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Quasi-idempotents in finite semigroup of full order-preserving transformations*

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Abstract. Let X_n be the finite set $\{1, 2, 3 \cdots, n\}$ and $\mathcal{O}_n$ defined by $O_n = \{\alpha \in T_n : (\forall x, y \in X_n),\ x \leqslant y \to x\alpha \leqslant y\alpha\}$ be the semigroup of full order-preserving mapping on X_n. A transformation α in $\mathcal{O}_n$ is called quasi-idempotent if $\alpha \neq \alpha^2 = \alpha^4$. We characterise quasi-idempotent in $\mathcal{O}_n$ and show that the semigroup $\mathcal{O}_n$ is quasi-idempotent generated. Moreover, we obtained an upper bound for quasi-idempotents rank of $\mathcal{O}_n$, that is, we showed that the cardinality of a minimum quasi-idempotents generating set for $\mathcal{O}_n$ is less than or equal to $\lceil \frac{3(n-2)}{2} \rceil$ where $\lceil x \rceil$ denotes the least positive integer m such that $x \leqslant m < x + 1$.

1. Introduction

Let $\mathcal{T}_n$ be the semigroup of full transformations of a finite set $X_n = \{1, 2, \cdots, n\}$. It is well known that every finite semigroup is realizable as a subsemigroup of $\mathcal{T}_n$. Hence, the importance of $\mathcal{T}_n$ to the theory of semigroups is similar to that of $\mathcal{S}_n$ to group theory. Since the work of Howie [8], establishing that every singular map in $\mathcal{T}_n$ is expressible as a product (that is composition) of idempotent singular maps ($\alpha \in \mathcal{T}_n \setminus \mathcal{S}_n$

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satisfying $\alpha^2 = \alpha$), there have been many articles concerned with this idea in $\mathcal{T}_n$ (see for example, [3, 6, 9–12, 14, 17]).

Idempotent elements have a significant role in semigroup theory, yet there are still some kinds of semigroups (such as inverse semigroups) where this role is of little or no significance. The set of idempotents cannot be a generating set in an inverse semigroup because it forms a subsemigroup (see [13]). Garba and Imam [5] used a special type of semigroup elements known as a "quasi-idempotents" (non-idempotent elements whose squares are idempotents) to generate the inverse semigroup of all partial one-to-one mappings of X_n. An element a in a semigroup S is called a *quasi-idempotent* if $a \neq a^2 = a^4$.

The first appearance of quasi-idempotent elements was in [18] where it was shown that the semigroup of all decreasing partial injections is quasi-idempotents generated and that the size of its minimal generating set is equal to $\frac{n(n+1)}{2}$. Madu and Garba [16] proved that each element in the semigroup $\mathcal{IO}_n$, of all order-preserving partial injections of X_n, is expressible as a product of quasi-idempotents of height $n-1$, and that the minimal size of such generating set is $2(n-1)$. In [4] the authors proved that $Sing_n$, the semigroup of singular full transformations of X_n is quasi-idempotent generated and that the minimal size of such generating set is $\frac{n(n-1)}{2}$. A more recent combinatorial study of quasi-idempotent generating elements is carried out in [15] for $Sing_n$. Sizes of minimal generating sets of quasi-idempotents in the semigroups $\mathcal{I}_n$ and its ideals were obtained by Bugay [1, 2].

In this paper, we generate the semigroup of all order-preserving full transformations of X_n using quasi-idempotents. We also obtain an upper bound for the minimum number of such quasi-idempotents required to generate the semigroup. Throughout the paper statements like $1 \leqslant i \leqslant n$ should be understood as 'all integer i from 1 to n'.

2. Preliminaries

For $n \geqslant 2$, we denote by $Sing_n = \mathcal{T}_n \setminus \mathcal{S}_n$ the semigroup of all singular mappings in $\mathcal{T}_n$, that is

$$Sing_n = \{\alpha \in \mathcal{T}_n : |\mathrm{im}\alpha| \leqslant n-1\}.$$

The subsemigroup of $Sing_n$ consisting of all order-preserving maps will be denoted by $\mathcal{O}_n$, that is

$$\mathcal{O}_n = \{\alpha \in Sing_n : (\forall x, y \in X_n)\; x \leqslant y \implies x\alpha \leqslant y\alpha\}.$$

The structure of $\mathcal{O}_n$ was described by Gomes and Howie [7] as follows: The semigroup $\mathcal{O}_n$ is easily seen to be a regular semigroup (i.e for all $\alpha \in \mathcal{O}_n$, there is $\beta \in \mathcal{O}_n$ such that $\alpha\beta\alpha = \alpha$). Hence, by [13, Proposition 1.4.11], the Green's $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ relations on $\mathcal{O}_n$ are defined as follows:

$$\begin{aligned} \alpha\mathcal{L}\beta \quad &\text{if and only if} \quad \operatorname{im}(\alpha) = \operatorname{im}(\beta), \\ \alpha\mathcal{R}\beta \quad &\text{if and only if} \quad \ker(\alpha) = \ker(\beta), \\ \alpha\mathcal{J}\beta \quad &\text{if and only if} \quad |\operatorname{im}(\alpha)| = |\operatorname{im}(\beta)|. \end{aligned}$$

The Grean's $\mathcal{H}$-relation in $\mathcal{O}_n$ is the identity relation, for once an $\operatorname{im}(\alpha)$ and $\ker(\alpha)$ are fixed, there is precisely one and only one order-preserving mapping $\alpha \in \mathcal{O}_n$ having the given image and kernel. Also, for $\alpha \in \mathcal{O}_n$, the $\ker(\alpha)$-classes are convex subsets of X_n, in the sense that

$$\forall x, y \in C \ (x \leqslant z \leqslant y \implies z \in C).$$

Thus, each $\alpha \in \mathcal{O}_n$, with image set $\operatorname{im}(\alpha) = \{b_1 < b_2 < \cdots < b_r\}$, can be written, in array notation, as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

where the $\ker(\alpha)$-classes $X_n/\ker(\alpha) = \{A_1, A_2, \cdots A_r\}$ are convex subsets of X_n satisfying $A_1 < A_2 < \cdots < A_r$. Here, we write $A_i < A_j$ to mean $a_i < a_j$ for all $a_i \in A_i$ and $a_j \in A_j$. Each $\ker(\alpha)$-class A_i is called a *block* of α.

Definition 1. For the map $\alpha \in \mathcal{O}_n$, a block A_i of α is called *stationary* if $A_i\alpha = b_i \in A_i$ otherwise it is called *non-stationary*.

If the equivalence on $X_n = \{1, 2, \cdots, n\}$ whose sole non singleton class is $\{i, j\}$ is denoted by $|i, j|$, we see that the $\mathcal{R}$-classes within the $\mathcal{J}$-class $J_{n-1} = \{\alpha \in \mathcal{O}_n : |\operatorname{im}(\alpha)| = n - 1\}$ are indexed by the $n - 1$ such equivalences

$$|n-1, n|, |n-2, n-1|, \ldots, |1, 2|.$$

The $\mathcal{L}$-classes within the J_{n-1} correspond to the n possible subsets of X_n of cardinality $(n - 1)$

$$X_n \setminus \{n\}, X_n \setminus \{n-1\}, \ldots, X_n \setminus \{1\}.$$

The unique element $\alpha \in \mathcal{O}_n$ in the $\mathcal{H}$-class determined by the equivalence $\operatorname{Ker}(\alpha) = |i, i+1|$ and the subset $\operatorname{im}(\alpha) = X_n \setminus \{k\}$ will be called *increasing* if $k \leqslant i$ and *decreasing* if $k \geqslant i+1$. The element α is an idempotent if and only if $k = i$ or $k = i+1$. There are $(n-1)$ increasing idempotents of the form

$$\begin{pmatrix} i \\ i+1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i, i+1 & i+2 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i+1 & i+2 & \cdots & n \end{pmatrix} \quad (1 \leqslant i \leqslant n-1)$$

and $(n-1)$ decreasing idempotents of the form

$$\begin{pmatrix} i+1 \\ i \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i, i+1 & i+2 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i & i+2 & \cdots & n \end{pmatrix} \quad (1 \leqslant i \leqslant n-1).$$

Thus, if $E_1 = \{\alpha \in \mathcal{O}_n : |\mathrm{im}(\alpha)| = n-1, \alpha^2 = \alpha\}$, then $|E_1| = 2(n-1)$.

We denote the increasing idempotents in J_{n-1} by $e_i = \binom{i}{i+1}$ $(1 \leqslant i \leqslant n-1)$, and the decreasing idempotents in J_{n-1} by $f_i = \binom{i}{i+1}$ $(1 \leqslant i \leqslant n-1)$. From Howie [9] it is known that $\mathcal{O}_n = \langle E_1 \rangle$ and in Gomes and Howie [7] we have $\mathrm{rank}(\mathcal{O}_n) = n$ and $\mathrm{idrank}(\mathcal{O}_n) = 2(n-1)$ (where $\mathrm{rank}(\mathcal{O}_n)$ denotes the minimum number of generators for $\mathcal{O}_n$ and $\mathrm{idrank}(\mathcal{O}_n)$ denotes the minimum number of idempotent generators for $\mathcal{O}_n$). Also, $\mathcal{O}_n = \langle e_1, \ldots, e_{n-1}, \gamma \rangle$ where $e_i = \binom{i}{i+1}$ $(i = 1, 2, \ldots n-1)$ and $\gamma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 2 & \cdots & n-1 \end{pmatrix}$. Thus, for any subset $A \subseteq J_{n-1}$ we have

$$\mathcal{O}_n = \langle A \rangle \quad \text{if and only if} \quad \{e_1, \ldots, e_{n-1}, \gamma\} \subseteq \langle A \rangle. \tag{1}$$

Our first result is a characterization of quasi-idempotent elements in $\mathcal{O}_n$.

Theorem 1. *An element $\alpha \in \mathcal{O}_n$ is a quasi-idempotent if and only if the image of each non-stationary block of α is contained in a stationary block of α.*

Proof. If $\alpha \in \mathcal{O}_n$ contains no non-stationary blocks, then the argument is trivial. Suppose $\alpha \in \mathcal{O}_n$ contains both stationary block $A_1, \ldots, A_s$ and non-stationary blocks $B_1, \ldots, B_t$ in which every non-stationary block is mapped into a stationary block. Let $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant t$ be indices such that $B_j\alpha \in A_i$. If $i < j$, then, for each index k satisfying $A_i < B_k < B_j$, we have $B_k\alpha \in A_i$, for otherwise α will fail to be order-preserving. Similar conclusion is made for each index k satisfying $B_j < B_k < A_i$ if $i > j$.

Now, for each $1 \leqslant i \leqslant s$, denote by $l(i) = l$, the number of non-stationary blocks B_j of α for which $B_j\alpha \in A_i$. Then the union $\tilde{A}_i = A_i \cup B_{j_1} \cup \cdots \cup B_{j_l}$ is a stationary block of α^2. In this case, $\mathrm{Ker}(\alpha^2)$-classes are $\tilde{A}_1, \ldots, \tilde{A}_s$ and all are stationary. Therefore α^2 is an idempotent.

Conversely, suppose α is a quasi-idempotent. Then α must contain atleast one stationary block and one non-stationary block. Let B be an arbitrary non-stationary block of α. If $B\alpha$ is not contained in a stationary block, it must then be contained in a non-stationary block of α say C. And so, the block of α^2 containing C will not be fixed by α^2, contradicting the choice of α as a quasi-idempotent. Thus, every non-stationary block of α must be mapped into a stationary block of α. □

Example 1. Let $\alpha = \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} & \{9,10\} & \{11\} \\ 1 & 2 & 5 & 6 & 9 & 10 \end{pmatrix} \in \mathcal{O}_{11}$. Then α is a quasi-idempotent in $\mathcal{O}_{11}$ since all its non-stationary blocks are mapped into its stationary blocks. In fact

$$\alpha^2 = \begin{pmatrix} \{1,2,3,4\} & \{5,6,7,8\} & \{9,10,11\} \\ 1 & 5 & 9 \end{pmatrix}$$

On the other, the map $\beta = \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} & \{9,10\} \\ 1 & 2 & 3 & 5 & 9 \end{pmatrix} \in \mathcal{O}_{10}$ is not a quasi-idempotent as its non-stationary blocks $\{5,6\}$, $\{7,8\}$ are mapped into non-stationary blocks. Here

$$\beta^2 = \begin{pmatrix} \{1,2,3,4\} & \{5,6\} & \{7,8\} & \{9,10\} \\ 1 & 2 & 3 & 9 \end{pmatrix}.$$

An immediate consequence of Theorem 1 is the following.

Corollary 1. *An element $\alpha \in \mathcal{O}_n$ of height $n-1$ is a quasi-idempotent if and only if $s(\alpha) = 2$, where $s(\alpha) = |\{x \in X_n : x\alpha \neq x\}|$.*

3. Products of quasi-idempotents

From Corollary 1 it is clear that, in $\mathcal{O}_n$, an element $\alpha \in J_{n-1}$ with $\ker(\alpha) = |i, i+1|$ and $\operatorname{im}(\alpha) = X_n \setminus \{k\}$ is a quasi-idempotent if and only if $k = i-1$ or $k = i+2$. In the former α is increasing and in the latter α is decreasing.

For each $i = 2, 3, \ldots, n-1$ we denote the increasing quasi-idempotent elements in J_{n-1} by

$$\delta_i = \begin{pmatrix} i-1 & i \\ i & i+1 \end{pmatrix},$$

and the decreasing quasi-idempotent elements in J_{n-1} by

$$\mu_i = \begin{pmatrix} i & i+1 \\ i-1 & i \end{pmatrix}.$$

Then, it is clear that $\ker(\delta_i) = |i, i+1|$, $\ker(\mu_i) = |i-1, i|$, $\mathrm{im}(\delta_i) = X_n \setminus \{i-1\}$, $\mathrm{im}(\mu_i) = X_n \setminus \{i+1\}$. Also, the set $QE_1 = \{\delta_i, \mu_i : 2 \leqslant i \leqslant n-1\}$, of all quasi-idempotents in J_{n-1}, is of cardinality

$$|QE_1| = 2(n-2). \tag{2}$$

Theorem 2. *For $n \geqslant 4$, the semigroup $\mathcal{O}_n$ is generated by quasi-idempotents of height $n-1$, that is $\mathcal{O}_n = \langle QE_1 \rangle$.*

Proof. According to (1), to prove this theorem it is enough to show that for each $i = 1, 2, \ldots, n-1$, $\epsilon_i, \gamma \in \langle QE_1 \rangle$. But then one quickly verifies that $\epsilon_i = \mu_{i+1}\delta_{i+1}$ $(i = 1, 2, \ldots, n-2)$, $\epsilon_{n-1} = \delta_{n-1}\mu_{n-2}$ and $\gamma = \mu_2\mu_3 \cdots \mu_{n-1}$. □

We end this section with the following observation on product of elements in J_{n-1}.

Lemma 1. *For $n \geqslant 4$, let $\alpha \in J_{n-1} \setminus (QE_1 \cup E_1)$ and $\beta \in QE_1 \cup E_1$ be such that $\alpha\beta \in J_{n-1}$. Then α and $\alpha\beta$ are either both decreasing or both increasing.*

Proof. Let $\mathrm{im}(\alpha) = X_n \setminus \{i\}$ and $\ker(\alpha) = |j, j+1|$. Suppose α is decreasing, then $4 \leqslant i \leqslant n$ and $j+1 \leqslant i$. Thus we may write

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & j, j+1 & j+2 & \cdots & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & j & j+1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix}.$$

Note that since $\alpha\beta \in J_{n-1}$, we must have

$$\ker(\beta) = \begin{cases} |i-1, i| \text{ or } |i, i+1| & \text{for } i = 4, 5, \ldots, n-1; \\ |n-1, n| & \text{for } i = n. \end{cases}$$

If $i = n$, then $\beta \in \{f_{n-1}, e_{n-1}, \delta_{n-1}\}$ and

$$\alpha\beta = \begin{cases} \begin{pmatrix} 1 & 2 & \cdots & j, j+1 & j+2 & \cdots & n \\ 1 & 2 & \cdots & j & j+1 & \cdots & n-1 \end{pmatrix} & \text{if } \beta = f_{n-1}, \\[2ex] \begin{pmatrix} 1 & 2 & \cdots & j, j+1 & j+2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & j & j+1 & \cdots & n-2 & n \end{pmatrix} & \text{if } \beta = e_{n-1}, \\[2ex] \begin{pmatrix} 1 & 2 & \cdots & j, j+1 & j+2 & \cdots & n-2 & n-1 & n \\ 1 & 2 & \cdots & j & j+1 & \cdots & n-3 & n-1 & n \end{pmatrix} & \text{if } \beta = \delta_{n-1}, \end{cases}$$

which are decreasing maps in J_{n-1}.

If $4 \leqslant i \leqslant n-1$, then $\beta \in \{f_{i-1}, f_i, e_{i-1}, e_i, \delta_{i-1}, \delta_i, \mu_i, \mu_{i+1}\}$ (with the exception of μ_{i+1} when $i = n-1$). Thus,

$$\alpha\beta = \begin{cases} \begin{pmatrix} 1 & 2 & \cdots & j, j+1 & j+2 & \cdots & i-1 & i & \cdots & n \\ 1 & 2 & \cdots & j & j+1 & \cdots & i-2 & i-1 & \cdots & n \end{pmatrix} & \text{if } \beta = f_{i-1}, \\ \begin{pmatrix} 1 & 2 & \cdots & j, j+1 & j+2 & \cdots & i+1 & i+2 & \cdots & n \\ 1 & 2 & \cdots & j & j+1 & \cdots & i & i+2 & \cdots & n \end{pmatrix} & \text{if } \beta = f_i, \\ \begin{pmatrix} 1 & 2 & \cdots & j, j+1 & j+2 & \cdots & i-1 & i & \cdots & n \\ 1 & 2 & \cdots & j & j+1 & \cdots & i-2 & i & \cdots & n \end{pmatrix} & \text{if } \beta = e_{i-1}, \\ \begin{pmatrix} 1 & 2 & \cdots & j, j+1 & j+2 & \cdots & i-1 & i & \cdots & n \\ 1 & 2 & \cdots & j & j+1 & \cdots & i-2 & i-1 & \cdots & n \end{pmatrix} & \text{if } \beta = e_i, \\ \begin{pmatrix} 1 & 2 & \cdots & j, j+1 & j+2 & \cdots & i-2 & i-1 & \cdots & n \\ 1 & 2 & \cdots & j & j+1 & \cdots & i-3 & i-1 & \cdots & n \end{pmatrix} & \text{if } \beta = \delta_{i-1}, \\ \begin{pmatrix} 1 & 2 & \cdots & j, j+1 & j+2 & \cdots & i-1 & i & \cdots & n \\ 1 & 2 & \cdots & j & j+1 & \cdots & i-2 & i & \cdots & n \end{pmatrix} & \text{if } \beta = \delta_i, \\ \begin{pmatrix} 1 & 2 & \cdots & j, j+1 & j+2 & \cdots & i+1 & i+2 & \cdots & n \\ 1 & 2 & \cdots & j & j+1 & \cdots & i & i+2 & \cdots & n \end{pmatrix} & \text{if } \beta = \mu_i, \\ \begin{pmatrix} 1 & 2 & \cdots & j, j+1 & j+2 & \cdots & i+2 & i+3 & \cdots & n \\ 1 & 2 & \cdots & j & j+1 & \cdots & i+1 & i+3 & \cdots & n \end{pmatrix} & \begin{array}{l} \text{if } \beta = \mu_{i+1} \\ (i \neq n-1), \end{array} \end{cases}$$

which are clearly decreasing maps in J_{n-1}.

Similarly, if α is increasing, then $1 \leqslant i \leqslant n-3$ and $j \geqslant i$. Thus we may write

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & \cdots & j, j+1 & j+2 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i+1 & \cdots & j+1 & j+2 & \cdots & n \end{pmatrix}.$$

Note that since $\alpha\beta \in J_{n-1}$, we must have

$$\ker(\beta) = \begin{cases} |1,2| & \text{for } i = 1, \\ |i-1, i| \text{ or } |i, i+1| & \text{for } i = 2, 3, \ldots, n-3. \end{cases}$$

If $i = 1$, then $\beta \in \{f_1, e_1, \mu_2\}$ and

$$\alpha\beta = \begin{cases} \begin{pmatrix} 1 & 2 & 3 & \cdots & j-1 & j, j+1 & j+2 & \cdots & n \\ 1 & 3 & 4 & \cdots & j & j+1 & j+2 & \cdots & n \end{pmatrix} & \text{if } \beta = f_1, \\ \\ \begin{pmatrix} 1 & 2 & 3 & \cdots & j-1 & j, j+1 & j+2 & \cdots & n \\ 2 & 3 & 4 & \cdots & j & j+1 & j+2 & \cdots & n \end{pmatrix} & \text{if } \beta = e_1, \\ \\ \begin{pmatrix} 1 & 2 & 3 & \cdots & j-1 & j, j+1 & j+2 & \cdots & n \\ 1 & 2 & 4 & \cdots & j & j+1 & j+2 & \cdots & n \end{pmatrix} & \text{if } \beta = \mu_2, \end{cases}$$

which are increasing maps in J_{n-1}. If $2 \leqslant i \leqslant n-3$, then $\beta \in \{\delta_{i-1}, \delta_i, \mu_i, \mu_{i+1}\}$ (with the exception of δ_{i-1} when $i = 2$). Thus,

$$\alpha\beta = \begin{cases} \begin{pmatrix} 1 & 2 & \cdots & i-3 & i-2 & \cdots & j, j+1 & j+2 & \cdots & n \\ 1 & 2 & \cdots & i-3 & i-1 & \cdots & j+1 & j+2 & \cdots & n \end{pmatrix} & \text{if } \beta = \delta_{i-1} \\ & \quad (i \neq 2), \\ \begin{pmatrix} 1 & 2 & \cdots & i-2 & i-1 & \cdots & j, j+1 & j+2 & \cdots & n \\ 1 & 2 & \cdots & i-2 & i & \cdots & j+1 & j+2 & \cdots & n \end{pmatrix} & \text{if } \beta = \delta_i, \\ \\ \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & j, j+1 & j+2 & \cdots & n \\ 1 & 2 & \cdots & i & i+2 & \cdots & j+1 & j+2 & \cdots & n \end{pmatrix} & \text{if } \beta = \mu_i, \\ \\ \begin{pmatrix} 1 & 2 & \cdots & i+1 & i+2 & \cdots & j, j+1 & j+2 & \cdots & n \\ 1 & 2 & \cdots & i+1 & i+3 & \cdots & j+1 & j+2 & \cdots & n \end{pmatrix} & \text{if } \beta = \mu_{i+1}, \end{cases}$$

which are all increasing maps in J_{n-1}. □

4. Upper Bound for Quasi-Idempotents Rank of $\mathcal{O}_n$

In this section, we attempt to find the minimum size of a subset of QE_1 generating $\mathcal{O}_n$. This is known as the quasi-idempotent rank of $\mathcal{O}_n$ denoted by $\text{qrank}(\mathcal{O}_n)$. That is

$$\text{qrank}(\mathcal{O}_n) = \min\{|A| : A \subseteq QE_1 \text{ and } \langle A \rangle = \mathcal{O}_n\}.$$

We first note that any generating set for $\mathcal{O}_n$ must cover both the $\mathcal{L}$-classes and the $\mathcal{R}$-classes in J_{n-1}. Also, since $\mathcal{O}_n$ has n $\mathcal{L}$-classes and $n-1$

$\mathcal{R}$-classes, it follows from (2) that

$$n \leqslant \text{qrank}(\mathcal{O}_n) \leqslant 2(n-2). \tag{3}$$

Theorem 3. *For $n = 4$ or 5; we have*

$$\text{qrank}(\mathcal{O}_n) = \begin{cases} 4 & \text{if } n = 4; \\ 6 & \text{if } n = 5. \end{cases}$$

Proof. In $\mathcal{O}_4$, $QE_1 = \{\mu_2, \mu_3, \delta_2, \delta_3\}$ Therefore, it follows from (3) that $\text{qrank}(\mathcal{O}_4) = 4$. In $\mathcal{O}_5$, $QE_1 = \{\mu_2, \mu_3, \mu_4, \delta_2, \delta_3, \delta_4\}$ and since, in this case, no proper subset of QE_1 covers both the $\mathcal{L}$-classes and the $\mathcal{R}$-classes in J_{n-1} we have $\text{qrank}(\mathcal{O}_5) = 6$. □

Theorem 4. *Let $n \geqslant 6$. Then*

$$\text{qrank}(\mathcal{O}_n) \leqslant \lceil \frac{3}{2}(n-2) \rceil$$

Proof. For $n \geqslant 6$, any subset of QE_1 that generates $\mathcal{O}_n$ must contain the six quasi-idempotents

$$\mu_2, \mu_{n-2}, \mu_{n-1}, \delta_2, \delta_3, \delta_{n-1},$$

since each of these is alone in either one of its $\mathcal{L}$-class or $\mathcal{R}$-class.

Now, for $n = 6, 8, 10, \ldots$, let

$$A = \{\delta_2, \delta_3, \ldots, \delta_{n-3}, \delta_{n-1}, \mu_2, \mu_4, \ldots, \mu_{n-4}, \mu_{n-2}, \mu_{n-1}\}$$

a set of quasi-idempotents in $QE_1(\mathcal{O}_n)$ with

$$|A| = (n-3) + (\frac{n}{2} - 1) + 1 = \frac{3}{2}(n-2).$$

To show that $\langle A \rangle = \mathcal{O}_n$ it suffices, by Theorem 2, to show that

$$QE_1(\mathcal{O}_n) \setminus A = \{\delta_{n-2}, \mu_3, \mu_5, \ldots, \mu_{n-3}\} \subseteq \langle A \rangle.$$

But then it is easy to see that

$$\begin{aligned} \delta_{n-2} &= \mu_{n-1}\delta_{n-1}\delta_{n-3}\mu_{n-4} \\ \mu_i &= \delta_{i-1}\mu_{i-1}\mu_{i+1}\delta_{i+2} \qquad (i = 2, 4, \ldots, n-4). \end{aligned}$$

Thus, $\langle A \rangle = \mathcal{O}_n$ and so

$$\text{qrank}(\mathcal{O}_n) \leqslant \frac{3}{2}(n-2) = \lceil \frac{3}{2}(n-2) \rceil.$$

Also, for $n = 7, 9, 11, \ldots$, let

$$B = \{\delta_2, \delta_3, \ldots, \delta_{n-4}, \delta_{n-2}, \delta_{n-1}, \mu_2, \mu_4, \ldots, \mu_{n-3}, \mu_{n-2}, \mu_{n-1}\}$$

a set of quasi-idempotents in $QE_1(\mathcal{O}_n)$ with

$$|B| = (n-3) + \left(\frac{n-1}{2}\right) + 1 = \frac{1}{2}(3n-5).$$

To show that $\langle B \rangle = \mathcal{O}_n$ it suffices, by Theorem 2, to show that

$$QE_1(\mathcal{O}_n) \setminus B = \{\delta_{n-3}, \mu_3, \mu_5, \ldots, \mu_{n-4}\} \subseteq \langle B \rangle.$$

But then again it is easy to see that

$$\begin{aligned} \delta_{n-3} &= \mu_{n-2}\delta_{n-2}\delta_{n-4}\mu_{n-5} \\ \mu_i &= \delta_{i-1}\mu_{i-1}\mu_{i+1}\delta_{i+2} \qquad (i = 2, 4, \ldots, n-3). \end{aligned}$$

Thus, $\langle B \rangle = \mathcal{O}_n$ and so

$$\text{qrank}(\mathcal{O}_n) \leqslant \frac{1}{2}(3n-5) = \frac{3}{2}(n-2) + \frac{1}{2} = \lceil \frac{3}{2}(n-2) \rceil. \qquad \square$$

Remark 1. We suspect that the upper bound for qrank$(\mathcal{O}_n)$ found in Theorem 4 is best possible, that is, no fewer quasi-idempotents will suffice in generating $\mathcal{O}_n$. Currently, we are unable to provide proof that this is the case.

References

[1] Bugay, L (2019). Quasi-idempotent ranks of some permutation groups and transformation semigroups. *Turkish Journal of Mathematics* 43:2390-2395.

[2] Bugay, L. (2021). Quasi-idempotent rank of proper ideals in finite symmetric inverse semigroup. *Turkish Journal of Mathematics* 45:281-287.

[3] Garba, G. U. (1994). On idempotent rank of certain semigroups of transformations. *Portugaliae Mathematica* 51:185-204.

[4] Garba, G. U.,Tanko, A. I., Madu, B. A. (2011). Products and rank of quasi-idempotents in finite full transformations semigroups. *JMI, International Journal of Mathematical Sciences* 2(1):12-19.

[5] Garba, G. U., Imam, A. T. (2016). Product of quasi-idempotents in finite symmetric inverse semigroups. *Semigroup Forum* 92(3):645-658.

[6] Gomes, G. M. S., Howie, J. M. (1987). On the ranks of certain semigroups of transformations. *Semigroup Forum* 45:272-282.

[7] Gomes, G. M. S., Howie, J. M. (1992). On the ranks of certain finite semigroups of transformations. *Proc. Math. Proc. Cambridge Phil. Soc.* 101:395-403.

[8] Howie, J. M. (1966). The subsemigroup generated by the idempotents of a full transformation semigroup. *J. London Math. Soc.* 41:707-716.

[9] Howie, J. M. (1971). Products of idempotents in certain semigroups of transformations. *Proc. Roy. Soc. Edinburgh* 17A:233-236.

[10] Howie, J. M. (1978). Idempotent generators in finite full transformation semigroups. *Proc. Roy. Soc. Edinburgh* 81A:317-323.

[11] Howie, J. M. (1980). Products of idempotents in a finite full transformation semigroup. *Proc. Roy. Soc. Edinburgh* 86A:243-254.

[12] Howie, J. M., McFadden, R. B. (1990). Idempotent rank in finite full transformation semigroups. *Proc. Roy. Soc. Edinburgh* 116A:161-167.

[13] Howie, J. M. (1995). Fundamentals of semigroup theory. *London Mathematical Society, New Series 12, The Clarendon Press, Oxford University Press*

[14] Howie, J. M., Robertson, R. B., Schein, B. M. (1988). A combinatorial property of finite full transformation semigroups. *Proc. Roy. Soc. Edinburgh* 109A:319-328.

[15] Imam, A. T., Ibrahim, M. J. (2022). On products of 3-paths in finite full transformation semigroups. *Algebra and Discrete Mathematics* 33(2):60-77.

[16] Madu, B. A., Garba, G. U. (2001). Quasi-idempotents in finite semigroups of order-preserving charts. *Research Journal of Sciences* 7(1 and 2):61-64.

[17] Saito, T. (1989). Products of idempotents in finite full transformation semigroups. *Semigroup forum* 39:295-309.

[18] Umar, A.. (1993). On the semigroup of partial one-one order-decreasing finite transformation. *Proc. Roy. Soc. Edinburgh* 123A:355-363.

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