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SURVEY ARTICLE

Groups satisfying certain rank conditions

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To Alexander Olshanskii, for his 70th birthday

Abstract. This is a survey of a number of recent results concerned with groups whose subgroups satisfy certain rank conditions.

Introduction

The concept of "rank" in group theory has its roots in vector space theory, where the analogous notion of dimension is so important. Groups also are associated with numerical characteristics, some of which in one way or another are analogs of the concept of the dimension of a vector space. These numerical invariants are not universal, and since they are introduced and employed in specific classes of groups, their nature is local. One of the earliest numerical characteristics is as follows.

Let G be a finitely generated group. Denote by $\mathbf{d}(G)$ the smallest number of generators of the group G.

Here have a complete analogy with vector spaces: the dimension of a vector space is precisely the number of elements in a minimal system of generators. However, this analogy ends at this point: a group may have minimum sets of generators with a different number of elements. Besides, finitely generated groups can have non-finitely generated subgroups. Moreover, a finitely generated group G, whose subgroups are finitely generated (that is the group satisfying the maximal condition for all subgroups), can

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include a subgroup H such that $\mathbf{d}(H) > \mathbf{d}(G)$. Thus, unlike dimension, the property of homogeneity is not observed in groups.

Now we consider other numerical invariants, in varying degrees preserving the analogy with the notion of dimension. One of the first extensions of the concept of dimension was the concept of the R-rank of a module A over a (commutative) ring R. Since every abelian group is a module over the ring $\mathbb{Z}$ of integers, the concept of $\mathbb{Z}$-rank (in the theory of abelian groups we use the term 0-rank) has been resourcefully used in the theory of abelian groups. Starting from the $\mathbb{Z}$-rank, first we define the 0-rank, or torsion-free rank, of a group. In the paper [48], A.I. Maltsev introduced the class of soluble A_1-groups, a class consisting of those groups having a finite subnormal series whose factors are periodic abelian or torsion-free and locally cyclic. These groups also possess finite subnormal series, the factors of which are periodic or infinite cyclic and it was observed that the number of infinite cyclic factors is invariant.

If G is any group which has an ascending series whose factors are either infinite cyclic or periodic and if the number of infinite cyclic factors is finite then the group G is said to have *finite* 0*-rank*. In this case the 0-rank of the group G is the number of infinite cyclic factor-groups in the series and is denoted by $\mathbf{r}_0(G)$. It is not hard to see that $\mathbf{r}_0(G)$ is an invariant of G. If the number of infinite cyclic factors is infinite, then we say that G has infinite 0-rank. If G has no such ascending series the 0-rank is undefined.

For an abelian group G, the cardinality of a maximal $\mathbb{Z}$-independent subset of G (i.e. the $\mathbb{Z}$-rank) is precisely the torsion-free rank of G. Of course, the 0-rank of a periodic group is 0.

For periodic abelian groups a very important role is played by the p-rank. It is defined as the dimension of the lower layer of its Sylow p-subgroup. One characterization of the p-rank introduces us to the following concept.

Let p be a prime. If every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section of order precisely p^r, then G is said to have *finite section p-rank* r, denoted by $\mathbf{sr}_p(G) = r$.

The ranks described above are connected with another important characteristic of a group, namely its *special rank*. It is based on the following property of dimension. If A is a vector space of finite dimension k over a field $\mathbb{F}$ and B is a subspace of A, then it is well-known that B is finite dimensional and that the dimension of B is at most k. We say that a group G is of *finite special rank* $\mathbf{r}(G) = r$ if every finitely generated subgroup can be generated by at most r elements, and r is the least integer with this property. This concept was introduced for arbitrary groups by A.I. Maltsev [47]. Very often the term "rank" is used instead of the "special rank", which is also sometimes called the *"Prüfer rank"*. The study of groups of finite ranks has been of central importance in infinite group theory.

In this paper we first give a brief history of the theory of groups satisfying the various rank conditions. Many classical theorems have been devoted to this topic. For more extensive papers on this topic the reader should consult [26] or [59]. In Section 2 we consider groups whose proper subgroups typically satisfy some rank conditions; this involves more recent results as well as certain of the classical results. In Section 3 we discuss recent work concerned with groups whose subgroups of infinite special rank satisfy some type of normality condition. In Section 4 we discuss results concerning groups whose subgroups are extensions of groups of finite special rank. Our notation is generally standard and can be found in [58].

1. Groups satisfying rank conditions

It is not our intention here to dwell too much on the history of groups satisfying rank conditions, but it is helpful to put some of the later results in context by briefly discussing some of the main results.

D.I. Zaitsev discussed the class of groups of finite 0-rank in a number of papers, including [67–69], where he demonstrated the usefulness of the concept. In the paper [66] D.I. Zaitsev discussed a special class of groups for which the 0-rank is finite, namely the class of *polyrational groups*, where a group is polyrational if it has a subnormal series whose factors are torsion-free locally cyclic groups, and hence are subgroups of the rationals. For groups of finite 0-rank, probably the most general theorem obtained so far occurs in [25] where the following theorem is proved.

We recall that a group is called *generalized radical* if it has an ascending series whose factors are either locally nilpotent or locally finite.

Theorem 1. *Let G be a locally generalized radical group of finite 0-rank. Then G has normal subgroups $T \leqslant L \leqslant K \leqslant S \leqslant G$ such that*

(i) *T is locally finite and G/T is soluble-by-finite of finite special rank;*
(ii) *L/T is a torsion-free nilpotent group;*
(iii) *K/L is a finitely generated torsion-free abelian group;*
(iv) *G/K is finite and S/T is the soluble radical of G/T.*

Moreover, if $\boldsymbol{r}_0(G) = r$*, then there are the functions* f_1*,* f_2 *such that* $|G/K| \leqslant f_1(r)$ *and* $\boldsymbol{dl}(S/T) \leqslant f_2(r)$*.*

A key role, in the proof of this and many other results mentioned here, is played by the following beautiful theorem due to A.I. Maltsev [48, Theorem 5].

Theorem 2. *Let* G *be a torsion-free locally nilpotent group. Suppose that every abelian subgroup of* G *has finite* 0*-rank. Then* G *is a nilpotent group of finite* 0*-rank. Moreover, if* A *is a maximal normal abelian subgroup in* G *and* $\boldsymbol{r}_0(A) = k$*, then* $\boldsymbol{r}_0(G) \leqslant \frac{1}{2}k(k+1)$ *and* G *is nilpotent of class at most* $2k$*.*

We note also the following local property of 0-rank, which has been proved in [25].

Theorem 3. *Let* G *be a group and suppose that* G *satisfies the following conditions:*

(i) *for every finitely generated subgroup* L *of a group* G *the factor-group* $L/\operatorname{Tor}(L)$ *is a locally generalized radical group;*

(ii) *there is a positive integer* r *such that* $\boldsymbol{r}_0(L) \leqslant r$ *for every finitely generated subgroup* L*.*

Then $G/\operatorname{Tor}(G)$ *includes a normal soluble subgroup* $D/\operatorname{Tor}(G)$ *of finite index. Moreover,* G *has finite* 0*-rank* r *and there is a function* f_3 *such that* $|G/D| \leqslant f_3(r)$*.*

For locally finite p-groups, G, it is well-known that the finiteness of $\mathbf{sr}_p(G)$ is equivalent to G being Chernikov, which itself is well-known to be equivalent to the finiteness of the special rank of G. Moreover, in [5] it has been proved that for a locally finite p-group G we have $\mathbf{sr}_p(G) = \mathbf{r}(G)$. This means more can be said concerning locally finite groups with finite section p-rank for all primes p. A result of V.V. Belyaev [6] shows that such groups are almost locally soluble and somewhat more of the structure of such groups can then be deduced (see [43], for example). For torsion-free locally nilpotent groups, G, with finite section p-rank r (for some prime p) it is proved in [5] that G is nilpotent of class at most r and $\mathbf{sr}_p(G) = \mathbf{r}_0(G)$.

For groups in general the following theorem is known (see [25]).

Theorem 4. *Let* G *be a locally generalized radical group of finite section* p*-rank* r_p *for some prime* p*. Then* G *has finite* 0*-rank at most* $2r_p$*. Furthermore,* G *has normal subgroups* $T \leqslant L \leqslant K \leqslant S \leqslant G$ *such that*

(i) *T is a locally finite group whose Sylow p-subgroups are Chernikov;*
(ii) *L/T is a torsion-free nilpotent group;*
(iii) *K/L is a finitely generated torsion-free abelian group;*
(iv) *G/K is finite and S/T is the soluble radical of G/T.*

Moreover, there are the functions f_4, f_5 such that $|G/K| \leqslant f_4(r_p)$, $\boldsymbol{dl}(S/T) \leqslant f_5(r_p)$.

Certainly, groups of finite special rank have finite section p-rank, for each prime p, so special cases of Theorems 1 and 4 hold for locally generalized radical groups of finite special rank. In these special cases more is known. When the main theorem of [6] is combined with a result of M.I. Kargapolov [41], for example, the following result can be obtained.

Theorem 5. *Let G be a locally generalized radical group of finite special rank r. Then its locally nilpotent radical L is hypercentral and G/L includes a normal abelian subgroup K/L such that G/K is finite. In particular, G is generalized radical, even almost hyperabelian. Moreover,* $\mathrm{Tor}(L)$ *is a direct product of its Chernikov Sylow p-subgroups, $L/\operatorname{Tor}(L)$ is nilpotent, $K/\operatorname{Tor}(L)$ has finite 0-rank at most r. In particular, G has finite 0-rank, moreover, $\boldsymbol{r}_0(G) \leqslant r$.*

Recall that a group G is *locally graded* if every finitely generated non-trivial subgroup of G has a finite non-trivial image. Denote by $\mathfrak{X}$ the class of groups, obtained from the class of periodic locally graded groups by using of the formation of local systems, subcartesian products and both ascending and descending normal series. In [11], N.S. Chernikov proved that the groups of finite special rank from this class are almost locally soluble.

In the papers [17, 21] groups that are residually of finite special rank have been considered. The main result of these papers are as follows.

Theorem 6. *Let G be a group and suppose that G has a family of normal subgroups $\mathfrak{S}$ such that $\bigcap \mathfrak{S} = \langle 1 \rangle$ and G/H is a locally (soluble-by-finite) group of special rank at most r for each $H \in \mathfrak{S}$, where r is a fixed positive integer. Then either G is locally (soluble-by-finite) or G includes a non-abelian free subgroup.*

It is known that not every residually finite p-group is locally finite. However,

Theorem 7. *Let G be a periodic group and suppose that G has a family of normal subgroups $\mathfrak{S}$ such that $\bigcap \mathfrak{S} = \langle 1 \rangle$ and G/H is a locally finite* *group of special rank at most r for each $H \in \mathfrak{S}$, where r is a fixed positive integer. Then G is locally finite.*

Theorem 8. *Let G be a finitely generated group and suppose that G has a family of normal subgroups $\mathfrak{S}$ such that $\bigcap \mathfrak{S} = \langle 1 \rangle$ and G/H is a soluble group of special rank at most r for each $H \in \mathfrak{S}$, where r is a fixed positive integer. Then G includes a normal abelian-by-nilpotent subgroup Q such that G/Q is subdirect product of finitely many linear groups. If G includes no non-abelian free subgroups then G is nilpotent-by-abelian-by-finite.*

Theorem 9. *Let p be a prime and G be a free group with a countable set of free generators. Then G has a descending series*

$$G = K_0 \geqslant K_1 \geqslant \ldots \geqslant K_n \geqslant K_{n+1} \geqslant \ldots \bigcap_{j \in \mathbb{N}} K_j = \langle 1 \rangle$$

of normal subgroups such that G/K_n is a finite p-group of special rank 9 for all $n \in \mathbb{N}$.

Theorem 6 generalizes the now well-known result of A. Lubotzky and A. Mann [46] and N.S. Chernikov [11] that a residually finite group of finite special rank is almost locally soluble.

When one leaves the class of generalized soluble groups, the theory of groups of finite special rank is beset with difficulties. It is well-known that A.Yu. Olshanskii [53] has constructed infinite 2-generator simple groups with all proper subgroups cyclic of order p, for large enough primes p. Such groups are of rank 2 and have finite section p-rank 1.

2. Groups in which some system of subgroups have finite rank

In this survey we are interested in groups in which certain proper subgroups satisfy one of the rank conditions described above. The theory of groups all of whose abelian subgroups satisfy some finiteness condition on their ranks is well documented. In particular, in [42], M.I. Kargapolov showed that a soluble group all of whose abelian subgroups have finite special rank itself has finite special rank and, in [49], Yu.I. Merzljakov showed that a locally soluble group in which all abelian subgroups have bounded special rank also has finite special rank. On the other hand, Yu.I. Merzljakov [50] showed that there exists a non-periodic locally polycyclic group, having infinite special rank, in which all abelian subgroups have finite (not bounded) special rank. However for periodic groups the situation is better. V.P. Shunkov [63] showed that a locally finite group all of whose abelian subgroups have finite special rank itself has finite special rank. A result of V.V. Belyaev [6] shows that a locally finite group whose abelian subgroups have finite section p-rank for all primes p itself has finite section p-rank for all primes p.

Next we recall that a group is said to have *finite abelian subgroup rank* if the rank of every abelian p-subgroup is finite for all primes p and if the 0-rank of every abelian subgroup is finite. The class of groups with finite abelian subgroup rank is not closed under taking homomorphic images, so that it is sometimes more useful to consider groups with finite abelian section rank (we say that G is a group of *finite abelian section rank*, if every elementary abelian section of G is finite). Such groups have been discussed in [59]. Of course if G is a group with all abelian subgroups of finite special rank then G has finite abelian subgroup rank. Furthermore, if G is a group with finite section p-rank for all primes p, then G has finite abelian subgroup rank and it is remarkable that these two classes of groups quite often coincide. The main result of a paper of R. Baer and H. Heineken [4] gives the structure of radical groups with finite abelian subgroup rank and shows that such groups have finite section p-rank for all primes p; [6] shows that locally finite groups with finite abelian subgroup rank have finite section p-rank, for all primes p.

In [15] a partial generalization of the theorem of N.S. Chernikov was obtained. It was proved in [15] that if G is a locally (soluble-by-finite) group in which every locally soluble subgroup of G has finite special rank, then G has finite special rank and is almost locally soluble. Furthermore if G is a locally (soluble-by-finite) group with all abelian periodic subgroups of finite special rank and all torsion-free abelian subgroups of bounded special rank, then G has finite special rank and is almost locally soluble (see [15]).

A group has *bounded torsion-free abelian subgroup rank* if there is a bound on the 0-ranks of the torsion-free abelian subgroups. If N is a soluble group with finite abelian subgroup rank and bounded torsion-free rank r_0, then [59, Theorem 10(iii)] shows that there exists an integer d depending on r_0 only such that $N^{(d)}$ is periodic. Then, by [59, Theorem 10(iv)], there is an integer $h(r_0)$ such that each torsion-free abelian factor of N has 0-rank at most $h(r_0)$. One consequence of these ideas is the theorem, from [20], that if G is locally (soluble-by-finite) with finite abelian subgroup rank and bounded torsion-free abelian subgroup rank, then G is almost locally soluble.

If G is a free group then all abelian, and indeed all locally soluble, subgroups of G have special rank 1. Thus a locally graded group with all locally soluble subgroups of finite special rank need not have finite special rank. It seems to be unknown whether the following question has an affirmative answer or not: if G is a periodic locally graded group and all locally soluble subgroups of G have finite special rank then does G have finite special rank? The structure of locally graded groups with all proper subgroups of finite special rank also seems to be unknown.

We conclude this section with the remarkable examples that can be constructed using the following theorem of A.Yu. Olshanskii (see [54, Theorem 35.1]).

Theorem 10. *Let $\{G_\lambda | \lambda \in \Lambda\}$ be a finite or countable set of non-trivial finite or countably infinite groups without involutions. Suppose $|\Lambda| \geqslant 2$ and that n is a sufficiently large odd number (at least $n \geqslant 10^{75}$). Suppose $G_\lambda \cap G_\mu = \langle 1 \rangle$ for $\lambda \neq \mu, \lambda, \mu \in \Lambda$. Then there is a countable simple group $G = \boldsymbol{OG}(G_\lambda | \lambda \in \Lambda)$, containing a copy of G for all $\lambda \in \Lambda$ with the following properties:*

(i) *if $x, y \in G$ and $x \in G_\lambda \setminus \langle 1 \rangle$, $y \notin G_\lambda$ for some $\lambda \in \Lambda$, then $G = \langle x, y \rangle$;*
(ii) *every proper subgroup of G is either a cyclic group of order dividing n or is contained in some subgroup conjugate to some G_λ.*

Hence, these groups of A.Yu. Olshanskii's are 2-generator and have subgroups which are restricted by the choice of the constituent groups G_λ. An application of this theorem allows us to construct the following examples.

Example 1.

(i) *There is a 2-generator group G of infinite special rank, all of whose proper subgroups have finite special rank.*
(ii) *There is a periodic 2-generator group G of infinite special rank with all proper subgroups abelian of finite special rank.*
(iii) *There is a group of infinite special rank, which is not finitely generated, with all proper subgroups of finite special rank.*
(iv) *Let p be a prime such that $p \geqslant 10^{75}$. Then there exist an uncountable p-group G of finite special rank.*

3. Groups with proper subgroups of infinite rank

Many recent papers have been concerned with groups of infinite special rank whose proper subgroups of infinite special rank satisfy some property; typical such properties include nilpotency, normality and their generalizations and we here give a brief summary of these ideas. For example, if the non-abelian subgroups of a group G have finite special rank when does it follow that all the subgroups of G have finite special rank, and what conclusion can then be drawn concerning G?

Let $\mathcal{P}$ be a group theoretical property or class of groups. As usual, we shall say that G is a $\mathcal{P}$-group, or that $G \in \mathcal{P}$, if G has the property $\mathcal{P}$ or belongs to the class $\mathcal{P}$. We let $\mathcal{P}^*$ denote the class of groups G in which every proper subgroup of G is a $\mathcal{P}$-group or a subgroup of finite special rank. As usual we let $\mathfrak{N}$, $\mathfrak{S}$ denote the classes of nilpotent and soluble groups respectively, with a subscript c denoting nilpotent of class at most c (respectively of derived length at most c). We also let $\mathbf{L}\mathfrak{N}$ denote the class of locally nilpotent groups.

Suppose G is in the class $\mathfrak{N}_1^*$, so all proper subgroups of G are abelian or of finite special rank. Clearly groups which themselves are abelian, or are of finite special rank, fall into this class. It was proved in [19] that if c is a positive integer and if $G \in \mathfrak{X} \cap \mathfrak{N}_c^*$, then either $G \in \mathfrak{N}_c$ or G has finite special rank. In either case G is almost locally soluble. The paper [19] contains other results of this nature. For example, if $G \in \mathfrak{X} \cap (\mathbf{L}\mathfrak{N})^*$, then either $G \in \mathbf{L}\mathfrak{N}$, or G has finite special rank. In either case G is almost locally soluble.

Of course there is no corresponding result if we replace $\mathfrak{N}_c$ by $\mathfrak{N}$ here since the Heineken-Mohamed groups [39] are locally nilpotent, non-nilpotent and of infinite special rank, but have all proper subgroups nilpotent. However the existence of such groups is essentially the only reason that we cannot replace $\mathfrak{N}_c$ by $\mathfrak{N}$ in the results mentioned above.

At this point it is natural to ask if one can obtain similar results for groups in the class $\mathfrak{S}_d^*$. One potential obstacle to such a classification is the existence of locally finite simple groups of infinite special rank in which every proper subgroup is finite or metabelian. For example, the group $PSL(2, \mathbb{F})$ has this property if $\mathbb{F}$ is a locally finite field with no proper infinite subfields (see [56]). In order to settle this question information concerning soluble groups with all proper subgroups soluble of derived length d is required. The results mentioned above require a well-known result of O.Yu. Schmidt [61] which states that a finite group with all proper subgroups nilpotent is soluble. For groups with all proper subgroups soluble of bounded derived length, a slight extension of a theorem of D.I. Zaitsev [65], that an infinite soluble group of derived length exactly d has a proper subgroup of derived length precisely d, is required.

Using this result the following theorem can be obtained (see [16]).

Theorem 11. *Let* $G \in \mathfrak{X} \cap (\boldsymbol{L}\mathfrak{S})^*$. *Then either*

(i) *G is locally soluble or*
(ii) *G has finite special rank or*
(iii) *G is isomorphic to one of $SL(2,\mathbb{F})$, $PSL(2,\mathbb{F})$ or $Sz(\mathbb{F})$ for some infinite locally finite field $\mathbb{F}$ in which every proper subfield is finite.*

Here $Sz(\mathbb{F})$ denotes the Suzuki group over the field $\mathbb{F}$. We note that each of the groups in (iii) actually is in the class under consideration.

A number of very recent papers concerning groups of infinite special rank whose proper subgroups of infinite special rank satisfy some normality condition have extended our knowledge in this area (see [12,13,28–37], for example). As is well-known a group in which every subgroup is normal is called a Dedekind group, and such finite groups were classified by R. Dedekind in [14]. What then happens if every subgroup of infinite special rank is a normal subgroup? In [38] a number of results were obtained in this direction. First the authors showed in [38, Theorem 2.3] that a locally graded group with all subgroups Chernikov or normal is itself either Chernikov or a Dedekind group, a result having a number of other consequences. They show in [38, Corollary 2.5] that if G is a locally graded group with all proper subgroups soluble of finite abelian section rank, then G is also soluble with finite abelian section rank. In [38, Theorem 2.7] the authors obtain the following theorem.

Theorem 12. *Let G be a locally graded group. Then every non-normal subgroup of G is soluble with finite abelian section rank if and only if G satisfies one of the following conditions:*

(i) *G is a Dedekind group;*
(ii) *G is soluble with finite abelian section rank;*
(iii) *G has finite abelian section rank and contains a finite normal minimal non-soluble subgroup N such that G/N is a Dedekind group. In particular, $[G,G]$ is finite.*

This result is used to obtain precise conditions for the structure of a locally graded group with all non-normal subgroups polycyclic.

In general the problem of deciding the structure of a locally graded group whose subgroups are normal or of finite special rank seems more complicated, again because our knowledge of locally graded groups with all proper subgroups of finite special rank is limited.

The papers [44,45] and [27] start to rectify this. A well-known theorem of Roseblade [60] asserts that if G is a group in which every subgroup is subnormal of defect at most d, then G is a nilpotent group (with class dependent on d). In [27] the authors study the class of $\mathfrak{X}$-groups all of whose proper subgroups are of finite special rank or subnormal of defect at most d. It is proved there that if G is an $\mathfrak{X}$-group of infinite special rank in which every proper subgroup of infinite special rank is subnormal of defect at most d, then G is nilpotent of nilpotency class bounded by a function of d. In particular, if every subgroup of infinite special rank is normal then G is a Dedekind group, so is nilpotent of class at most 2. This result can also be deduced from the results of paper [44].

Earlier in [45] it was proved that if G is a soluble group of infinite special rank whose proper subgroups of infinite special rank are subnormal, then G is a Baer group. This result was taken a step further in [44] where it is shown that a locally (soluble-by-finite) group of infinite special rank, all of whose proper subgroups of infinite special rank are subnormal, is in any case soluble therefore generalizing a famous result of W. Möhres [51], who showed that a group in which all subgroups are subnormal is soluble. Indeed, for torsion-free locally (soluble-by-finite) groups G of infinite special rank whose proper subgroups of infinite special rank are subnormal, the conclusion is that G is nilpotent [44]. In this case, however, an example is exhibited of a metabelian locally nilpotent group G of infinite special rank, whose torsion subgroup T has finite special rank and which contains a non-subnormal subgroup. It turns out though that every subgroup of infinite special rank is subnormal.

There are other natural generalizations of normality such as permutability. It is well-known that O. Ore [55] proved that in a finite group G all permutable subgroups of G are subnormal in G. Permutable subgroups have also been called quasinormal subgroups by some authors, and during his investigations of permutable subgroups, S.E. Stonehewer [64] proved that all permutable subgroups of a group G are ascendant in G.

Groups with all subgroups permutable, so called *quasihamiltonian groups*, were classified quite precisely by K. Iwasawa [40]. The following theorem summarizes some of the main results and proofs can be found in [62], for example.

Theorem 13. *Let G be a non-abelian quasihamiltonian group. Then*

(i) *G is locally nilpotent;*

(ii) *G is metabelian;*

(iii) *if $\operatorname{Tor}(G)$ is the torsion subgroup of G and if G is not periodic then $\operatorname{Tor}(G)$ is abelian and $G/\operatorname{Tor}(G)$ is a torsion-free abelian group of special rank* 1.

Groups all of whose subgroups of infinite special rank are permutable were studied in [24] where it was shown that, for the class $\mathfrak{X}$, if G is an $\mathfrak{X}$-group having infinite special rank and every subgroup of infinite special rank is permutable then G is quasihamiltonian. This result was generalized in a certain subclass of $\mathfrak{X}$ in [32], where it was also proved that if G is a periodic locally graded group of infinite special rank and if all subgroups of infinite special rank are modular, then G has modular subgroup lattice.

A group G is called *metahamiltonian* if the non-normal subgroups are abelian. Recently groups whose non-abelian subgroups of infinite special rank are normal have been studied (see [34]) and the following theorem obtained.

Theorem 14. *Let G be a locally (soluble-by-finite) group of infinite special rank in which every non-abelian subgroup of infinite special rank is normal. Then G is metahamiltonian.*

4. Groups whose subgroups are (finite rank)-by-soluble

Now let $\mathfrak{R}$ denote the class of groups of finite special rank. Finite groups with all subgroups nilpotent were studied in [61] and shown to be soluble, but again the examples of A.Yu. Olshanskii [53] show that this is not true in general. In a series of papers B. Bruno [7–9] and B. Bruno and R. Phillips [10] showed that a locally graded, non-perfect group, which is not nilpotent-by-finite but all of whose proper subgroups are nilpotent-by-finite must be periodic. This continued a line of research concerned with the notion of minimal non-$\mathfrak{Y}$-groups, where $\mathfrak{Y}$ is some class of groups. A *minimal non-$\mathfrak{Y}$-group* is a group with all proper subgroups in $\mathfrak{Y}$, but the group itself is not. In [57], locally graded groups all of whose proper subgroups are Chernikov-by-nilpotent and locally graded groups with all proper subgroups nilpotent-by-Chernikov were considered and, in a follow-up paper, F. Napolitani and E. Pegoraro [52] proved that a locally graded group with all proper subgroups nilpotent-by-Chernikov is either nilpotent-by-Chernikov or is a perfect locally finite p-group in which all proper subgroups are nilpotent. In [2] it was proved that a locally nilpotent p-group in which every proper subgroup is nilpotent-by-Chernikov is itself nilpotent-by-Chernikov. In particular this meant that the following theorem is true.

Theorem 15. *Let G be a locally graded group all of whose proper subgroups is nilpotent-by-Chernikov. Then G is nilpotent-by-Chernikov and hence is soluble.*

This result has inspired many generalizations. For example, in [22], groups with all proper subgroups (finite rank)-by-nilpotent were considered. The following result was obtained.

Theorem 16. *Let G be a locally (soluble-by-finite) group and suppose that every proper subgroup of G belongs to $\mathfrak{RN}$.*

(i) *If G is a p-group for some prime p then either $G \in \mathfrak{RN}$ or $G/[G,G] \cong C_{p^\infty}$ and every proper subgroup of G is nilpotent.*

(ii) *If G is not a p-group then $G \in \mathfrak{RN}$.*

When a bound is placed on the nilpotency classes things are more straightforward (see [18]). In this case, if G is a locally (soluble-by-finite) group and every proper subgroup of G belongs to the class $\mathfrak{RN}_c$, then $G \in \mathfrak{RN}_c$.

This theorem was generalized in [23].

Theorem 17. *Let G be a locally (soluble-by-finite) group and suppose that every proper subgroup of G belongs to $\mathfrak{SR}$. Then either*

(i) *G is locally soluble, or*

(ii) *G is soluble-by-(finite rank) and almost locally soluble, or*

(iii) *G is soluble-by-$PSL(2,\mathbb{F})$, or*

(iv) *G is soluble-by-$Sz(\mathbb{F})$,*

where $\mathbb{F}$ is an infinite locally finite field with no infinite proper subfields.

There are a number of other results of the types discussed here, including [1, 3], for example.

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