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The socle of Leavitt path algebras over a semiprime ring

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Abstract. The Reduction Theorem in Leavitt path algebra over a commutative unital ring is very important to prove that the Leavitt path algebra is semiprime if and only if the ring is also semiprime. Any minimal ideal in the semiprime ring and line point will construct a left minimal ideal in the Leavitt path algebra. Vice versa, any left minimal ideal in the semiprime Leavitt path algebra can be found both minimal ideal in the semiprime ring and line point that generate it. The socle of semiprime Leavitt path algebra is constructed by minimal ideals of the semiprime ring and the set of all line points.

Introduction

In [2] the authors thoroughly discuss Leavitt path algebras over field K on a graph E, denoted $L_K(E)$. The Leavitt path algebra is the extension of the path algebra, which can be studied in detail in [7]. Many pro-perties of $L_K(E)$ have been discussed, among others, the simplicity and primeness. The necessary and sufficient conditions on a graph have been found so that

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the corresponding Leavitt path algebra $L_K(E)$ is simple [2], semisimple [1], and a prime one [3]. In addition, any $L_K(E)$ is non-degenerate. In other words, the Leavitt path algebra over field is always semiprime ([4],[5]).

Tomforde [12] generalized $L_K(E)$ to Leavitt path algebras over a commutative unital ring R, that is denoted as $L_R(E)$. There is a similarity of the necessary and sufficient conditions of $L_R(E)$, which is basically simple, with the simplicity of $L_K(E)$. Basically simple is more general than simple. It means that every simple Leavitt path algebra $L_K(E)$ is basically simple, but the basically simple $L_R(E)$ is not necessarily simple. The definition of $L_R(E)$ can be expanded from the definition of the path algebra over a commutative ring described by [13].

Based on Proposition 1.1 of [5], any Leavitt path algebra over field K, $L_K(E)$ is semiprime. We know that any field is semiprime, but not every semiprime ring is a field. It is easy to get a counter example of non semiprime Leavitt path algebra over a commutative unital ring. Consider the commutative unital ring $\mathbb{Z}_4$ and the graph F follows

$$u_4 \bullet \leftarrow \bullet^{u_1} \rightarrow \bullet^{u_2} \rightarrow \bullet^{u_3}$$

Then Leavitt path algebra $L_{\mathbb{Z}_4}(F) \cong M_2(\mathbb{Z}_4) \oplus M_3(\mathbb{Z}_4)$ is not semiprime. We know that the commutative unital ring $\mathbb{Z}_4$ is also not semiprime. The first focus of this paper is to show that $L_R(E)$ is semiprime if and only if R is semiprime. To prove that, it needs the Reduction Theorem on $L_R(E)$.

The (left) socle of an algebra A, denoted $\mathrm{Soc}(A)$, is the sum of all its minimal left ideals. $\mathrm{Soc}(A)$ is said to be zero in such a case that left ideals do not exist. The concept of socle can be widely studied in [9]. It is well known that for the semiprime algebras $A, \mathrm{Soc}(A)$ coincides with the sum of all minimal right ideals of A (or it is zero if there is no right ideal). The second discussion in this paper is that any minimal ideal $\Im$ in the semiprime $L_R(E)$ can be found a minimal left ideal $I \subset R$ such that $\Im = L_R(E)Iv$ for a line point $v \in P_l(E)$.

It is obviously different construction of a left minimal ideal in the Leavitt path algebra over field having no nontrivial ideal. It implies that the socle of semiprime $L_R(E)$ will be different from the socle of $L_K(E)$ discussed in [6]. Therefore, the final topic in this paper will elaborate on how to determine the socle of Leavitt path algebra over a semiprime commutative unital ring.

1. Preliminaries

We start with the basic definitions. For further notions on graphs, path algebras, and Leavitt path algebras over a commutative ring, we refer to [2], [8], [12], [13], and the references therein.

A *directed graph* is a 4-tuple $E = (E^0, E^1, r_E, s_E)$ consisting of two disjoint sets E^0, E^1, and two maps $r_E, s_E : E^1 \to E^0$. The elements of E^0 are called the *vertices* of E, and the elements of E^1 the *edges* of E, while for $e \in E^1$, $r_E(e)$ and $s_E(e)$ are called the *range* and the *source* of e, respectively. If there is no confusion concerning the graph, we simply write them as $r(e)$ and $s(e)$.

A *path* μ in a graph E is a finite sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) := s(e_1)$ and $r(\mu) := r(e_n)$ are the *source* and *range* of μ, respectively, and n is the *length* of μ. We also say that μ is *a path from* $s(e_1)$ *to* $r(e_n)$ and denote by μ^0 the set of its vertices, i.e., $\mu^0 := \{s(e_1), r(e_1), \dots, r(e_n)\}$. By μ^1 we denote the set of edges appearing in μ, i.e., $\mu^1 := \{e_1, \dots, e_n\}$. We view the elements of E^0 as paths of length 0. The set of all paths of a graph E is denoted by $\mathrm{Path}(E)$.

Given a (directed) graph E and a commutative unital ring R. The *path R-algebra* of E, denoted by RE, is defined as the free-associative R-algebra generated by the set of paths of E with relations:

(V) $vw = \delta_{v,w} v$ for all $v, w \in E^0$.

(E1) $s(e)e = er(e) = e$ for all $e \in E^1$.

If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. If E^0 is finite and E is row-finite, then E^1 must necessarily be finite as well; in this case, we say that E is *finite*. A vertex that emits no edges is called a *sink*. A vertex v is called an *infinite emitter* if $s^{-1}(v)$ is an infinite set, and a *regular vertex* if it is neither a sink nor an infinite emitter. The set of infinite emitters will be denoted by E^0_{inf}, while $\mathrm{Reg}(E)$ will denote the set of regular vertices.

The *extended graph of* E is defined as the new graph

$$\widehat{E} = (E^0, E^1 \cup (E^1)^*, r_{\widehat{E}}, s_{\widehat{E}}),$$

where $(E^1)^* = \{e_i^* \mid e_i \in E^1\}$ and the functions $r_{\widehat{E}}$ and $s_{\widehat{E}}$ are defined as $r_{\widehat{E}|_{E^1}} = r$, $s_{\widehat{E}|_{E^1}} = s$, $r_{\widehat{E}}(e_i^*) = s(e_i)$, and $s_{\widehat{E}}(e_i^*) = r(e_i)$.

The elements of E^1 will be called *real edges*, while for $e \in E^1$ we will call e^* a *ghost edge*.

The *Leavitt path algebra of E with coefficients in R*, denoted $L_R(E)$, is the path algebra $R\widehat{E}$ generated by the relations:

(CK1) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.

(CK2) $v = \sum_{\{e \in E^1 | s(e)=v\}} ee^*$ for every $v \in \mathrm{Reg}(E)$.

Observe that in $R\widehat{E}$ the relations (V) and (E1) remain valid and that the following is also satisfied:

(E2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.

Based on the relation (V), every vertex in E is an idempotent element in $L_R(E)$. Moreover, every two vertices are orthogonal idempotents. However, not every vertex is primitive idempotent because of the relation $(CK2)$. For example, if we have a vertex v such that $s^{-1}(v) = \{e, f\}$ then v is not primitive since $v = ee^* + ff^*(CK2)$, where both ee^* and ff^* are nonzero orthogonal idempotents. The definition of a primitive idempotent refers to [7], that an idempotent u in algebra A is said to be *primitive* if it cannot be written as a sum $u = u_1 + u_2$ where u_1 and u_2 are nonzero orthogonal idempotents of A.

Note that if E is a finite graph, then $L_R(E)$ is unital with $\sum_{v \in E^0} v = 1_{L_R(E)}$. Otherwise, $L_R(E)$ is a ring with a set of local units consisting of sums of distinct vertices (For a ring A, the assertion A *has local units*). It means that each finite subset of A is contained in a *corner* of A, that is, a subring of the form eAe, where e is an idempotent of A). Note that since every Leavitt path algebra $L_R(E)$ has local units, it is the directed union of its corners.

The Leavitt path algebra $L_R(E)$ is a $\mathbb{Z}$-graded R-algebra, spanned as an R-module by $\{\alpha\beta^* \mid \alpha, \beta \in \mathrm{Path}(E)\}$. In particular, for each $n \in \mathbb{Z}$, the degree n component $L_R(E)_n$ is spanned by the set

$$\{\alpha\beta^* \mid \alpha, \beta \in \mathrm{Path}(E) \text{ and } \mathrm{length}(\alpha) - \mathrm{length}(\beta) = n\}.$$

Denote by $h(L_R(E))$ the set of all homogeneous elements in $L_R(E)$, that is,

$$h(L_R(E)) := \cup_{n \in \mathbb{Z}} L_R(E)_n.$$

If μ is a path in E, and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at* v. If $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*. A graph that contains no cycles is called *acyclic*. For $\mu = e_1 \dots e_n \in \mathrm{Path}(E)$ we write μ^* for the element $e_n^* \dots e_1^*$ of $L_R(E)$.

An edge e is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists i in $\{1, \dots, n\}$ such that $s(e) = s(e_i)$ and $e \neq e_i$. We denote by $P_l(E)$, the set of all *line points*, i.e., the vertices whose tree does not contain neither bifurcations nor cycles. Based on the relation $(CK2)$, every vertex in $P_l(E)$ is primitive idempotent.

It is easy to see that $P_l(E)$ is hereditary subsets, although they are not necessarily saturated. For any subset X of vertices, $\overline{X}$ will denote the hereditary and saturated closure of X, that is, the smallest saturated hereditary subset of E^0 containing X.

In any Leavitt path algebra over a commutative unital ring, $L_R(E)$ we have:

$$L_R(E) = \operatorname{span}_R \{\alpha\beta^* \mid \alpha, \beta \in \operatorname{Path}(E), r(\alpha) = r(\beta)\}$$

and the multiplication is given by the following rule:

$$(\alpha\beta^*)(\gamma\delta^*) = \begin{cases} \alpha\gamma'\delta^* & \text{if} \quad \gamma = \beta\gamma' \\ \alpha\delta^* & \text{if} \quad \beta = \gamma \\ \alpha\beta'^*\delta^* & \text{if} \quad \beta = \gamma\beta' \\ 0 & \text{otherwise} \end{cases}$$

2. On minimal left ideals in Leavitt path algebras over a semiprime ring

It has been said that not every Leavitt path algebra over a commutative unital ring $L_R(E)$ is semiprime. Then, the first discussion is the necessary and sufficient condition of the semiprime $L_R(E)$. The important theorem here is the Reduction Theorem, whose proof needs the following lemma.

Lemma 1. *Let E be a graph and R a commutative unital ring. For any vertex v in a cycle without exits c and any ideal I of R, we have:*

$$vL_R(E)Iv = \left\{ \sum_{i=-m}^{n} k_i c^i \mid k_i \in I, m, n \in \mathbb{N} \right\} \cong I[x, x^{-1}],$$

where c^0 denotes the vertex v and $c^{-t} = (c^*)^t$ for every $t \geqslant 1$.

Proof. Since the cycle c has no exits, it is easy to see that if $u \in T(v)$ and $f, g \in E^1$ are such that $s(f) = s(g) = u$, then $f = g$. Write $c = e_1e_2...e_n$, with $e_i \in E^1$. For every $\mu \in \operatorname{Path}(E)$ such that $s(\mu) = u \in T(v)$ there is a $k \in \mathbb{N}$ with $1 \leqslant k \leqslant n$ and a path μ' such that $\mu = e_k\mu'$ and $s(e_k) = u$.

Now, let $x \in vL_R(E)Iv$. We may write $x = \sum_{i=1}^{p} k_i\alpha_i\beta_i^* + kv$, where $k, k_i \in I$, $s(\alpha_i) = r(\beta_i^*) = s(\beta_i) = v$. Then, taking into account what we have explained in the first paragraph and following the proof of [6, Lemma 1.5], for any i we have $\alpha_i\beta_i^* = c^{d_i}$ for some $d_i \in \mathbb{Z}$. Define an R-linear map $\varphi : vL_R(E)Iv \to R[x, x^{-1}]$ by setting $\varphi(kv) = k$, $\varphi(kc) = kx$, and $\varphi(kc^*) = kx^{-1}$. Then φ is a monomorphism with $\varphi(vL_R(E)Iv) = I[x, x^{-1}]$. Hence, $vL_R(E)Iv \cong I[x, x^{-1}]$. □

Let E be an arbitrary graph and R a commutative unital ring. Then the Reduction Theorem in [2, Theorem 2.2.11] is also valid in this context, that for any non zero element $x \in L_R(E)$ there exist $\alpha, \beta \in \text{Path}(E)$ such that:

1) $0 \neq \alpha^* x\beta \in Rv$, for a vertex $v \in E^0$, or
2) $0 \neq \alpha^* x\beta = p(c)$ where c is a cycle without exits and $p(c)$ is a non zero polynomial in $R[x, x^{-1}]$.

Lemma 1 is used to prove the Reduction Theorem in $L_R(E)$. However, it is not discussed here because the steps of proof can be followed by [2, Theorem 2.2.11].

Recall that a ring R is said to be *semiprime* if it has no nonzero ideals of zero square. For a commutative ring R, this is equivalent to say that R has no nonzero elements of zero square. Such a ring is also called *reduced*.

Leavitt path algebras over fields are semiprime (see, for example, the Proposition 2.3.1 in [2]). Over commutative unital rings, they are when the underlying ring is, as we show.

Theorem 1. *Let E be an arbitrary graph and R a commutative unital ring. The Leavitt path algebra $L_R(E)$ is semiprime if and only if the ring R is semiprime.*

Proof. Assume $L_R(E)$ is semiprime. Suppose that R is not semiprime, then there is a non zero $k \in R$ such that $k^2 = 0$. Since $k \neq 0$, then $kv \neq 0$ for any $v \in E^0$. Apply that $L_R(E)$ is semiprime to get $\mu \in L_R(E)$ such that $0 \neq kv\mu kv$. Write $\mu = \sum_{i=1}^{m} k_i\alpha_i\beta_i^*$ for some $k_i \in R, \alpha_i, \beta_i \in \text{Path}(E), m \in \mathbb{N}$. Then,

$$\begin{aligned} kv\mu kv &= kv\left(\sum_{i=1}^{m} k_i\alpha_i\beta_i^*\right)kv = v\left(\sum_{i=1}^{m} (kk_ik)\alpha_i\beta_i^*\right)v \\ &= v\left(\sum_{i=1}^{m} (k^2k_i)\alpha_i\beta_i^*\right)v = 0, \end{aligned}$$

which is a contradiction.

Conversely, assume that R is semiprime. Suppose that there is a non zero ideal $\Im \subseteq L_R(E)$ such that $\Im^2 = \{0\}$. Let $0 \neq x \in \Im$; by the Reduction Theorem there exist $\alpha, \beta \in \text{Path}(E)$ such that

1) $0 \neq \alpha^* x\beta = kv$, for some $k \in R$ and $v \in E^0$, or
2) $0 \neq \alpha^* x\beta = p(c)$, where c is a cycle without exits and $p(c)$ is a non zero polynomial in $R[x, x^{-1}]$.

In the first case, $kv \in \Im$. By the semiprimeness of R we have $k^2 \neq 0$, hence $0 \neq k^2 v = (kv)(kv) \in \Im^2 = \{0\}$, a contradiction. In the second case, $0 \neq p(c) \in \Im$. Since R is semiprime, $R[x, x^{-1}]$ is also semiprime, in particular $p(c)^2 \neq 0$. Hence, $0 \neq p(c)^2 \in \Im^2 = \{0\}$, a contradiction. In any case we get a contradiction. Hence, $L_R(E)$ must be semiprime. □

Lemma 2. *Let I be a minimal ideal of a semiprime commutative unital ring R. Then $I = Re$ for $e = e^2 \in R$. Morever, Re is a field.*

Proof. Since any minimal ideal in a semiprime ring is generated by an idempotent (see, for example [10, Subsection 3.4]), $I = Re = eR$ where $e = e^2 \in R$. The minimality implies that $I = eRe$ is division ring, hence it is a field, as R is commutative. □

The left minimal ideal of $L_K(E)$ in [5] is in the form of $L_K(E)v$ for some line point v. It is different from a left minimal ideal in $L_R(E)$, which requires a minimal ideal of the semiprime ring R beside the line point. To construct the left minimal ideal of $L_R(E)$, we need to redefine preorder as follows.

Definition 1 ([2]). For every $u, v \in E^0$, we define $u \geqslant v$ if there exists $\alpha \in \text{Path}(E)$ such that $s(\alpha) = u$ and $r(\alpha) = v$. In this case, we say that α is a path *joining u to v*.

Lemma 3. *Let E be an arbitrary graph and I a minimal ideal of a semiprime commutative unital ring R. Let any vertices v, w be such that $v \geqslant w$. If the path joining v to w contains no bifurcations then $L_R(E)Iv \cong L_R(E)Iw$ as left $L_R(E)$-modules.*

Proof. Let $\alpha \in \text{Path}(E)$ be the only path such that $s(\alpha) = v, r(\alpha) = w$. Define the maps

$$\varphi : L_R(E)Iv \to L_R(E)Iw$$
$$kxv \mapsto kxv\alpha$$

and

$$\psi : L_R(E)Iw \to L_R(E)Iv$$
$$kyw \mapsto kyw\alpha^*$$

for every $k \in I$, $x, y \in L_R(E)$. It is easy to see that φ and ψ are homomorphisms of left $L_R(E)$-modules. Now, we see that $\psi = \varphi^{-1}$ as follows.

Take $kxv \in L_R(E)Iv$ and $kyw \in L_R(E)Iw$. Then:

$$\psi(\varphi(kxv)) = \psi(kxv\alpha) = \psi(kxv\alpha w) = kxv\alpha w\alpha^* = kxv$$

and

$$\varphi(\psi(kyw)) = \varphi(kyw\alpha^*) = \varphi(kyw\alpha^* v) = kyw\alpha^* v\alpha = kyw.$$

Hence, $\psi\varphi = 1_{L_R(E)Iv}$ and $\varphi\psi = 1_{L_R(E)Iw}$, where $1_{L_R(E)Iv}$ and $1_{L_R(E)Iw}$ denote the identity maps in $L_R(E)Iv$ and $L_R(E)Iw$, respectively. □

Proposition 1. *Let E be an arbitrary graph, R be a commutative unital ring and I be an ideal of R. Assume that u is a vertex, which is not a sink.*
1) $\bigoplus_{f\in s^{-1}(u)} L_R(E)Iff^* \subseteq L_R(E)Iu$.
2) *If u is not an infinite emitter then*
   (a) $\bigoplus_{f\in s^{-1}(u)} L_R(E)Iff^* = L_R(E)Iu$.
   (b) *Denote $v_f := r(f)$. If $r(f) \neq r(f')$ for every $f \neq f'$, with f, f' in $s^{-1}(u)$, then $L_R(E)Iu \cong \bigoplus_{f\in s^{-1}(u)} L_R(E)Iv_f$.*

Proof. 1) Follows immediately, taking into account that the sum of the left ideals $L_R(E)If_if_i^*$ is direct as $f_i^* f_j = 0$ for every $i \neq j$.

2) (a) follows from 1) and the (CK2) relation that says

$$u = \sum_{f\in s^{-1}(u)} ff^*.$$

(b) There is an isomorphism of left $L_R(E)-$module

$$\mu : L_R(E)Iv_f \to L_R(E)Iff^*$$
$$kxv_f \mapsto kxff^*$$

for every $k \in I, x \in L_R(E)$. □

Corollary 1. *Let $w \in E^0$. If $T(w)$ contains some bifurcation, then the left ideal $L_R(E)Iw$ is not minimal.*

Lemma 4. *If there is some closed path based at $u \in E^0$, then the left ideal $L_R(E)Iu$ is not minimal.*

Proof. The proof is similar to Proposition 2.5 in [5]. □

Proposition 2. *Let E be an arbitrary graph and I be a minimal ideal of the semiprime commutative unital ring R denoted by $I \lhd_m R$. If $v \in P_l(E)$ then $L_R(E)Iv$ is a minimal left ideal of $L_R(E)$.*

Proof. Take any non zero $\alpha \in L_R(E)Iv$ with $v \in E^0$ then

$$\alpha = \left(\sum_{i=1}^{m} k_i\alpha_i\beta_i^*\right)sv = \sum_{i=1}^{m} k_i s\alpha_i\beta_i^* v = \sum_{i=1}^{m} s_i'\alpha_i\beta_i^* v$$

where $\alpha_i, \beta_i \in \text{Path}(E)$, $r(\alpha_i) = r(\beta_i)$ and $0 \neq s_i' = k_i s \in I$ for every $i \in \{1, 2, ..., m\}$. By Lemma 2, I is a field so that I is semiprime then $L_R(E)I$ is also semiprime (Proposition 1). Based on [10, Proposition 2 in § 3.4], it is sufficient to show that $vL_R(E)Iv$ is a division ring. Take any non zero element $\delta \in vL_R(E)Iv$ then

$$\delta = v\left(\sum_{i=1}^{m} k_i\alpha_i\beta_i^*\right)av = \sum_{i=1}^{m} ak_i(v\alpha_i\beta_i^* v)$$

for $a \in I$, $\alpha_i, \beta_i \in \text{Path}(E)$ such that $s(\alpha_i) = r(\beta_i) = v$. Since v is a line point, α_i and β_i must have the same length, so that $\delta = \sum_{i=1}^{m} s_i v \in Iv$ where $s_i = ak_i \in I$ for all i. It means that $vL_R(E)Iv = Iv \cong I$ with I a field (Lemma 2). Hence, $L_R(E)Iv$ is a minimal left ideal of $L_R(E)$. □

Definition 2. For every $u, v \in P_l(E)$, we define $u \equiv v \Leftrightarrow \exists I \triangleleft_m R$ such that $(Iu) = (Iv)$ where (Iu) is an ideal of $L_R(E)$ generated by Iu.

Let $u, v \in P_l(E)$ such that $(Iu) = (Iv)$ then $tu \in L_R(E)vL_R(E)$ for any $t \in I$. The form of tu is $tu = \sum_{i=1}^{n} x_i v y_i$ with $x_i, y_i \in L_R(E)$, so that

$$tu = \sum_{i=1}^{n} ux_i v y_i u = \sum_{i=1}^{n} u\left(\sum k_j\alpha_j\beta_j^*\right)v\left(\sum k_l\lambda_l\mu_l^*\right)u.$$

Then there exist $\alpha, \beta \in \text{Path}(E)$ such that $0 \neq u\alpha\beta^* v$. Since u, v are line points,

$$\alpha\beta^* = \begin{cases} \lambda, & \lambda \in \text{Path}(E), \\ \mu^*, & \mu \in \text{Path}(E). \end{cases}$$

It means that $(Iu) = (Iv)$ then there is $\alpha \in \text{Path}(E)$ such that $\alpha = u\alpha v$ or $\alpha = v\alpha u$. Since $\alpha\alpha^* = u$ and $\alpha^*\alpha = v$ in the first case, $\alpha\alpha^* = v$ and $\alpha^*\alpha = u$ in the second case, we have $(Ju) = (Jv)$ for every $J \triangleleft_m R$.

Lemma 5. *Let $[v]$ is an equivalence class of $\equiv$ in Definition 2. Then $[v]$ is hereditary.*

Proof. Let $u \in [v]$ and let $w \in E^0$ such that there is an edge e with $s(e) = u, r(e) = w$. By $CK1$ and $CK2$, we have $u = e^*we$ and $w = eue^*$. Hence, $(Iw) = (Iu) = (Iv)$ so that $w \in [v]$ □

Proposition 3. *Let $u, v \in P_l(E)$. Then $u \equiv v$ if and only if $L_R(E)Iu \cong L_R(E)Iv$ as left $L_R(E)$-modules, for every $I \lhd_m R$.*

Proof. Let $u, v \in P_l(E)$ such that $u \equiv v$. Then $(Iu) = (Iv)$ for every $I \lhd_m R$, so that there is $\alpha \in \text{Path}(E)$ such that $\alpha = u\alpha v$. By Lemma 3, we have $L_R(E)Iu \cong L_R(E)Iv$ as left $L_R(E)$-modules.

Conversely, let any minimal ideal I of R and $u, v \in \text{Path}(E)$ such that $L_R(E)Iu \cong L_R(E)Iv$ as left $L_R(E)$-modules. By Proposition 2, $L_R(E)Iu$ and $L_R(E)Iv$ are minimal left ideals of $L_R(E)$. Then $Iu, Iv \subset \text{Soc}(L_R(E))$. In fact, they are in the same homogeneous component. □

Theorem 2. *Let $v \in E^0$ and I is a minimal ideal of the semiprime commutative unital ring R. Then v is a line point if and only if $L_R(E)Iv$ is a minimal left ideal of $L_R(E)$.*

Proof. Let $v \in E^0$ be a line point; by Proposition 2 we have $L_R(E)Iv$ is the minimal left ideal of $L_R(E)$. Conversely, assume that $L_R(E)Iv$ is simple, we will show that the vertex v is a line point, that is, no vertex in $T(v)$ has bifurcations, nor any vertex in $T(v)$ is the base of a cycle. For any $u \in T(v)$ and let $\mu \in \text{Path}(E)$ such that $s(\mu) = v$, $r(\mu) = u$. Then the map $\varphi_\mu : L_R(E)Iv \to L_R(E)Iu$ given by $\alpha v \mapsto \alpha v\mu$ is a non zero epimorphism of $L_R(E)$-modules, as for every $\beta u \in L_R(E)Iu$ we find $\beta u = \beta\mu^*\mu = \varphi_\mu(\beta\mu^*)$. Then $L_R(E)Iu$ must be simple because of the simplicity of $L_R(E)Iv$. Suppose there is a vertex $w \in T(v)$ having a bifurcation and take edges $e, f \in s^{-1}(w)$ with $e \neq f$. Then $L_R(E)Iw = L_R(E)Iee^* \oplus \Im$ where $\Im = \{\alpha - \alpha ee^* | \alpha \in L_R(E)I\} \neq \{0\}$ because of $f = f - fee^* \in \Im$. It means that $L_R(E)Iw$ is not simple, which is a contradiction. In the second case, suppose there is $w \in T(v)$, the base of a cycle c. According to Lemma 1, we have $wL_R(E)Iw \cong I[x, x^{-1}]$, which is not a division ring. Then $L_R(E)Iw$ is not minimal, there is a contradiction. Hence, the vertex v must be a line point. □

Proposition 4. *Let $\Im$ be any minimal left ideal of $L_R(E)$, where R is a semiprime commutative unital ring. Then there is a minimal ideal $I \subset R$ such that $\Im = L_R(E)Iv$ for v some line point.*

Proof. Since R is semiprime, $L_R(E)$ is also semiprime (by Proposition 1). By [10, Proposition 2. in Section 3.4], $\Im = L_R(E)\mu$ for a non zero idempotent $\mu \in L_R(E)$. Based on the Reduction Theorem, we have two cases. The first case, there exist $\alpha, \beta \in \text{Path}(E)$ and a non zero $k \in R$ such that $0 \neq \alpha^*\mu\beta = kv$ for some $v \in E^0$.

We have $L_R(E)(\alpha^*\mu\beta) = L_R(E)(kv) = L_R(E)\mu = \Im$ for the minimality of $L_R(E)\mu$. Then $L_R(E)(kv)$ is a minimal left ideal of $L_R(E)$. In other hand, the form of any non zero $x \in L_R(E)(kv)$ is

$$x = \left(\sum_{i=i}^{m} k_i\alpha_i\beta_i^*\right)kv = \sum_{i=i}^{m} kk_i\alpha_i\beta_i^* v,$$

where $kk_i \in kR \subsetneq R, \alpha_i, \beta_i \in \text{Path}(E), r(\alpha_i) = r(\beta_i)$ for every i. By the minimality of $L_R(E)(kv)$, we have $L_R(E)\mu = L_R(E)(kv) = L_R(E)Iv$ for an ideal $I = kR = Rk$ in the semiprime R. We will see that $I = Rk$ is minimal. Let any non zero $rk \in Rk$. Then $rkv \neq 0$ so that $0 \neq L_R(E)rkv = L_R(E)kv$ for the minimality of $L_R(E)kv$. It implies that $kv = \alpha rkv$ for some $\alpha \in L_R(E)$. We have $v = r\alpha v$ for $k \neq 0$, where αv has zero degree. Then $\alpha v = lv$ with $l \in R$, so that $kv = krlv$ and $k = krl$. It must be $rl = 1$. Hence, $Rkrl = Rk = I$. In other words, the ideal I is minimal. Since the left ideal $L_R(E)Iv$ is minimal in $L_R(E)$, the vertex v is a line point (by Theorem 2).

The second one, suppose there exist paths $\alpha, \beta \in \text{Path}(E)$ such that $0 \neq \alpha^*\mu\beta = p(c)$ for a cycle without exit c based at v. Then $p(c) \in vL_R(E)v \cong R[x, x^{-1}]$. Since the left ideal $L_R(E)\mu$ is mi-nimal, we have $L_R(E)p(c) = L_R(E)\alpha^*\mu\beta = L_R(E)\mu$, so that $L_R(E)p(c)$ is a minimal left ideal of $L_R(E)$. Define a map $\psi : R[x, x^{-1}] \to L_R(E)$ given by $\psi(1) = v, \psi(x) = c, \psi(x^{-1}) = c^*$. Then ψ is a monomorphism of R-algebras with $\psi(R[x, x^{-1}]) = vL_R(E)v$. Consider that $vL_R(E)p(c)$ is a minimal left ideal of $vL_R(E)v$ then $\psi^{-1}(vL_R(E)p(c))$ is a minimal left ideal of $R[x, x^{-1}]$, which is a contradiction. Hence, the second case is not possible. □

Corollary 2. *Let $v \in E^0$ and I be an ideal of the semiprime commutative unital ring R. Then $L_R(E)IvL_R(E) = (Iv)$ is a minimal two-sided ideal of $L_R(E)$ if and only if $v \in P_l(E)$, and the ideal I is minimal.*

Lemma 6. *Let E be an arbitrary graph, R any commutative unital ring and an ideal I of R. Let H be a hereditary subset of E^0. Then, the ideal (IH) of $L_R(E)$ consists of elements of $L_R(E)$ of the form*

$$(IH) = \left\{\sum_{i=1}^{n} a_i\alpha_i\beta_i^* \mid a_i \in I, \alpha_i, \beta_i \in \text{Path}(E), r(\alpha_i) = r(\beta_i) \in H\right\} \quad (1)$$

Morever $(IH) = (I\bar{H})$, where $\bar{H}$ is a saturated closure of H.

Proof. Let J denote the set presented in (1). We will see that J is an ideal in $L_R(E)$, so that we need to show that for every element of the form $\alpha\beta^*$, where $r(\alpha) = r(\beta) = u \in H$, and for every $x, y \in L_R(E)$, $a \in I$ we have $ax\alpha u\beta^* y \in J$. By Lemma 5 and the multiplication in (1), it is enough to show that $a\gamma\delta^* u\mu\eta^* \in J$ for every $a \in I, \gamma, \delta, \mu, \eta \in \text{Path}(E)$ and $u \in H$.

If $a\gamma\delta^* u\mu\eta^* = 0$ then it is finished. Suppose that $a\gamma\delta^* u\mu\eta^* \neq 0$. By the multiplication in (1), we have $a\gamma\delta^* u\mu\eta^* = a\gamma\mu'\eta^*$ if $\mu = \delta\mu'$ or $a\gamma\delta^* u\mu\eta^* = a\gamma\eta^*$ if $\delta = \mu$ or $a\gamma\delta^* u\mu\eta^* = a\gamma\delta'\eta^*$ if $\delta = \mu\delta'$. Note that $u = s(\mu) \in H$ and H is hereditary. Then we have $r(\mu) = r(\mu') \in H$ in the first case, $r(\mu) = r(\delta) \in H$ in the second case, and $r(\delta) = r(\delta') \in H$ in the last one for $s(\mu) = s(\delta) = u$. Hence, $0 \neq a\gamma\delta^* u\mu\eta^* \in J$ in all cases.

It is clear that $(IH) \subseteq (I\bar{H})$ for $H \subseteq \bar{H}$. Conversely, let any monomial $a\alpha\beta^* \in (I\bar{H})$, where $a \in I$, $\alpha, \beta \in \text{Path}(E), r(\alpha) = r(\beta) \in \bar{H}$. Based on [2, Lemma 1.2.4] $\bar{H} = \bigcup_{n \geqslant 0} H_n$, where $H_0 = T(H) = H$ as H is hereditary and $H_n = \{v \in Reg(E) \mid r(s^{-1}(v)) \subset H_{n-1}\} \cup H_{n-1}$. We wil see that $a\alpha\beta^* \in (IH)$, by mathematics induction in n. For $n = 0$, it is clear that $a\alpha\beta^* \in (IH)$, for $r(\alpha) = r(\beta) \in H_0 = T(H) = H$. Suppose it is true that $a\alpha\beta^* \in (IH)$ for $r(\alpha) = r(\beta) \in H_{n-1}$. Let $u = r(\alpha) = r(\beta) \in H_n$ then $u \in H_{n-1}$ or $r(s^{-1}(u) \subseteq H_{n-1}$. By hypothesis, $a\alpha\beta^* \in (IH)$ if $u \in H_{n-1}$. Otherwise, if $r(s^{-1}(u) \subseteq H_{n-1}$ then we have $a\alpha\beta^* \in (IH)$ for $u = \sum_{e \in s^{-1}(u)} ee^* = \sum_{e \in s^{-1}(u)} er(e)e^*$ and $r(e) \in H_{n-1}$. □

Lemma 7. *Let E be any graph, R any commutative unital ring and I be an ideal of R. Let $\{H_i\}_{i \in \Gamma}$ be a family of hereditary subsets of E^0 such that $H_i \cap H_j = \varnothing$ for every $i \neq j$. Then*

$$\left(I\overline{\bigcup_{i \in \Gamma} H_i} \right) = I \bigcup_{i \in \Gamma} H_i = \bigoplus_{i \in \Gamma} (IH_i) = \bigoplus_{i \in \Gamma} (I\overline{H_i}).$$

Proof. Assume that $H = \bigcup_{i \in \Gamma} H_i$. It is clear that H is hereditary. By Lemma 6, the first and the last equality are obviously fulfilled.

Based on (1), for every $x \in (IH)$ can be expressed by $x = \sum_{l=1}^{n} a_l \alpha_l \beta_l^*$, where $a_l \in I$, $\alpha_l, \beta_l \in \text{Path}(E)$ and $r(\alpha_l) = r(\beta_l) \in H$. Furthermore, we separate $r(\alpha_l)$ depending on the H_i's they belong to and we have $x = \sum_{l=1}^{n} a_l \alpha_l \beta_l^* \in \sum_{i \in \Gamma} (IH_i)$, so that $(IH) \subseteq \sum_{i \in \Gamma} (IH_i)$. It is clear that $\sum_{i \in \Gamma} (IH_i) \subseteq (IH)$ as $(IH_i) \subset (IH)$ for every i. Hence, $(IH) = \sum_{i \in \Gamma} (IH_i)$.

Suppose there is $j \in \Gamma$ such that $(IH_j) \cap \sum_{j \neq i \in \Gamma} (IH_i) \neq \{0\}$. Let $0 \neq y \in (IH_j) \cap \sum_{j \neq i \in \Gamma} (IH_i)$. By (1), $y = \sum_{k=1}^{t} s_k \gamma_k \delta_k^*$ with $s_k \in I$, $\gamma_k, \delta_k \in \text{Path}(E)$, $r(\gamma_k) = r(\delta_k) \in H_j$ and $r(\gamma_k) = r(\delta_k) \in \bigcup_{j \neq i \in \Gamma} H_i$. Hence, $H_j \cap \bigcup_{j \neq i \in \Gamma} H_i \neq \varnothing$, which is a contradiction. □

Proposition 5. *Let E be an arbitrary graph, R be any commutative unital ring and I be an ideal of R. Let $v \in P_l(E)$ and Λ_v denote the set of paths $\alpha \in \mathrm{Path}(E)$ such that $r(\alpha)$ meets $T(v)$ for the first time $r(\alpha)$. Then $(Iv) \cong M_n(I)$, where n is the cardinality of Λ_v.*

Proof. Let $v \in P_l(E)$ and a sequence $T(v) = \{v_1, v_2, ...\}$, where $v = v_1$ and for all $i \in \mathbb{N}$ there exists a unique $e_i \in E^1$ such that $s(e_i) = v_i, r(e_i) = v_{i+1}$. Consequently, for each pair $v_i, v_j \in T(v)$ with $i < j$ there exists a unique path $p_{i,j} \in \mathrm{Path}(E)$ for which $s(p_{i,j}) = v_i$ and $r(p_{i,j}) = v_j$. Since v is a line point then for every i, v_i is a line point. By $(CK2)$, we have $p_{i,j}p^*_{i,j} = v_i$ for every pair $v_i, v_j \in T(v)$ with $i < j$. Let I be an ideal of R and $T_I = \mathrm{span}_I(\{\varepsilon_{i,j}\})$ be an ideal of an R-algebra A, where the subset $\{\varepsilon_{i,j} \mid i, j \in \Gamma\}$ of T_I is called a set of matrix units of T_I with $\varepsilon_{i,j}\varepsilon_{k,l} = \delta_{j,k}\varepsilon_{i,l}$ for all i, j, k, l. Then $T_I \cong M_{|\Gamma|}(I)$ as R-algebras via an isomorphism sending $\varepsilon_{i,j}$ into $e_{i,j}$, where $e_{i,j} \in M_{|\Gamma|}(I)$ having all the entries equal zero except that in row i, column j, where the entry is 1.

We will construct a set of matrix units in (Iv) indexed by $|\Lambda_v|$, where Λ_v is the set of paths $\alpha \in \mathrm{Path}(E)$, such that $r(\alpha)$ meets $T(v)$ for the first time $r(\alpha)$. By Lemma 6, every element in (Iv) is a linear combination of elements of the form $a\alpha x_{i,j}\beta^*$, where $a \in I$, $\alpha, \beta \in \Lambda_v$, $x_{i,j} = p_{i,j}$ if $i \leqslant j$ and$x_{i,j} = p^*_{j,i}$ if $j \leqslant i$. Denote $\alpha x_{i,j}\beta^* = e_{\alpha,\beta}$ and $\varepsilon = \{e_{\alpha,\beta} \mid \alpha, \beta \in \Lambda_v\}$. Since the set $\{x_{i,j} \mid i, j \in \mathbb{N}\}$ has the multiplicative property $x_{i,j}x_{k,l} = \delta_{j,k}x_{i,l}$, then ε is a set of matrix units of (Iv). □

Lemma 8. *Let E be an arbitrary graph, R be any semiprime commutative unital ring, and I be a minimal ideal of R. Then, there is a family $\{[v]\}_{v\in P_l(E)}$ of hereditary subsets of E^0 such that $P_l(E) = \coprod_{v\in P_l(E)}[v]$. Furthermore, we have $(I[v]) = (Iv)$ for every $v \in P_l(E)$.*

Proof. Let $v \in P_l(E)$, we proved that $[v]$ is hereditary (Lemma 5). Let any $u \notin [v]$ then $u \not\equiv v$ so that $[u] \cap [v] = \varnothing$. Hence, $P_l(E) = \coprod_{v\in P_l(E)}[v]$. Furthermore, take any $w \in [v]$, then $(Iw) = (Iv)$, so that $(I[v]) = (Iv)$. □

3. The Socle of the semiprime Leavitt path algebra $L_R(E)$

According to Theorem 1, the Leavitt path algebras $L_R(E)$ is semiprime if the commutative unital ring R is semiprime. Based on [10] Remark 2.6.6, the two sided ideal $\mathrm{Soc}(L_R(E))$ denotes the sum of minimal left (right) ideals of $L_R(E)$.

Proposition 6. *For any graph E and minimal ideal I of the semiprime commutative unital ring R, we have*

1) $\sum_{u\in P_l(E), I\triangleleft_m R} L_R(E)Iu \subset \mathrm{Soc}(L_R(E))$
2) $(IP_l(E)) = (I\overline{P_l(E)}) \cong \bigoplus_{v\in P_l(E)} M_{n_v}(I)$, *where* $n_v = |\Lambda_v|$ *and* Λ_v *is the set of paths* $\alpha \in \mathrm{Path}(E)$ *such that* $r(\alpha)$ *meets* $T(v)$ *for the first time* $r(\alpha)$.

Proof. For the first, becaused that every $u \in P_l(E), I\triangleleft_m R$ then $L_R(E)Iu$ is a minimal left ideal of $L_R(E)$. The second, $(IP_l(E)) = (I\overline{P_l(E)})$ for hereditary $P_l(E)$ (Lemma 6). Furthermore, based on Lemma 7, Proposition 5, and Lemma 8 we have

$$(IP_l(E)) = \Bigg(I\Bigg(\coprod_{v\in P_l(E)}[v]\Bigg)\Bigg) = \bigoplus_{v\in P_l(E)}(Iv) \cong \bigoplus_{v\in P_l(E)} M_{n_v}(I). \qquad \square$$

The following example shows that

$$\mathrm{Soc}(L_R(E)) \neq \sum_{u\in P_l(E), I\triangleleft_m R} L_R(E)Iu.$$

In addition, it gives an overview of the minimal ideals of $L_R(E)$ and the two-sided ideal generated by Iv with $I \triangleleft_m R$, $v \in P_l(E)$.

Example 1. Consider the commutative unital ring $\mathbb{Z}_6$ and the graph F that follows

$$_{u_4}\bullet \leftarrow \bullet^{u_1} \rightarrow \bullet^{u_2} \rightarrow \bullet^{u_3}.$$

Then $L_{\mathbb{Z}_6}(F) \cong M_2(\mathbb{Z}_6) \oplus M_3(\mathbb{Z}_6)$ and $\mathrm{Soc}(L_{\mathbb{Z}_6}(F)) = L_{\mathbb{Z}_6}(F)$ since $\mathrm{Soc}(M_2(\mathbb{Z}_6) \oplus M_3(\mathbb{Z}_6)) = M_2((\bar{2})) \oplus M_2((\bar{3})) \oplus M_3((\bar{2})) \oplus M_3((\bar{3})) = M_2(\mathbb{Z}_6) \oplus M_3(\mathbb{Z}_6)$, where $(\bar{2}) = \{\bar{0}, \bar{2}, \bar{4}\}$ and $(\bar{3})$ are minimal ideals of $\mathbb{Z}_6$. We have

$$\begin{aligned}
&\sum_{u\in P_l(E), I\triangleleft_m \mathbb{Z}_6} L_{\mathbb{Z}_6}(F)Iu \\
&\quad = L_{\mathbb{Z}_6}(F)(\bar{2})u_2 + L_{\mathbb{Z}_6}(F)(\bar{2})u_3 + L_{\mathbb{Z}_6}(F)(\bar{2})u_4 + L_{\mathbb{Z}_6}(F)(\bar{3})u_2 \\
&\qquad\quad + L_{\mathbb{Z}_6}(F)(\bar{3})u_3 + L_{\mathbb{Z}_6}(F)(\bar{3})u_4 \\
&\quad = L_{\mathbb{Z}_6}(F)((\bar{2}) + (\bar{3}))u_2 + L_{\mathbb{Z}_6}(F)((\bar{2}) + (\bar{3}))u_3 + L_{\mathbb{Z}_6}(F)((\bar{2}) + (\bar{3}))u_4 \\
&\quad \subsetneq \mathrm{Soc}(L_{\mathbb{Z}_6}(F)),
\end{aligned}$$

where

$$L_{\mathbb{Z}_6}(F)(\bar{2})u_2 \cong 0 \oplus \begin{pmatrix} 0 & \bar{2} & 0 \\ 0 & \bar{2} & 0 \\ 0 & \bar{2} & 0 \end{pmatrix}, \qquad L_{\mathbb{Z}_6}(F)(\bar{2})u_3 \cong 0 \oplus \begin{pmatrix} 0 & 0 & \bar{2} \\ 0 & 0 & \bar{2} \\ 0 & 0 & \bar{2} \end{pmatrix},$$

$$L_{\mathbb{Z}_6}(F)(\bar{2})u_4 \cong \begin{pmatrix} 0 & \bar{2} \\ 0 & \bar{2} \end{pmatrix}, \qquad L_{\mathbb{Z}_6}(F)(\bar{3})u_2 \cong 0 \oplus \begin{pmatrix} 0 & \bar{3} & 0 \\ 0 & \bar{3} & 0 \\ 0 & \bar{3} & 0 \end{pmatrix},$$

$$L_{\mathbb{Z}_6}(F)(\bar{3})u_3 \cong 0 \oplus \begin{pmatrix} 0 & 0 & \bar{3} \\ 0 & 0 & \bar{3} \\ 0 & 0 & \bar{3} \end{pmatrix}, \qquad L_{\mathbb{Z}_6}(F)(\bar{3})u_4 \cong \begin{pmatrix} 0 & \bar{3} \\ 0 & \bar{3} \end{pmatrix}$$

Note that $e_1^* \notin \sum_{u \in P_l(F), I \triangleleft_m \mathbb{Z}_6} L_{\mathbb{Z}_6}(F)Iu$.

Suppose that $e_1^* \in \sum_{u \in P_l(F), I \triangleleft_m \mathbb{Z}_6} L_{\mathbb{Z}_6}(F)Iu$, then $e_1^* = \alpha_1 u_2 + \alpha_2 u_3 + \alpha_3 u_4$ for some $\alpha_i \in L_{\mathbb{Z}_6}(F)((\bar{2}) + (\bar{3}))$. But $e_1^* = e_1^* u_1 = (\alpha_1 u_2 + \alpha_2 u_3 + \alpha_3 u_4)u_1 = 0$ which is a contradiction. Furthermore, based on the definition of the ideal in $L_R(E)$ generated by Iu, we have ideals in $L_{\mathbb{Z}_6}(F)$:

$$((\bar{2})u_2) = ((\bar{2})u_3) \cong M_3((\bar{2})); ((\bar{2})u_4) \cong M_2((\bar{2})),$$
$$((\bar{3})u_2) = ((\bar{3})u_3) \cong M_3((\bar{3})); ((\bar{3})u_4) \cong M_2((\bar{3})),$$

which are minimal.

Theorem 3. *Let E be an arbitrary graph and R be a semiprime commutative unital ring. Then* $\mathrm{Soc}(L_R(E)) = \bigoplus_{I \triangleleft_m R}(IP_l(E)) = \bigoplus_{I \triangleleft_m R}(I\overline{P_l(E)})$.

Proof. It is obvious that $\bigoplus_{I \triangleleft_m R}(IP_l(E)) = \bigoplus_{I \triangleleft_m R}(I\overline{P_l(E)})$ from Proposition 6. Furthermore, we show that $\mathrm{Soc}(L_R(E)) = \bigoplus_{I \triangleleft_m R}(IP_l(E))$. Let any minimal left ideal $\Im \in L_R(E)$. The Leavitt path algebra $L_R(E)$ is semiprime for the semiprimeness of R (Proposition 1). By [10, Subsection 3.4. Proposition 2], there is a non zero $\mu = \mu^2 \in L_R(E)$ such that $\Im = L_R(E)\mu$. Based on Proposition 4, we have $\Im = L_R(E)\mu \cong L_R(E)Iv$ where I is a minimal ideal of R and $v \in P_l(E)$. Assume that $\phi : L_R(E)\mu \to L_R(E)Iv$ is an $L_R(E)$-module isomorphism and write $\phi(\mu) = kxv$ and $\phi^{-1}(kv) = k'y\mu$ where $k, k' \in I$, $x, y \in L_R(E)$. Then

$$\mu = \phi^{-1}(\phi(\mu)) = \phi^{-1}(kxv^2) = xv\phi^{-1}(kv) = (xv)(k'y\mu) = (k'xv)(y\mu)$$

$$kv = \phi(\phi^{-1}(kv)) = \phi(k'y\mu^2) = k'y\mu\phi(\mu) = (k'y\mu)(kxv) = k(y\mu)(k'xv)$$

If $\alpha = k'xv, \beta = y\mu$ then $\alpha, \beta \in L_R(E)$ such that $\mu = \alpha\beta$ and $kv = k\beta\alpha$. We find that $v = \beta\alpha$ for $kv \neq 0, k(v - \beta\alpha) = 0,$ so that $\mu = \mu^2 = \alpha(\beta\alpha)\beta = \alpha v\beta = k'xv\beta = x(k'v)\beta$. Hence, $\mu \in (Iv) \subset (IP_l(E))$ for $x, \beta \in L_R(E)$.

Conversely, take any minimal ideal I of R and $v \in P_l(E)$. By Proposition 6, we have $L_R(E)Iv \subset \mathrm{Soc}(L_R(E))$. Since the socle is always a two-sided ideal, then $L_R(E)IvL_R(E) \subset \mathrm{Soc}(L_R(E))$ so that $(IP_l(E)) \subset \mathrm{Soc}(L_R(E))$. Hence, $\bigoplus_{I \triangleleft_m R}(IP_l(E)) \subseteq \mathrm{Soc}(L_R(E))$. □

Conclusion

The Reduction Theorem in [2, Theorem 2.2.11] can be applied to the Leavitt path algebra $L_R(E)$ over a commutative unital ring R on a (directed) graph E. It is very important to prove that $L_R(E)$ is semiprime if and only if R is semiprime. Therefore, every Leavitt path Algebra $L_K(E)$ over field K is always semiprime, since every field is semiprime.

We denote $P_l(E)$ be the set of all line points, i.e., the vertices whose tree contains neither bifurcations nor cycles. It is quite a role in determining socle of the semiprime $L_R(E)$. For every minimal left ideal $\Im$ in the semiprime $L_R(E)$, then there is a minimal ideal I in the semiprime commutative unital R such that $\Im = L_R(E)Iv$ for some $v \in P_l(E)$. On the other hand, for any minimal ideal I in the semiprime commutative unital R, $L_R(E)Iv$ is a minimal left ideal of $L_R(E)$ if and only if $v \in P_l(E)$. Furthermore, let E be an arbitrary graph and R a semiprime commutative unital ring, then we have:

1) $\sum_{u \in P_l(E), I \triangleleft_m R} L_R(E)Iu \subset \mathrm{Soc}(L_R(E))$
2) $(IP_l(E)) = (I\overline{P_l(E)}) \cong \bigoplus_{v \in P_l(E)} M_{n_v}(I)$, where $n_v = |\Lambda_v|$ and Λ_v is the set of paths $\alpha \in \mathrm{Path}(E)$ such that $r(\alpha)$ meets $T(v)$ for the first time $r(\alpha)$.
3) $\mathrm{Soc}(L_R(E)) = \bigoplus_{I \triangleleft_m R}(IP_l(E)) = \bigoplus_{I \triangleleft_m R}(I\overline{P_l(E)})$.

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