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On the direct sum of dual-square-free modules

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Abstract. A module M is called square-free if it contains no non-zero isomorphic submodules A and B with $A \cap B = 0$. Dually, M is called dual-square-free if M has no proper submodules A and B with $M = A + B$ and $M/A \cong M/B$. In this paper we show that if $M = \oplus_{i \in I} M_i$, then M is square-free iff each M_i is square-free and M_j and $\oplus_{j \neq i \in I} M_i$ are orthogonal. Dually, if $M = \oplus_{i=1}^{n} M_i$, then M is dual-square-free iff each M_i is dual-square-free, $1 \leqslant i \leqslant n$, and M_j and $\oplus_{i \neq j}^{n} M_i$ are factor-orthogonal. Moreover, in the infinite case, we show that if $M = \oplus_{i \in I} S_i$ is a direct sum of non-isomorphic simple modules, then M is a dual-square-free. In particular, if $M = A \oplus B$ where A is dual-square-free and $B = \oplus_{i \in I} S_i$ is a direct sum of non-isomorphic simple modules, then M is dual-square-free iff A and B are factor-orthogonal; this extends an earlier result by the authors in [2, Proposition 2.8].

Let R be an associative ring with identity. A right R-module M is called square-free if it contains no non-zero isomorphic submodules A and B with $A \cap B = 0$. This notion was dualized in [1] as follows: a right R-module M is called dual-square-free if M has no proper submodules A and B with $M = A + B$ and $M/A \cong M/B$, and a ring R is called right DSF-ring if R as a right R-module is a DSF-module. Subsequently, a thorough investigation of DSF-modules was carried out in [2], where it was shown that every DSF-module M is Dedekind finite, and if in addition M satisfies the finite-exchange property, then M satisfies the substitution

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property and its endomorphism ring has stable range 1. Moreover, a DSF-module M has the finite exchange property iff M is clean, iff M has the full exchange property. It was also shown in [2] that, maximal submodules of a DSF-module are fully invariant; in particular a ring R is a right DSF-ring iff every maximal right ideal of R is two-sided. The latter class of rings is known in the literature by right quasi-duo rings, and it is an open question if every right quasi-duo ring is left quasi-duo.

In [3, Lemma 2.17], the authors proved that an orthogonal direct sum of two square-free modules is square-free. However, with a direct induction argument we can show that if $M = \oplus_{i=1}^{n} M_i$, then M is square-free iff each M_i is square-free, $1 \leqslant i \leqslant n$, and M_j and $\oplus_{i\neq j}^{n} M_i$ are orthogonal, where two right R-modules M and N are called orthogonal if, no nonzero submodule of M is isomorphic to a submodule of N. Moreover, in [2, Proposition 2.8], the authors proved that if $M = A \oplus B$ where A is a dual-square-free module and B is a finite direct sum of non-isomorphic simple modules, then M is a dual-square-free module if and only if A & B are factor-orthogonal, where two right R-modules L and N are called factor-orthogonal if, no nonzero factor of L is isomorphic to a factor of N.

In this paper we extend the aforementioned result on direct sums of SF-modules to the infinite case, and present a partial dualization in the dual-square-free case. More precisely, we prove that if $M = \oplus_{i=1}^{n} M_i$, then M is dual-square-free iff each M_i is dual-square-free, $1 \leqslant i \leqslant n$, and M_j and $\oplus_{i\neq j}^{n} M_i$ are factor-orthogonal. Moreover, while the infinite case still remains open, we show that if $M = \oplus_{i\in I} S_i$ is a direct sum of non-isomorphic simple modules, then M is a dual-square-free module. In particular, if $M = A \oplus B$ where A is a dual-square-free module and $B = \oplus_{i\in I} S_i$ is a direct sum of non-isomorphic simple modules, then M is a dual-square-free module if and only if A & B are factor-orthogonal.

Theorem 1. *If $M = \oplus_{i\in I} M_i$, then the following conditions are equivalent:*

1) *M is a square-free module;*
2) *Each M_i is square-free, $i \in I$, and M_j and $\oplus_{j\neq i\in I} M_i$ are orthogonal.*

Proof. Observe first that, by [2, Lemma 2.17] and a straightforward induction on n, one can show that if $N = \oplus_{i=1}^{n} N_i$, then N is a square-free module iff each N_i is square-free, $1 \leqslant i \leqslant n$, and N_j and $\oplus_{i\neq j}^{n} N_i$ are orthogonal.

(2) $\Rightarrow$ (1). Let $A \cong B$ with $A \cap B = 0$, where $A, B \subseteq M$. If $x \in A$, then $xR \cong yR$ for some $y \in B$. But this means that, there is a finite subset $F \subseteq I$ such that $xR, yR \subseteq \oplus_{i\in F} M_i$. By the aforementioned observation,

since $\oplus_{i\in F} M_i$ is square-free, $xR = yR = 0$. This shows that $A = B = 0$, and M is square-free.

(1) $\Rightarrow$ (2). Obvious, since the class of square-free modules is closed under direct summands. □

Corollary 1 ([2, Lemma 2.17]). *If $M = \oplus_{i=1}^{n} M_i$, then the following conditions are equivalent:*

1) *M is a square-free module;*
2) *Each M_i is square-free, $1 \leqslant i \leqslant n$, and M_j and $\oplus_{i\neq j}^{n} M_i$ are orthogonal.*

Observe that if $f : X \longrightarrow Y$ is a homomorphism and A is a submodule of X, then f induces a homomorpism $\bar{f} : X/A \longrightarrow Y/f(A)$ given by $\bar{f}(x + A) = f(x) + f(A)$ with $\ker \bar{f} = (A + \ker f)/A$. Moreover if f is an epimorphism (monomorphism, isomorphism, resp.), then so is $\bar{f}$. Now, the next lemma is well-known and therefore we don't include a proof.

Lemma 1. *Let $M = M_1 \oplus M_2$, $A \subseteq M$ and $f : M \to M_1$ be the projection map of M onto M_1. Then $A + M_2 = f(A) + M_2$. In particular if $f(A) = M_1$, then $M = A + M_2$.*

Lemma 2. *Let $M = \oplus_{i\in I} M_i$ with each M_i a DSF-module, $i \in I$, and M_j and $\oplus_{i\neq j} M_i$ are factor-orthogonal for every $j \in I$. For $j \in I$, let $f_j : M \to M_j$ be the projection map of M onto M_j. If A and B are submodules of M with $M = A + B$ and $\frac{M}{A} \cong \frac{M}{B}$, then $f_j(A) = f_j(B) = M_j$ and $M = A + (\oplus_{i\neq j} M_i) = B + (\oplus_{i\neq j} M_i)$.*

Proof. We will only show that $f_j(A) = M_j$, as the other equality $f_j(B) = M_J$ can be done with a similar argument. Clearly, we have the following epimorphism:

$$\frac{M}{B} \cong \frac{M}{A} \xrightarrow{\bar{f}_j} \frac{M_j}{f_j(A)}.$$

But then, $\frac{M_j}{f_j(A)} \cong \frac{M}{X}$, where $X \subseteq M$ and $B \subseteq X$. Next, consider the following epimorphism:

$$\frac{M_j}{f_j(A)} \cong \frac{M}{X} \xrightarrow{\bar{f}_j} \frac{M_j}{f_j(X)}.$$

As before, $\frac{M_j}{f_j(X)} \cong \frac{M_j}{Y}$, with $Y \subseteq M_j$ and $f_j(A) \subseteq Y$. Now, since $M = A + B = A + X$, $M_j = f_j(M) = f_j(A) + f_j(X) = Y + f_j(X)$.

Inasmuchas M_j is a DSF-module, we infer that $f_j(X) = M_j$. Now, by Lemma 1, $M = X + (\oplus_{i\neq j} M_i)$, and consequently

$$\frac{M_j}{f_j(A)} \cong \frac{M}{X} = \frac{X + (\oplus_{i\neq j} M_i).}{X} \cong \frac{(\oplus_{i\neq j} M_i)}{X \cap (\oplus_{i\neq j} M_i)}.$$

Since M_j and $(\oplus_{i\neq j} M_i)$ are factor-orthogonal, $f_j(A) = M_j$, as required. The last statement now follows from Lemma 1. □

Theorem 2. *Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. Then the following are equivalent:*

1) *M is a DSF-module.*
2) *Each M_i is a DSF-module, $1 \leqslant i \leqslant n$, and M_j and $\oplus_{i\neq j}^n M_i$ are factor-orthogonal.*

Proof. (1) $\Rightarrow$ (2). This is clear, since the class of DSF-modules is closed under direct summands.

(2) $\Rightarrow$ (1). We proceed by induction on n. Nothing need to be done when $n = 1$. Assume that $n \geqslant 2$ and $K = M_1 \oplus M_2 \oplus \cdots \oplus M_{n-1}$ is a DSF-module. Then $M = K \oplus M_n$ with K and M_n are DSF-modules which are factor-orthogonal. We need to show that, if $M = A + B$ with $\frac{M}{A} \cong \frac{M}{B}$, then $M = A = B$. We will only show that $M = A$, as the other equality can be verified the same way. By Lemma 2, $M = A + K = A + M_n = B + K = B + M_n$. Now, we have:

$$\frac{K}{A \cap K} \cong \frac{A + K}{A} = \frac{M}{A} \cong \frac{M}{B} = \frac{B + M_n}{B} \cong \frac{M_n}{B \cap M_n}.$$

Since K and M_n are factor-orthogonal, we get $K = A \cap K$ and so $K \subseteq A$. Therefore, $M = A + K = A$, as required. □

Lemma 3. *If $M = \oplus_{i\in I} S_i$ is a direct sum of non-isomorphic simple modules, then M is a DSF-module.*

Proof. Let $M = A + B$ with $M/A \cong M/B$. We need to show that $M = A = B$. We will only show that $M = A$. Since M is semisimple, $A \cap B \subseteq^{\oplus} M$. Now, write, $M = (A \cap B) \oplus T$ for a submodule $T \subseteq M$. Therefore, $A = (A \cap B) \oplus (A \cap T)$ and $B = (A \cap B) \oplus (B \cap T)$. Consequently,

$$\begin{aligned} M &= A + B = [(A \cap B) \oplus (A \cap T)] + [(A \cap B) \oplus (B \cap T)] \\ &= (A \cap B) \oplus (A \cap T) \oplus (B \cap T) = A \oplus (B \cap T) = B \oplus (A \cap T) \end{aligned}$$

Since M is semisimple, we have $A \subseteq^{\oplus} M$ and $B \subseteq^{\oplus} M$, with $(A \cap T) \cong M/B \cong M/A \cong (B \cap T)$. Now, if $S_i \nsubseteq A$, for some $i \in I$, then $S_i \cap A = 0$.

Thus if $f : A \oplus (B \cap T) \longrightarrow (B \cap T)$ is the projection map of M onto $(B \cap T)$, then $S_i \cong f(S_i) \subseteq (B \cap T)$. But since $(A \cap T) \cong (B \cap T)$, we infer that $S_i \cong X \subseteq A$, for some submodule X in A. Since S_i is a fully invariant submodule of M, $X = S_i$, which is a contradiction. This shows that $S_i \subseteq A$ for every $i \in I$, and $M = A$ as required. □

The next result extends the work of the authors in [2, Proposition 2.8].

Proposition 1. *Let $M = A \oplus B$ where A is a DSF-module and $B = \oplus_{i \in I} S_i$ is a direct sum of non-isomorphic simple modules. Then M is a DSF-module if and only if A and B are factor-orthogonal.*

Proof. Follows from both Theorem 2 and Lemma 3. □

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