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RESEARCH ARTICLE

Jacobson Hopfian modules

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Abstract. The study of modules by properties of their endomorphisms has long been of interest. In this paper we introduce a proper generalization of that of Hopfian modules, called Jacobson Hopfian modules. A right R-module M is said to be Jacobson Hopfian, if any surjective endomorphism of M has a Jacobson-small kernel. We characterize the rings R for which every finitely generated free R-module is Jacobson Hopfian. We prove that a ring R is semisimple if and only if every R-module is Jacobson Hopfian. Some other properties and characterizations of Jacobson Hopfian modules are also obtained with examples. Further, we prove that the Jacobson Hopfian property is preserved under Morita equivalences.

1. Introduction

Throughout this paper, R denotes an associative ring with identity and modules M are unitary right R-modules. We use the notations $\subseteq$, $\leqslant$ and $\leqslant^{\oplus}$ to denote inclusion, submodule and direct summand, respectively, and $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$, $\mathrm{End}(M)$ will denote the radical, the socle and the ring of endomorphisms of a module M. A submodule N of M is called a small submodule of M if whenever $N + K = M$ for some submodule K of M, we have $M = K$, and in this case we write $N \ll M$. A nonzero module M is called hollow, if every proper submodule of M is small in M. The study of modules by properties of their endomorphisms is a classical research subject. Hopfian and co-Hopfian groups, rings and modules have

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been studied by many authors since 1963 ([3], [4], [7], [8], [9], [11]). Recall that a module M is called co-Hopfian (resp. Hopfian) if any injective (resp. surjective) endomorphism of M is an isomorphism. Note that any Artinian module is co-Hopfian, and any Noetherian module is Hopfian, but the converse is not true in general. The additive group $\mathbb{Q}$ of rational numbers is a non-Noetherian non-Artinian $\mathbb{Z}$-module, which is Hopfian and co-Hopfian. In [7], a proper generalization of Hopfian modules, called generalized Hopfian modules, was given. A right R-module M is called generalized Hopfian, if any surjective endomorphism of M has a small kernel. Recall that the module M is called Dedekind finite if $fg = 1$ implies $gf = 1$ for each $f, g \in \mathrm{End}(M)$. Consequently, M is Dedekind finite if and only if M is not isomorphic to any proper direct summand of itself. In [7, Corollary 1.4], it is shown that the concepts of Dedekind finite modules, Hopfian modules and generalized Hopfian modules equivalent for every (quasi-)projective module. The socle of M is defined as the sum of all its simple submodules and can be shown to coincide with the intersection of all the essential submodules of M. It is a fully invariant submodule of M. Note that M is semisimple precisely when $M = \mathrm{Soc}(M)$ (see [1]). The radical of an R-module M defined as a dual of the socle of M, is the intersection of all maximal submodules of M, taking $\mathrm{Rad}(M) = M$ when M has no maximal submodules. A submodule K of M is said to be Jacobson-small in M $(K \ll_J M)$, in case $M = K + L$ with $\mathrm{Rad}(M/L) = M/L$, implies $M = L$ (see [10]). It is clear that if A is a small submodule of M, then A is a Jacobson-small submodule of M, but the converse is not true in general. By [10], if $\mathrm{Rad}(M) = M$ and $K \leqslant M$, then K is small in M if and only if K is Jacobson-small in M.

In [5], Ebrahimi, Khoramdel and Dolati introduced and studied the concept of δ-Hopfian modules. A right R-module M is called δ-Hopfian if any surjective endomorphism of M has a δ-small kernel. In [5, Theorem 3], it is shown that a ring R is semisimple if and only if every R-module is δ-Hopfian. In [6], El Moussaouy and Ziane introduced and studied the concept of μ-Hopfian modules. A right R-module M is called μ-Hopfian if any surjective endomorphism of M has a μ-small kernel. In [6, Theorem 2.10], it is shown that a ring R is semisimple if and only if every R-module is μ-Hopfian.

Motivated by the above-mentioned works, we are interested in introducing a new generalization of Hopfian modules namely Jacobson Hopfian modules. We call a module Jacobson Hopfian if any its surjective endomorphism has a Jacobson-small kernel. In Theorem 5 we prove that a ring R is semisimple if and only if every R-module is Jacobson Hopfian. Then for a semisimple ring, the properties δ-Hopfian, μ-Hopfian and Jacobson Hopfian are all equivalent.

The notion of Jacobson Hopfian modules form a proper generalization of that of Hopfian modules (Example 1), and in particular Noetherian modules. It is clear that every generalized Hopfian module is Jacobson Hopfian. Example 2 shows that the converse is not true, in general. Also, it shows that a Jacobson Hopfian module need not be Dedekind finite.

We discuss the following questions:

1) When does a module have the property that every of its surjective endomorphisms has a Jacobson-small kernel?

2) How can Jacobson Hopfian modules be used to characterize the base ring itself?

We present some equivalent properties of these modules (Theorem 1), and provide a characterization of semisimple rings in terms of Jacobson Hopfian modules by proving that a ring R is semisimple if and only if every R-module is Jacobson Hopfian (Theorem 5). Also we prove that the Jacobson Hopfian property is Morita invariant (Theorem 3).

In [7], Ghorbani and Haghany proved that if ACC holds on nonsmall submodules of M, then M is generalized Hopfian. In [6], El Moussaouy and Ziane proved that if ACC holds on non μ-small submodules of M, then M is μ-Hopfian. Also we know that Noetherian modules are Hopfian modules. Thus it is natural to show that if ACC or DCC holds on non Jacobson-small submodules of M, then M is Jacobson Hopfian (Proposition 2 and Proposition 3).

Finally, we characterize the rings R for which every finitely generated free R-module is Jacobson Hopfian. We prove that every finitely generated free R-module is Jacobson Hopfian if and only if every finitely generated projective R-module is Jacobson Hopfian if and only if $M_n(R)$ is Jacobson Hopfian as an $M_n(R)$-module for each $n \geqslant 1$ (Corollary 3).

We list some properties of Jacobson-small submodules that will be used in the paper.

Lemma 1. *[10]. Let M be an R-module.*

(1) *Let $A \leqslant B \leqslant M$. Then $B \ll_J M$ if and only if $A \ll_J M$ and $B/A \ll_J M/A$.*
(2) *Let A, B be submodules of M, then $A + B \ll_J M$ if and only if $A \ll_J M$ and $B \ll_J M$.*
(3) *Let A_1, A_2, ...,A_n are submodules of M. Then $A_i \ll_J M$, $\forall\ i = 1, ..., n$ if and only if $\sum_{i=1}^{n} A_i \ll_J M$.*
(4) *Let A, B be submodules of M with $A \leqslant B$, if $A \ll_J B$, then $A \ll_J M$.*

(5) *Let* $f : M \to N$ *be a homomorphism such that* $A \ll_J M$*, then* $f(A) \ll_J N$*.*
(6) *Let* $M = M_1 \oplus M_2$ *be an* R*-module and let* $A_1 \leqslant M_1$ *and* $A_2 \leqslant M_2$*. Then* $A_1 \oplus A_2 \ll_J M_1 \oplus M_2$ *if and only if* $A_1 \ll_J M_1$ *and* $A_2 \ll_J M_2$*.*

2. Jacobson Hopfian modules

Motivated by the concept of Hopfian modules and the notion of generalized Hopfian modules, we define a Jacobson Hopfian module as follows.

Definition 1. *Let* M *be an* R*-module. We say that* M *is Jacobson Hopfian (JH for short) if any surjective endomorphism of* M *has a Jacobson-small kernel.*

The next result gives several equivalent conditions for a Jacobson Hopfian module.

Theorem 1. *Let* M *be an* R*-module. The following are equivalent:*
(1) M *is JH.*
(2) *For any surjective endomorphism* f *of* M*, if* $K \ll_J M$*, then* $f^{-1}(K) \ll_J M$*.*
(3) *For any epimorphism* $f : M/K \to M$*, we have* $K \ll_J M$*.*
(4) *If* M/K *is nonzero and* $\mathrm{Rad}(M/K) = M/K$ *for some* $K \leqslant M$ *and if* f *is a surjective endomorphism of* M *then* $f(K) \neq M$*.*

Proof. (1) $\Rightarrow$ (2) Assume that $f : M \to M$ is a surjective endomorphism and $K \ll_J M$. Let $f^{-1}(K) + N = M$ for some $N \leqslant M$, where $\mathrm{Rad}(M/N) = M/N$. Hence $K + f(N) = M$. As $\mathrm{Rad}(M/N) = M/N$ and $f(\mathrm{Rad}(M/N)) \subseteq \mathrm{Rad}(M/f(N))$. Hence $f(M/N) = M/f(N) \subseteq \mathrm{Rad}(M/f(N))$. Then $\mathrm{Rad}(M/f(N)) = M/f(N)$. Hence $K + f(N) = M$ and $K \ll_J M$, giving $f(N) = M$. So $N + Ker(f) = M$. Since M is JH, $Ker(f) \ll_J M$. Hence $\mathrm{Rad}(M/N) = M/N$ implies that $N = M$. Thus $f^{-1}(K) \ll_J M$.

(2) $\Rightarrow$ (3) Let $f : M/K \to M$ be an epimorphism. It is clear that $K \leqslant Ker(f\pi)$, where $\pi : M \to M/K$ is the canonical epimorphism. By (2), $Ker(f\pi) = (f\pi)^{-1}(0) \ll_J M$. Hence by Lemma 1, $K \ll_J M$.

(3) $\Rightarrow$ (4) Let K be a proper submodule of M such that $\mathrm{Rad}(M/K) = M/K$ and $f : M \to M$ a surjective endomorphism with $f(K) = M$. Then $M = Ker(f) + K$, moreover $g : M/Ker(f) \to M$ is an epimorphism, hence $Ker(f) \ll_J M$ by (3). Thus $M = K$, contradiction.

(4) $\Rightarrow$ (1) Let $f : M \to M$ be an epimorphism. If $M = K + Ker(f)$, with $\mathrm{Rad}(M/K) = M/K$, hence $M = f(M) = f(K)$. Then $K = M$ by (4). Therefore $Ker(f) \ll_J M$. □

Corollary 1. *Let M be a JH module, $h \in \operatorname{End}(M)$ a surjective endomorphism and $N \leqslant M$. Then $N \ll_J M$ if and only if $h(N) \ll_J M$ if and only if $h^{-1}(N) \ll_J M$.*

The following example shows that Hopfian modules form a proper subclass of Jacobson Hopfian modules.

Example 1. Let $M = \mathbb{Z}_{p^\infty}$. As any submodule of M is Jacobson-small in M, it is clear M is JH while M is not Hopfian. Note that multiplication by p induces an $\mathbb{Z}$-epimorphism of M which is not an isomorphism.

Remark 1. According to the definitions, any hollow module is JH, but the converse is not true in general. Note that $M = \mathbb{Z}_6$ is a semisimple $\mathbb{Z}$-module which is not hollow. Since for any semisimple module M we have $\operatorname{Rad}(M) = 0$, so any proper submodule is Jacobson-small in M while M has non nonzero small submodule.

Lemma 2. *Let M be an R-module and $K \leqslant M$. The following are equivalent.*

(1) *$K \ll_J M$.*

(2) *If $X + K = M$, then $M = X \oplus L$ for a semisimple submodule L of M.*

Proof. (1) $\Rightarrow$ (2) Let $L \leqslant M$ such that $\operatorname{Rad}(M/L) = M/L$. Then by [10, Proposition (2.2)] $\operatorname{Rad}(M/(X \oplus L)) = M/(X \oplus L)$. Since $X + L + K = M$ and $K \ll_J M$, hence $X \oplus L = M$. To prove that L is semisimple, let A be a submodule of L . Then $X + A + K = M$. Arguing as above with $X + A$ replacing X, we have that $X + A = X \oplus A$ is a direct summand of M, thus A is a direct summand of L, hence L is semisimple.

(2) $\Rightarrow$ (1) Let $N \leqslant M$ such that $N + K = M$ and $\operatorname{Rad}(M/N) = M/N$. By (2) M/N is semisimple, hence $\operatorname{Rad}(M/N) = 0$. Therefore $M/N = 0$. Thus $M = N$ and $K \ll_J M$. □

Theorem 2. *The following are equivalent for an R-module M:*

(1) *M is JH.*

(2) *For any right module L, if there is an epimorphism $M \to M \oplus L$, then L is semisimple.*

Proof. (1) $\Rightarrow$ (2) Let $f : M \to M \oplus L$ be an epimorphism, and $\pi : M \oplus L \to M$ the natural projection. It is clear that $Ker(\pi f) = f^{-1}(0 \oplus L)$. Since M is JH, $Ker(\pi f) \ll_J M$. By Lemma 1, $0 \oplus L = f[f^{-1}(0 \oplus L)] = f(Ker(\pi f)) \ll_J M \oplus L$. Therefore $L \ll_J L$ by Lemma 1. So, by Lemma 2, L is semisimple.

(2) $\Rightarrow$ (1) Let $f : M \to M$ be a surjective endomorphism and $Ker(f) + T = M$ for some $T \leqslant M$, where $\text{Rad}(M/T) = M/T$. Since $\frac{M}{Ker(f)\cap T} = \frac{Ker(f)}{Ker(f)\cap T} \oplus \frac{T}{Ker(f)\cap T} \cong \frac{M}{T} \oplus \frac{M}{Ker(f)} \cong \frac{M}{T} \oplus M$, the epimorphism $M \to M \oplus \frac{M}{T}$ exists. By (6), M/T is semisimple, then $\text{Rad}(M/T) = 0$. Therefore $M/T = 0$. Thus $M = T$ and $Ker(f) \ll_J M$. □

The following result shows JH property is preserved under Morita equivalences.

Theorem 3. *JH is a Morita invariant property.*

Proof. Let R and S be Morita equivalent rings with inverse category equivalences

$$\alpha : \text{Mod-}R \to \text{Mod-}S, \;\; \beta : \text{Mod-}S \to \text{Mod-}R.$$

Suppose $M \in \text{Mod-}R$ is JH. To show that $\alpha(M)$ is JH in Mod-S, let $f : \alpha(M) \to \alpha(M) \oplus X$ be an S-module epimorphism where X is an S-module. Since any category equivalence preserves direct sums and epimorphisms, we obtain $\beta(f) : \beta\alpha(M) \to \beta\alpha(M) \oplus \beta(X)$, as an epimorphism in Mod-R.

Since $\beta\alpha(M) \cong M$, we obtain an epimorphism $M \to M \oplus \beta(X)$ in Mod-R, which by Theorem 2 implies that $\beta(X)$ is semisimple as an R-module. Since any category equivalence preserves semisimple properties, X is semisimple as an S-module. Thus by Theorem 2, $\alpha(M)$ is JH. □

Corollary 2. *Let R be a ring and $n \geqslant 2$. Then the following are equivalent:*

(1) Every n-generated R-module is JH.

(2) Every cyclic $M_n(R)$-module is JH.

Proof. Let $L = R^n$ and $S = \text{End}(L)$. Then, it is known that

$$Hom_R(L, .) : N_R \to Hom({}_S L_R, N_R)$$

defines a Morita equivalence between Mod-R and Mod-S with the inverse equivalence.

$$- \otimes_S L : M_S \to M \otimes L.$$

Moreover, if N is an n-generated R-module, then $Hom_R(L, N)$ is a cyclic S-module and for any cyclic S-module M, $M \otimes_S L$ is an n-generated R-module. By Theorem 3, a Morita equivalence preserves the JH property of modules. Therefore, every cyclic S-module is JH if and only if every n-generated R-module is JH. □

Theorem 4. *Let M be a quasi-projective R-module. Then the following conditions are equivalent:*
(1) *M is JH.*
(2) *$Ker(f)$ is semisimple for every surjective endomorphism f of M.*

Proof. (1) $\Rightarrow$ (2) Let $f \in \mathrm{End}(M)$ be a surjective endomorphism. Then by (1), $Ker(f) \ll_J M$. Since M is quasi-projective, there exists $g : M \to M$, such that $fg = 1$. It is clear that $Ker(f) = (1 - gf)M$ and $M = Ker(f) \oplus (gf)M$. So, by Lemma 2, $Ker(f)$ is semisimple.

(2) $\Rightarrow$ (1) Let $f \in \mathrm{End}(M)$ be a surjective endomorphism. Then by (2), $Ker(f)$ is a semisimple. We show that $Ker(f) \ll_J M$. Let $Ker(f) + L = M$ for some $L \leqslant M$. Since $Ker(f)$ is semisimple, $(Ker(f) \cap L) \oplus T = Ker(f)$ for some $T \leqslant Ker(f)$. Therefore $T \oplus L = M$. As T is semisimple, $Ker(f) \ll_J M$, by Lemma 2. □

Next, we characterize the class of rings R for which every (free) R-module is JH.

Theorem 5. *Let R be a ring. Then the following conditions are equivalent:*
(1) *Every R-module is JH.*
(2) *Every projective R-module is JH.*
(3) *Every free R-module is JH.*
(4) *R is semisimple.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Evident.

(3) $\Rightarrow$ (4) By (3), $R^{(\mathbb{N})}$ is JH. As $R^{(\mathbb{N})} \cong R^{(\mathbb{N})} \oplus R^{(\mathbb{N})}$, by Theorem 2, $R^{(\mathbb{N})}$ is semisimple. Hence R is semisimple.

(4) $\Rightarrow$ (1) Let R be a semisimple ring and M be an R-module. Hence M is projective and for each surjective endomorphism f of M, $Ker(f)$ is semisimple. Hence by Theorem 4, M is JH. □

It is clear that every generalized Hopfian module is JH. The following example shows that the converse is not true, in general. Also, it shows that a JH module need not be Dedekind finite.

Example 2. Let R be a semisimple ring. Hence by Theorem 5, $M = R^{(\mathbb{N})}$ is a JH R-module. As $M \cong M \oplus M$ and $M \neq 0$, M is not a generalized Hopfian module and not a Dedekind finite module by [7, Corollary 1.4].

Theorem 6. *Let M be an R-module. The following statements are equivalent:*
(1) *M is JH.*

(2) *There exists a fully invariant Jacobson-small submodule K of M such that M/K is JH.*

Proof. (1) $\Rightarrow$ (2) Clear, just take $K = 0$.

(2) $\Rightarrow$ (1) Assume that K is a fully invariant Jacobson-small submodule of M such that M/K is JH. Let $f : M \to M$ be a surjective endomorphism. Then $g : M/K \to M/K$ given by $g(m + K) = f(m) + K$ is a well-defined epimorphism. Since M/K is JH then $Ker(g) \ll_J M/K$. Suppose $Ker(g) = L/K$ for some appropriate submodule L of M. Then $L/K \ll_J M/K$ and since $K \ll_J M$, then by Lemma 1, $L \ll_J M$. Since $Ker(f)$ is a submodule of L, we obtain $Ker(f) \ll_J M$ and M is JH. □

Proposition 1. *Let M be an R-module and let N be a JH fully invariant submodule of M such that M/N is Hopfian. Then M is JH.*

Proof. Let $f : M \to M$ be a surjective endomorphism. Since the induced map $g : M/N \to M/N$ is surjective, it must be an isomorphism, thus $N = f^{-1}(N)$. Therefore $f|_N : N \to N$ is an epimorphism. Now if N is Jacobson Hopfian, $Ker(f) \cap N \ll_J N$. Since $Ker(f)$ is a submodule of N, then $Ker(f) \ll_J N \leqslant M$. Hence by Lemma 1, $Ker(f) \ll_J M$ and M is JH. □

Lemma 3. *Let P be a property of modules preserved under isomorphism. If a module M has the property P and satisfies ACC on non Jacobson-small submodules N such that M/N has the property P, then M is JH.*

Proof. Suppose M is not JH. Then there exists a submodule N_1 with N_1 not Jacobson-small in M and $M/N_1 \simeq M$. Thus M/N_1 is not JH but satisfies P. Hence there exists a submodule $N_2 \supseteq N_1$ with N_2/N_1 not Jacobson-small in M/N_1 and $M/N_2 \simeq M/N_1$. So we get $N_1 \subseteq N_2$ and both non Jacobson-small in M with $M/N_i \simeq M$ for $i = 1, 2$. Repeating the process yields a chain of submodules of the type that contradicts our hypothesis. Hence M is JH. □

Proposition 2. *Let M be an R-module with ACC on non Jacobson-small submodules. Then M is JH.*

Proof. We may assume M is nonzero with ACC on non Jacobson-small submodules and that P is the property of being nonzero. By Lemma 3, M is JH. □

Proposition 3. *Let M be an R-module. If M satisfies DCC on non Jacobson-small submodules, then M is JH.*

Proof. Assume that M satisfies DCC on non Jacobson-small submodules and M is not JH. Hence there exists an epimorphism $f : M \to M$ such that $K = Ker(f)$ is not a Jacobson-small submodule of M. Therefore each submodule L of M, which contains K, is not a Jacobson-small submodule of M. As M is not JH, then it is not generalized Hopfian and it is not Artinian by [7, Remarks 1.19(i)]. Hence $M/K \cong M$ is not Artinian and there is a descending chain $L_1/K \supset L_2/K \supset L_3/K \supset ...$ of submodules of M/K. Thus $L_1 \supset L_2 \supset L_3 \supset ...$ is a descending chain of non Jacobson-small submodule of M, a contradiction. □

Definition 2. *[2]. An R-module M is said to be a Fitting module if for any endomorphism f of M, there exists a positive integer $n \geqslant 1$ such that $M = Kerf^n \oplus Imf^n$.*

Remark 2. The following facts are well known:
(1) Every Artinian and Noetherian R-module is Fitting. [1]
(2) Every Fitting R-module is Hopfian and co-Hopfian. [1]
(3) An R-module M is Fitting if and only if $End(M)$ is strongly π-regular. (i.e., for every $f \in End(M)$ there exists $g \in End(M)$ and an integer n such that $f^n = gf^{n+1} = f^{n+1}g$). [2]

Proposition 4. *Let M be an R-module with the property that for any endomorphism f of M there exists an integer $n \geqslant 1$ such that $Kerf^n \cap Imf^n \ll_J M$. Then M is JH.*

Proof. Let $f : M \to M$ be an endomorphism. There exists $n \geqslant 1$ such that $Kerf^n \cap Imf^n \ll_J M$. If f is surjective then so is f^n, i.e., $Imf^n = M$, so we get that $Kerf^n \ll_J M$. But $Kerf$ is a submodule of $Kerf^n$, so by Lemma 1 $Kerf \ll_J M$, and M is JH. □

Example 3. (1) Every proper submodule of semisimple module M is Jacobson-small, then for any endomorphism f of M there exists an integer $n \geqslant 1$ such that $Kerf^n \cap Imf^n \ll_J M$. Hence M is JH.
(2) If M is Noetherian, then for any endomorphism f of M there exists an integer $n \geqslant 1$ such that $Kerf^n \cap Imf^n = 0$. Hence M is JH.

Proposition 5. *Any direct summand of a JH module M is JH.*

Proof. Let L be a direct summand of M. Then there is a submodule N of M such that $M = L \oplus N$. Let $f : L \to L$ be a surjective endomorphism of L, then f induces a surjective endomorphism of M, $f \oplus 1_N : M \to M$ with $(f \oplus 1_N)(l + n) = f(l) + n$, where $l \in L$ and $n \in N$. Since M is

JH, then $Ker(f \oplus 1_N) \ll_J M$, hence $Kerf \ll_J L$ by Lemma 1, and L is JH. □

Proposition 6. *Let $M = M_1 \oplus M_2$. If for every $i \in \{1, 2\}$, M_i is a fully invariant submodule of M, then M is JH if and only if M_i is JH for each $i \in \{1, 2\}$.*

Proof. $\Rightarrow$) Clear from Proposition 5.

$\Leftarrow$) Let $f = (f_{ij})$ be a surjective endomorphism of M, where $f_{ij} \in Hom(M_i, M_j)$ and $i, j \in \{1, 2\}$. As M_i is a fully invariant submodule of M, then $Hom(M_i, M_j) = 0$ for every $i, j \in \{1, 2\}$ with $i \neq j$. Since f is a surjective endomorphism, f_{ii} is a surjective endomorphism of M_i for each $i \in \{1, 2\}$. As M_i is JH for each $i \in \{1, 2\}$, $Ker(f_{ii}) \ll_J M_i$. Then $Ker(f) = Ker(f_{11}) \oplus Ker(f_{22}) \ll_J M_1 \oplus M_2 = M$ by Lemma 1. Hence M is JH. □

Definition 3. *Let M and N be two R-modules. M is called Jacobson Hopfian relative to N, if for each epimorphism $f : M \to N$, $Ker(f) \ll_J M$.*

By the above definition, an R-module M is Jacobson Hopfian if and only if M is Jacobson Hopfian relative to M.

In the following Proposition, we characterize the Jacobson Hopfian modules in terms of their direct summands and factor modules.

Proposition 7. *Let M and N be two R-modules. Then the following are equivalent:*

(1) M is Jacobson Hopfian relative to N.

(2) For each $L \leqslant^{\oplus} M$, L is Jacobson Hopfian relative to N.

(3) For each $L \leqslant M$, M/L is Jacobson Hopfian relative to N.

Proof. (1) $\Rightarrow$ (2) Let $L \leqslant^{\oplus} M$ say $M = L \oplus K$, where $K \leqslant M$ and $f : L \to N$ an epimorphism. Let $\pi : M \to L$ be the natural projection. Then $f\pi : M \to N$ is an epimorphism and so $Ker(f\pi) \ll_J M$ by (1). It is clear that $Ker(f\pi) = Ker(f) \oplus K$. Thus $Ker(f) \oplus K \ll_J M$. By Lemma 1, $Ker(f) \ll_J L$.

(2) $\Rightarrow$ (1) Take $L = M$.

(1) $\Rightarrow$ (3) Let $L \leqslant M$ and $f : M/L \to N$ be an epimorphism. Then $f\pi : M \to N$ is an epimorphism, where $\pi : M \to M/L$ is the natural homomorphism. As $Ker(f\pi) = \pi^{-1}(Ker(f))$ and $Ker(f\pi) \ll_J M$, $\pi(Ker(f\pi)) = Ker(f) \ll_J M/L$ by Lemma 1. Therefore M/L is Jacobson Hopfian relative to N.

(3) $\Rightarrow$ (1) Take $L = 0$. □

In the following Corollary, we characterize the rings R for which every finitely generated free R-module is JH.

Corollary 3. *Let R be a ring. Then the following statements are equivalent:*
(1) *Every finitely generated free R-module is JH.*
(2) *Every finitely generated projective R-module is JH.*
(3) *$M_n(R)$ is JH as an $M_n(R)$-module for each $n \geqslant 1$.*

Proof. (1) $\Rightarrow$ (2) It is clear from Proposition 5.

(2) $\Rightarrow$ (1) It is clear.

(1) $\Leftrightarrow$ (3) Let n be a positive integer and $S = M_n(R)$. By the proof of Corollary 2 and Theorem 3, if R^n is JH, then $Hom_R(R^n, R^n)$ is JH as an S-module. Conversely, if S is JH as an S-module, then $S \otimes_S R^n$ is JH as an R-module. □

Definition 4. *[3]. A module M is called semi Hopfian if any surjective endomorphism of M has a direct summand kernel, i.e. any surjective endomorphism of M splits.*

Example 4. [3].
(1) Every semisimple module is semi Hopfian.
(2) By [8, Theorem 16(ii)], a vector space V over a field F is Hopfian if and only if it is finite dimensional. Thus an infinite-dimensional vector space over a field is semi Hopfian, but it is not Hopfian.
(3) Every module with D2 is semi Hopfian. (Recall that a module M has D2 if any submodule N such that M/N is isomorphic to a direct summand of M is a direct summand of M).
(4) Every quasi-projective module is semi Hopfian.

Proposition 8. *Every semi Hopfian co-Hopfian R-module is JH.*

Proof. Let M be a semi Hopfian co-Hopfian R-module and let $f : M \to M$ be a surjective endomorphism. Since M is semi Hopfian, there exists $g : M \to M$, such that $fg = 1$, then g is a injective endomorphism, since M is co-Hopfian, so g is automorphism, which shows that f is an automorphism, then M is JH. □

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