappa(G_1, \Gamma_1)$ and $\kappa_2 = \kappa(G_2, \Gamma_2)$. Then:

$$\frac{\kappa_1^2}{4d^2} \leqslant \kappa_2 \leqslant 2d\sqrt{\kappa_1}.$$

Moreover, if $\kappa_1, \kappa_2 \leqslant \sqrt{2}$, then:

$$\sqrt{2 - \sqrt{4 - \frac{\kappa_2^4}{4d^4}}} \leqslant \kappa_1 \leqslant d\sqrt{2}\left(4 - (2 - \kappa_2^2)^2\right)^{1/4}.$$

3.2. Examples with equality of Kazhdan constants

Cycle graphs. For $n \geqslant 3$, we have that

$$\mathrm{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}) \cong \mathrm{Cay}(D_n, \{s, sr\}).$$

Indeed, both graphs are cycle graphs of length $2n$. A straightforward computation shows that any pair (G, Γ) for which $\mathrm{Cay}(G, \Gamma)$ is an even cycle graph of length $\geqslant 6$ must be GS isomorphic to $(\mathbb{Z}_{2n}, \{\pm 1\})$ or $(D_n, \{s, sr\})$. Our main theorem in this section shows that $\kappa(\mathbb{Z}_{2n}, \{\pm 1\}) = \kappa(D_n, \{s, sr\}) = 2\sin(\pi/2n)$. Indeed, then, this implies that $\kappa(G, \Gamma) = 2\sin(\pi/2n)$ whenever $\mathrm{Cay}(G, \Gamma)$ is a cycle graph of length $2n \geqslant 6$.

Theorem 3.4. We have that $\kappa(\mathbb{Z}_{2n}, \{\pm 1\}) = \kappa(D_n, \{s, sr\}) = 2\sin\frac{\pi}{2n}$.

Remark 3.5. The fact that $\kappa(D_n, \{s, sr\}) = 2\sin\frac{\pi}{2n}$ appears as Proposition 4 in [3]. Here we give an independent proof of this result.

Proof. First we compute $\kappa(\mathbb{Z}_{2n}, \{\pm 1\})$. Note that

$$\kappa(\mathbb{Z}_{2n}, \{\pm 1\}) = \kappa(\mathbb{Z}_{2n}, \{1\}),$$

because adding or deleting inverses has no effect on the Kazhdan constant. For $\mathbb{Z}_{2n}$, up to rescaling of the inner product (which has no effect on κ), the nontrivial irreducible unitary representations are of the form $\rho_a(k) = \xi^k$, where $\xi = e^{\pi i a/n}$ and $a = 1, \ldots, 2n-1$. Here we identify the nonzero complex number ξ^k with its action by multiplication on the complex plane with the standard inner product, so that $\rho_a\colon \mathbb{Z}_{2n} \to GL(1,\mathbb{C})$. It follows that $\kappa(\mathbb{Z}_{2n}, \{1\}, \rho_a) = 2\sin(a\pi/2n)$. This is minimized when $a = 1$, so

$$\kappa(\mathbb{Z}_{2n}, \{\pm 1\}) = 2\sin\frac{\pi}{2n}.$$

Next we calculate $\kappa(D_n, \{s, sr\})$. We recall that the irreducible representations of the dihedral group are as follows. Let $\xi = e^{\frac{2\pi i}{n}}$. For each integer j with $1 \leqslant j < n/2$, define

$$R_j = \begin{pmatrix} \xi^j & 0 \\ 0 & \xi^{-j} \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then define $\rho_j : D_n \to GL(2,\mathbb{C})$ by $\rho_j(r^a) = R_j^a$ and $\rho_j(s) = S$. These give us representations ρ_j of D_n for each j with $1 \leqslant j < n/2$. These maps ρ_j are unitary with respect to the usual inner product on $\mathbb{C}^2$. Up to rescaling of the inner product, these are all of the two-dimensional irreducible unitary representations of D_n.

We give the one-dimensional irreducible representation of D_n below. The first table gives the nontrivial irreducible representations when n is even.

χ	r^k	sr^k
χ_2	1	-1
χ_3	$(-1)^k$	$(-1)^k$
χ_4	$(-1)^k$	$(-1)^{k+1}$

The next table gives the nontrivial irreducible representations when n is odd.

χ	r^k	sr^k
χ_1	1	-1

To find $\kappa(D_n, \{s, sr\})$, we first find the values $\kappa(D_n, \{s, sr\}, \rho_j)$. We have:

$$\rho_j(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \rho_j(sr) = \begin{pmatrix} 0 & \xi^{-j} \\ \xi^j & 0 \end{pmatrix}.$$

Let $v = (a,b)^T, |a|^2 + |b|^2 = 1$, where $a, b \in \mathbb{C}$. Then $\|\rho_j(s)v - v\| = \|(b-a, a-b)\| = \sqrt{2}|b-a|$ and

$$\begin{aligned}\|\rho_j(sr)v - v\| &= \|(b\xi^j - a, a\xi^{-j} - b)\| \\ &= \sqrt{|\xi^j b - a|^2 + |\xi^{-j}a - b|^2} = \sqrt{2}|\xi^j b - a|.\end{aligned}$$

The angle from a to $a\xi^{-j}$ is $2j\pi/n$. We replace a with the image of a under a rotation of the complex plane such that the positive imaginary axis bisects the angle formed by the rays from the origin to a and from the origin to $a\xi^{-j}$. Therefore, the angle from the positive real axis to $a\xi^{-j}$ is $\pi/2 - j\pi/n = (n\pi - 2j\pi)/2n$. In other words, $a = re^{[(n\pi+2j\pi)/2n]i}$, where $1 \leqslant j < n/2$ for some positive real number r.

We can now see that if $\gamma \in \{s, sr\}$, then

$$\max \|\rho(\gamma)v - v\| = \begin{cases} \sqrt{2}|b - \xi^{-j}a| & \text{when } \mathrm{Re}(b) \leqslant 0, \\ \sqrt{2}|b - a| & \text{when } \mathrm{Re}(b) \geqslant 0. \end{cases}$$

Without loss of generality we may assume that $\mathrm{Re}(b) \geqslant 0$; if not, simply reflect about the imaginary axis throughout the subsequent argument.

We know $a = re^{i\theta}$ where $\theta = \frac{n\pi+2j\pi}{2n}$ and $1 \leqslant j < n/2$, for some $r > 0$. (We trust that there will be no confusion between this real number r and the element r of D_n.) Let $b = x + iy$ where $x, y \in \mathbb{R}$ and $x^2 + y^2 + r^2 = 1$. We seek to minimize $|a - b|^2$.

Let $f(r, x, y) = |re^{i\theta} - (x + iy)|^2 = |a - b|^2$. Then

$$\begin{aligned} f(r, x, y) &= |(r\cos\theta - x) + (r\sin\theta - y)i|^2 = (r\cos\theta - x)^2 + (r\sin\theta - y)^2 \\ &= r^2\cos^2\theta - 2rx\cos\theta + x^2 + r^2\sin^2\theta - 2ry\sin\theta + y^2 \\ &= 1 - 2rx\cos\theta - 2ry\sin\theta. \end{aligned}$$

Let $g(r, x, y) = r^2 + x^2 + y^2 - 1$. By invoking the Lagrange multiplier method we get the following equations, for some real number λ:

$$x\cos\theta + y\sin\theta = -r\lambda \tag{2}$$

$$r\cos\theta = -x\lambda \tag{3}$$

$$r\sin\theta = -y\lambda \tag{4}$$

Multiplying (2) by r, (3) by $-x$, and (4) by $-y$, then adding, we get:

$$0 = -r^2\lambda + x^2\lambda + y^2\lambda.$$

Adding $2r^2\lambda$ to both sides and using that $g(r, x, y) = 0$, we get that $2r^2\lambda = \lambda$. We now have two cases to consider: $\lambda = 0$ or $r = \frac{\sqrt{2}}{2}$.

Case $\lambda = 0$: If $\lambda = 0$ then $r\cos\theta = 0$, so $r = 0$ or $\cos\theta = 0$. Note that $\frac{\pi}{2} < \theta < \pi$, which means $\cos\theta$ is never zero. If $r = 0$ then $a = 0$ and $|b - a|^2 = 1$.

Case $r = \frac{\sqrt{2}}{2}$: Then from (3) and (4) we get that $y = x \tan\theta$. Using that $g(r, x, y) = 0$, we get $x = \frac{\sqrt{2}}{2}|\cos\theta|$. Then

$$f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}|\cos\theta|, \frac{\sqrt{2}}{2}|\cos\theta|\tan\theta) = 1 - |\cos\theta|\cos\theta - |\cos\theta|\tan\theta\sin\theta.$$

We have that $\cos\theta < 0$. Therefore the right-hand side reduces to 2.

From the Lagrange multiplier technique, we now know that our minimum is at most 1. We now find the minimum on the boundary, that is, when $x = 0$.

Let us consider the function $f(r, 0, y) = 1 - 2ry\sin\theta$ where $r^2 + y^2 = 1$. Let $g(r, 0, y) = r^2 + y^2$. Then we get the following equations for some real number λ:

$$y\sin\theta = -r\lambda \tag{5}$$
$$r\sin\theta = -y\lambda \tag{6}$$

Multiplying (5) by r and (6) by y and then adding the two equations yields $r = \frac{\sqrt{2}}{2}$. Solving for y gives $y = \pm\frac{\sqrt{2}}{2}$. This shows that our minimum is $\min\{1 - \sin\theta, 1 + \sin\theta, 1, 2\} = 1 - \sin\theta$. Therefore $|b - a| = \sqrt{1 - \sin\theta}$.

We now find the minimum amongst all j. Recall $\theta = \frac{\pi}{2} + \frac{j\pi}{n}$. Since $\sin\theta$ is decreasing on the interval $(\frac{\pi}{2}, \pi)$ therefore our minimum occurs when $j = 1$, that is, when $\theta = \frac{\pi}{2} + \frac{\pi}{n}$. Therefore, the minimum Kazhdan constant for the two-dimensional representations is

$$\kappa(D_n, \{s, sr\}, \rho_1) = \sqrt{2(1 - \cos\frac{\pi}{n})}.$$

A routine check shows that

$$\kappa(D_n, \{s, sr\}, \chi_k) = 2$$

for any nontrivial one-dimensional representation χ_k.

Therefore, $\kappa(D_n, \{s, sr\}) = \sqrt{2(1 - \cos\frac{\pi}{n})} = 2\sin\frac{\pi}{2n}$. □

Cyclic groups In this section, we construct a family of examples for which (1) holds, where both groups are finite cyclic groups.

Theorem 3.6. Let n be a positive multiple of 8 with $n \geqslant 16$. Let $\Gamma_1 = \{1, 2, \frac{n}{2}-1, \frac{n}{2}+1, n-2, n-1\}$ and $\Gamma_2 = \{2, \frac{n}{4}-1, \frac{n}{4}+1, \frac{3n}{4}-1, \frac{3n}{4}+1, n-2\}$. Then $\mathrm{Cay}(\mathbb{Z}_n, \Gamma_1) \cong \mathrm{Cay}(\mathbb{Z}_n, \Gamma_2)$ but $(\mathbb{Z}_n, \Gamma_1)$ is not GS isomorphic to $(\mathbb{Z}_n, \Gamma_2)$. Moreover, $\kappa(\mathbb{Z}_n, \Gamma_1) = \kappa(\mathbb{Z}_n, \Gamma_2)$.

We remark that the case $n = 16$ precisely gives us Elspas and Turner's counterexample [5] to Àdàm's conjecture for undirected graphs. Also, note that when $n = 8$ we have that $\Gamma_1 = \Gamma_2$ and so $(\mathbb{Z}_n, \Gamma_1)$ is GS isomorphic to $(\mathbb{Z}_n, \Gamma_2)$ under the identity mapping in that case; hence we impose the condition $n \geqslant 16$.

Proof. The function $f\colon \mathbb{Z}_n \to \mathbb{Z}_n$ given by

$$f(a) = \begin{cases} a & \text{if } a \text{ is even} \\ a + \frac{n}{4} & \text{if } a \text{ is odd} \end{cases}$$

defines a graph isomorphism from $\mathrm{Cay}(\mathbb{Z}_n, \Gamma_1)$ to $\mathrm{Cay}(\mathbb{Z}_n, \Gamma_2)$.

To see that $(\mathbb{Z}_n, \Gamma_1)$ is not GS isomorphic to $(\mathbb{Z}_n, \Gamma_2)$, suppose otherwise. So there is a group automorphism of $\mathbb{Z}_n$ taking Γ_1 to Γ_2. This must be of the form $a \mapsto ca$, where c is a fixed integer relatively prime to n. Because n is even, we have that c is odd, and therefore the even element $2 \in \Gamma_1$ must map either to 2 or to $n-2$. First take the case where $2 \mapsto 2$. Then $c = 1$ or $c = n/2 + 1$. First take the subcase $c = 1$. Because $1 \in \Gamma_1$ and Γ_1 maps to Γ_2, therefore $1 = c \cdot 1 \in \Gamma_2$. Because 1 is odd, therefore $1 = \frac{n}{4} - 1$ or $1 = \frac{n}{4} + 1$ or $1 = \frac{3n}{4} - 1$ or $1 = \frac{3n}{4} + 1$. However, because n is a positive multiple of 8 with $n \geqslant 16$, none of those are possible. One finds similarly that every other case also leads to a contradiction.

Finally, we compute the Kazhdan constants. We begin with Γ_1. Note that

$$\kappa(\mathbb{Z}_n, \Gamma_1) = \kappa\left(\mathbb{Z}_n, \{1, 2, \frac{n}{2} - 1\}\right),$$

because adding or deleting inverses has no effect on the Kazhdan constant. As we noted earlier, for $\mathbb{Z}_n$, up to rescaling of the inner product (which has no effect on κ), the nontrivial irreducible unitary representations are of the form $\rho_j(k) = \xi^k$, where $\xi = e^{2\pi i j/n}$ and $j = 1, \ldots, n-1$. As before, we identify the nonzero complex number ξ^k with its action by multiplication on the complex plane with the standard inner product, so that $\rho_j\colon \mathbb{Z}_n \to GL(1, \mathbb{C})$. So

$$\begin{aligned} \kappa\left(\mathbb{Z}_{2n}, \{1, 2, \frac{n}{2} - 1\}, \rho_j\right) &= \min_{\theta \in \mathbb{R}} \max_{\gamma \in \{1, 2, \frac{n}{2} - 1\}} ||\xi^\gamma e^{2\pi i \theta/n} - e^{2\pi i \theta/n}|| \\ &= \max_{\gamma \in \{1, 2, \frac{n}{2} - 1\}} ||\xi^\gamma - 1|| \\ &= \max_{\gamma \in \{1, 2, \frac{n}{2} - 1\}} 2\left|\sin\left(\frac{\gamma j \pi}{n}\right)\right|, \end{aligned}$$

from which it follows that

$$\kappa(\mathbb{Z}_n, \Gamma_1) = \min_{1 \leqslant j \leqslant n} \max_{\gamma \in \{1,2,\frac{n}{2}-1\}} 2\left|\sin\left(\frac{\gamma j \pi}{n}\right)\right|$$
$$= \min_{1 \leqslant j \leqslant n/2} \max_{\gamma \in \{1,2,\frac{n}{2}-1\}} 2\left|\sin\left(\frac{\gamma j \pi}{n}\right)\right|.$$

When $j = 1$, by taking $\gamma = \frac{n}{2} - 1$ we have that

$$\max_{\gamma \in \{1,2,\frac{n}{2}-1\}} 2\left|\sin\left(\frac{\gamma j \pi}{n}\right)\right| \geqslant 2\sin\left(\frac{4\pi}{n}\right).$$

When $j = 2$, we find that

$$\max_{\gamma \in \{1,2,\frac{n}{2}-1\}} 2\left|\sin\left(\frac{\gamma j \pi}{n}\right)\right| = 2\sin\left(\frac{4\pi}{n}\right).$$

When $3 \leqslant j \leqslant n/2 - 2$, by taking $\gamma = 2$ we have that

$$\max_{\gamma \in \{1,2,\frac{n}{2}-1\}} 2\left|\sin\left(\frac{\gamma j \pi}{n}\right)\right| \geqslant 2\sin\left(\frac{4\pi}{n}\right).$$

When $j = n/2 - 1$ or $j = n/2$, by taking $\gamma = 1$ we have that

$$\max_{\gamma \in \{1,2,\frac{n}{2}-1\}} 2\left|\sin\left(\frac{\gamma j \pi}{n}\right)\right| \geqslant 2\sin\left(\frac{4\pi}{n}\right).$$

Therefore

$$\kappa(\mathbb{Z}_n, \Gamma_1) = 2\sin\left(\frac{4\pi}{n}\right).$$

A similar argument shows that

$$\kappa(\mathbb{Z}_n, \Gamma_2) = 2\sin\left(\frac{4\pi}{n}\right). \qquad \square$$

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