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Online list coloring for signed graphs

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Abstract. We generalize the notion of online list coloring to signed graphs. We define the online list chromatic number of a signed graph, and prove a generalization of Brooks' Theorem. We also give necessary and sufficient conditions for a signed graph to be degree paintable, or degree choosable. Finally, we classify the 2-list-colorable and 2-list-paintable signed graphs.

1. Introduction

One of the most classical results in graph coloring is Brooks' Theorem:

Theorem 1 ([1]). Let G be a connected graph. Then $\chi(G) \leqslant \Delta(G)$, unless G is an odd cycle or a clique.

Recently, an analogue of Brooks' theorem for signed graphs was proven in [3]. Recall that a signed graph is a graph where each edge is labeled positive or negative. Signed graph coloring was originally introduced by Zaslavsky [6]. Recall that an unbalanced cycle is a cycle with an odd number of negative edges, and a signed graph is unbalanced if it contains an unbalanced cycle. Otherwise, it is balanced.

Theorem 2 ([3]). Let Σ be a simple connected signed graph. Then $\chi(\Sigma) \leqslant \Delta(\Sigma)$, unless Σ is a balanced odd cycle, unbalanced even cycle, or a balanced clique.

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This theorem only applies to simple signed graphs; however, many signed graphs in the literature allow parallel edges and loops. In this paper, we prove a generalization of Brooks' Theorem for signed graphs with parallel edges. We go further: we generalize the notion of online list coloring to signed graphs, and prove the natural generalization of Brooks' Theorem for the online list chromatic number (or paintability number) of a signed graph. The definitions of list coloring, the list chromatic number $\chi_\ell(\Sigma)$, online list coloring, and the online list chromatic number $\chi_p(\Sigma)$ are given in Section 3. Definitions of families of signed graphs, such as double clique, are given in Section 2.

Theorem 1. Let Σ be a connected signed graph without loops. Then $\chi_p(\Sigma) \leqslant \Delta(\Sigma)$, unless Σ is a balanced odd cycle, unbalanced cycle, odd double cycle, balanced clique, or a double clique. Moreover, if Σ is an unbalanced odd cycle, then $\chi_\ell(\Sigma) = 2$.

Some of the new graphs described in the theorem appear in Figure 1. A *double graph* consists of taking a simple graph and replacing every edge with two edges, where one is positive and one is negative. We also discuss the version of the theorem where we allow Σ to have negative loops: however, the resulting classification is not as nice.

In Section 3, we generalize the notions of degree choosable and degree paintable, and describe the signed graphs that fail to be degree choosable or degree paintable. Recall that a block is a maximal biconnected subgraph. A *signed Gallai tree* is a signed graph where each block is an odd cycle, a balanced even cycle, and odd double cycle, a balanced clique, or a double clique. Then we generalize a result of Erdős, Rubin, and Taylor [2]:

Theorem 2. Let Σ be a connected signed graph.

1) Σ is not degree choosable if and only if Σ is a signed Gallai tree with no unbalanced odd cycle block.
2) Σ is not degree paintable if and only if Σ is a signed Gallai tree with no intersecting unbalanced odd cycle blocks, and such that no vertex v in an unbalanced odd cycle block B has a negative loop.

For signed graphs, there is a distinction between degree choosability and degree paintability, where for ordinary graphs there is no distinction: degree choosable graphs must be degree paintable.

The paper is organized as follows: first, we review terminology about signed graphs, and then introduce online list coloring of signed graphs. Then we prove some preliminary lemmas about paintability and choosability. We focus on several families of signed graphs. Then we prove our result about degree paintability. Then we deduce our version of Brooks' Theorem. Then, we classify all 2-paintable signed graphs, relying on the similar classification of 2-list-choosable graphs appearing in [2]. Finally, we end with some open questions about paintability.

2. Signed graphs

A *signed graph* is a pair $\Sigma = (G, \sigma)$, where $G = (V, E)$ is a graph and $\sigma : E \to \{+, -\}$. Given $u, v \in V(\Sigma)$, and $\epsilon \in \{+, -\}$, we use uv^ϵ to denote an edge e between u and v such that $\sigma(e) = \epsilon$. A cycle is *balanced* if the product of all the edge labels are positive. If all cycles in a signed graph are balanced, then the graph itself is balanced. A signed graph that is not balanced is *unbalanced*. Equivalently, a signed graph is balanced if there exists a bipartition of the vertex set V into sets A and B such that an edge uv^ϵ is positive if and only if u, v both belong to the same set of the bipartition (that is, $u, v \in A$ or $u, v \in B$). We call such a bipartition a *balanced bipartition*.

Switching a signed graph at a vertex v is done by negating the label on every non-loop edge incident to v. We denote the switched graph by Σ^v. Switching is a common operation in the theory of signed graphs because many interesting properties of signed graphs are invariant under switching operations: for instance, Σ is balanced if and only if Σ^v is balanced. In Proposition 3, we show that Σ if f-choosable (or f-paintable) if and only if Σ^v is f-choosable (respectively, f-paintable). Invariance under switching operations often allows us to reduce the number of cases in various proofs. Recall that Σ is balanced if and only if it is switch equivalent to a graph with all positive edges. The *underlying graph*, $\underline{\Sigma}$ is obtained by removing parallel edges, and removing signs.

There are few types of signed graphs that are important for this paper, some of which appear in Figure 1 or Figure 3. We use dashed lines to denote negative edges, and solid lines to denote positive edges. A *balanced clique* is a simple balanced signed graph whose underlying graph is complete. A *double complete graph* is a graph that has both positive and negative edges between every pair of vertices. A *double cycle* is a signed graph Σ obtained from a cycle graph by replacing every edge with two edges, where one edge is positive and one is negative. Also, given a sequence of positive integers $a_1, \ldots, a_k$, a *theta graph* $\theta_{a_1,\ldots,a_k}$ is a graph with two distinguished vertices, x, and y, and k disjoint paths from x to y, of lengths a_1, $a_2, \ldots, a_k$, respectively. A *signed theta graph* is a signed graph whose underlying graph is a theta graph. In particular, parallel edges are allowed.

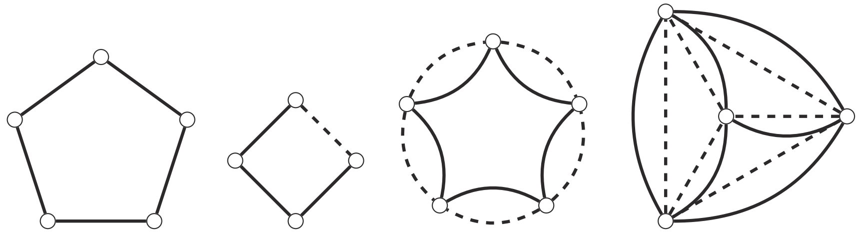

FIGURE 1. Signed graphs, where dashed lines represent negative edges.

The *degree* of a vertex v is the number of edges incident to that vertex, where loops are only counted once each. So the degree of a vertex on a double cycle is 4, while the degree of a vertex on the double complete graph on n vertices is $2n-2$.

3. Online list coloring

First, we review the definition of signed chromatic number from [3]. For $n \in \mathbb{N}$, let $M_{2n+1} = \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n\}$, and $M_{2n} = M_{2n+1} \setminus \{0\}$. A coloring of Σ is a function $f : V \to M_n$, for some n, such that, for every edge uv^{ϵ} we have $f(u) \neq \epsilon f(v)$. The chromatic number $\chi(\Sigma)$ of a signed graph is the minimum n such that there is a proper coloring $f : V \to M_n$.

We extend the definition of list coloring to signed graphs. A *list assignment* $L : V \to 2^{\mathbb{Z}}$ assigns a finite non-empty set of integers to each vertex. A proper coloring $c : V \to \mathbb{Z}$ is an *L-coloring* if $c(v) \in L(v)$ for all v. A graph is *L-colorable* if it has an L-coloring.

Given a function $f : V \to \mathbb{N}$, a list assignment is *f-compatible* if $|L(v)| \geqslant f(v)$ for all $v \in V$. A signed graph is *f-choosable* if there is an L-coloring for every f-compatible list assignment L. A signed graph is *k-choosable* if it is f-choosable, where $f(v) = k$ for all v. The minimum k such that this holds is the *list chromatic number* $\chi_{\ell}(\Sigma)$. Similarly, a signed graph is *degree choosable* if it is d-choosable, where d is the degree function of Σ.

Now, we extend the notion of online list coloring to signed graphs. Online list coloring, or graph painting, was first introduced in [5] and [7]. Given a function $f : V \to \mathbb{N}$, consider the following two player game, with players Lister and Painter. In round 0, Lister presents the set of all vertices whose lists contain color 0. Painter must then use color 0 on some independent subset of these vertices, and cannot change this set in the future. In each subsequent round k, Lister chooses some subset A of the vertices to contain color k in their lists, and some subset B of the vertices to contain the color $-k$ in their lists, subject to the constraint that no vertex v has more than $f(v)$ colors in its list. Then Painter chooses some subset $A' \subseteq A$ to color k, and some subset $B' \subseteq B$ to color $-k$. Painter wins if they succeed in obtaining a proper coloring. Alternatively, Lister wins if, after the end of some round, there is an uncolored vertex v whose list has size $f(v)$.

If Painter has a winning strategy, then Σ is *f-paintable*. It is not hard to see that f-paintability implies f-choosability: given an f-compatible list assignment L, and a winning strategy, we can just simulate the game, by having Lister add colors ϵi to $L(v)$ at the $(i+1)$st round if and only if $\epsilon i \in L(v)$, with $\epsilon \in \{+,-\}$. Then the game will terminate with a proper G-coloring. A signed graph Σ is *k-paintable* if it is f-paintable, where $f(v) = k$ for all $v \in V$. Similarly, a signed graph is *degree paintable* if it is d-paintable, where d is the degree function.

Now we show that f-choosability and f-paintability are invariant under switching operations. For choosability, we can perform switching operations on list assignments. Given a list $L(v)$, we define $-L(v) = \{-x : x \in L(v)\}$. Likewise, fix a list assignment L on a signed graph Σ. Given a vertex v, we define the *switching* of L with respect to v to be the list assignment L^v given by $L^v(u) = L(u)$ when $u \neq v$, and $L^v(v) = -L(v)$.

Proposition 3. Let Σ be a signed graph with list assignment L, and let v be a vertex. Then Σ is L-colorable if and only if Σ^v is L^v-colorable.

In particular, Σ is f-choosable if and only if Σ^v is f-choosable. Finally, Σ is f-paintable if and only if Σ^v is f-paintable.

Proof. For each claim, it suffices to prove that 'if Σ satisfies P, then Σ^v satisfies P'. This is because we can apply this statement to Σ^v to obtain 'if Σ^v satisfies P, then $(\Sigma^v)^v$ satisfies P'. Since $(\Sigma^v)^v = \Sigma$, we obtain the converse statement.

Fix a signed graph Σ, and let L be a list assignment for Σ. Suppose that Σ is L-colorable. Let $c : V \to \mathbb{Z}$ be an L-coloring. Fix a vertex $v \in V$. Define $c^v : V \to \mathbb{Z}$ by $c^v(u) = c(u)$ if $u \neq v$, and $c^v(v) = -c(v)$. Then $c^v(u) \in L^v(u)$ for all u. Then c^v is an L^v-coloring of Σ^v.

For the second result, suppose that Σ is f-choosable. Let L be a list assignment for Σ^v such that $|L(v)| \geqslant f(v)$ for all v. Then L^v is f-compatible, and Σ is L^v-colorable. Thus Σ^v is $(L^v)^v$-colorable. Since $(L^v)^v = L$, we have shown that Σ^v is f-choosable.

Note that for f-choosability, the idea was to switch the list assignment, choose a coloring, and then switch the coloring. For f-paintability, the strategy is similar, but happens at each step: at each step, switch the list assignment, then choose the coloring, then switch back. Now suppose that Σ is f-paintable. Consider the following strategy for Σ^v for Painter: Painter creates an auxiliary game with Σ. Every time Lister places $\epsilon i \in L(u)$ in Σ^v when $u \neq v$, then Painter puts $\epsilon i \in L^v(u)$ in Σ. When Lister places $\epsilon i \in L(v)$, then Painter puts $-\epsilon i \in L^v(v)$. Then Painter copies the winning strategy for Σ, choosing subsets A and B for Σ. If the vertex v does not appear in $A \cup B$, then Painter chooses those same subsets for Σ^v. If v appears in one of those sets, then Painter moves it to the other set, creating A' and B' for Σ^v. This strategy ensures that Painter is creating a proper coloring for Σ^v, while also creating a proper coloring for Σ using lists of the same size. Since we know that the Painter is using a winning strategy for Σ, the auxiliary game must end with $|L^v(u)| < f(u)$ for all $v \in \Sigma$. Then $|L(u)| < f(u)$ for all $v \in \Sigma^v$, and thus the Painter has a winning strategy to f-paint Σ^v. $\square$

4. Lemmas on painting

In this section, we prove lemmas for determining whether or not a signed graph is f-paintable or f-choosable. Given any signed graph Σ, the graph Σ° is obtained from Σ by deleting all negative loops. Note that, for $A \subset V$, we define $e_A(v)$ to be the number of edges between v and A.

Lemma 4. Let Σ be a signed graph, and let $f : V \to \mathbb{N}$. Let $A \subset V$. If $\Sigma[A]$ is f-paintable, and $\Sigma \setminus A$ is $(f - e_A)$-paintable, then Σ is f-paintable.

Proof. We describe the winning strategy for Painter. By the hypothesis, Painter has a winning strategy for f-painting $\Sigma[A]$, and a winning strategy for $(f - e_A)$-painting $\Sigma \setminus A$. Painter's strategy is going to consist of creating a list assignment L' on $\Sigma \setminus A$ such that $L'(v) \subseteq L(v)$ for all $v \in V \setminus A$, and $|L'(v)| \leqslant f(v) - e_A(v)$.

At the ith step, Lister adds color $\pm i$ to various lists. First, Painter applies the winning strategy for $\Sigma[A]$ to determine which vertices of A get colored i or $-i$. Then, for each $v \in V \setminus A$, and $\epsilon \in \{+, -\}$ if $\epsilon i \in L(v)$, and there is no signed edge uv^η with $u \in A$ that has been colored $\eta\epsilon i$, then Painter adds ϵi to $L'(v)$. Then Painter applies the winning strategy for $\Sigma \setminus A$ with respect to the list assignment L'. Due to how we choose L', Painter always creates a proper coloring. Also, since Painter only removes colors from lists when they correspond to edges between A and $V \setminus A$, we see that $|L'(v)| \leqslant f(v) - e_A(v)$ for all $v \in V \setminus A$. Thus Painter successfully paints both A and $V \setminus A$. $\square$

We apply our lemma to prove the following result:

Proposition 5. Let Σ be a connected signed graph, and let $U \subset V$ be nonempty. Let $\delta_U : V \to \{0,1\}$ be defined by $\delta_U(x) = 1$ if and only if $x \in U$. Then Σ is $(d + \delta_U)$-paintable.

Proof. We prove the result by induction on $|V(\Sigma)|$. Let $x \in U$. Let $U' = (U \cup N(x)) \setminus \{x\}$. Then $\Sigma' = \Sigma - x$ has components. Let C be a component, and define $V = U' \cap C$. Then by induction, C is $(d_C + \delta_V)$-paintable. For each vertex $u \in C \setminus (N(x) \setminus U)$, we have $d_\Sigma(u) \geqslant d_C(u)$, and $\delta_V(u) = \delta_U(u)$. For $u \in (N(x) \cap C) \setminus U$, we have $d_\Sigma(u) = d_C(u) - 1$ and $\delta_V(u) = \delta_U(u) + 1$. Thus C is $(d_\Sigma + \delta_U)$-paintable. This is true for every component, so Σ' is $(d_\Sigma + \delta_U)$-paintable. Since x is also 1-paintable, the result follows from Lemma 4 with $A = V \setminus \{x\}$. $\square$

Note that our proof gives a winning strategy: choose $x \in U$, and order vertices as $v_1, \ldots, v_n$ so that $\Sigma[v_i, \ldots, v_n]$ is connected for all i, and $v_n = x$. When Lister gives new color classes, Painter greedily colors according to the ordering: coloring a vertex i when it is possible, and otherwise coloring the vertex with $-i$.

First, we begin with two fundamental lemmas.

Lemma 6. Let Σ be a 2-connected graph, and let i be the first round where Lister has added colors to some of the lists. Suppose that there exists a non-loop signed edge uv^ϵ such that $L(u) \setminus \epsilon L(v) \neq \varnothing$. Then Painter has a winning strategy.

Note that this lemma forces Σ to be a regular graph. Moreover, if Σ is unbalanced, then $L(u) = L(v)$ for all u, v, and $-L(u) = L(u)$ as well.

Proof. Let $i \in L(u) \setminus \epsilon L(v)$. We assign the vertex u the color i. Then $\Sigma - u$ is connected. If we set $U = \{v\}$, then by Proposition 5, we see that $\Sigma - u$ is $(d_{\Sigma-u} + \delta_U)$-paintable. We see that $d_{\Sigma-u} + \delta_U = d_\Sigma$ on $\Sigma - u$, so the result follows by applying Lemma 4 with $A = \{u\}$. $\square$

We see that a similar proof can be used to establish the following:

Lemma 7. Let Σ be a 2-connected graph, and suppose that there is a degree-satisfiable list assignment L such that there exists a non-loop edge uv^ϵ with $L(u) \setminus \epsilon L(v) \neq \varnothing$. Then Σ is L-colorable.

Given a list assignment L, if $L(u) = \epsilon L(v)$ for every edge uv^ϵ, we refer to L as a *satisfied* list assignment. In light of Lemma 7, we see that it suffices to show that a graph Σ is f-choosable by showing that it has an L-coloring for any f-compatible satisfied list assignment L. The situation for f-paintability is a little more complicated: we have to show that Painter has a winning strategy to win any game where Lister creates a satisfied list assignment on the first round that Lister adds colors to lists. Note that Lister is not required to continue to maintain the property of the list assignment being satisfied on later rounds.

Lemma 8. Let Σ be a connected signed graph. Let B be a connected subgraph. If B is degree paintable, then Σ is degree paintable. If B is degree choosable, then Σ is degree choosable.

Proof. Let B be a connected subgraph of Σ. Let C be a component of $\Sigma \setminus B$, and set $U = N(B) \cap C$. Then $U \neq \varnothing$. Thus, by Proposition 5, we see that C is $(d_C + \delta_U)$-paintable. We also see that $d_C(v) + \delta_U(v) \leqslant d_\Sigma(v)$. Thus, each component of $\Sigma \setminus B$ is d_Σ-paintable. Hence $\Sigma \setminus B$ is d_Σ-paintable. The first result follows from Lemma 4 by setting $A = V(\Sigma \setminus B)$.

For the latter result, we see that $\Sigma \setminus B$ is d_Σ-paintable, and hence is d_Σ-choosable. Let L be a degree compatible list assignment for Σ. Let f be an L-coloring of $\Sigma \setminus B$, which exists since $\Sigma \setminus B$ is d_Σ-choosable. For $v \in B$, let $L'(v) = L(v) \setminus \{\sigma(e)f(u) : e = uv \in E(\Sigma)\}$. Then $|L'(v)| \geqslant d_B(v)$. Thus, since B is degree choosable, we extend f to B such that $f(v) \in L'(v)$ for $v \in B$. □

Lemma 9. Let Σ be a connected signed graph. If Σ contains two parallel edges e and f such that $\sigma(e) = \sigma(f)$, then Σ is degree paintable.

Let Σ° be obtained from Σ by removing all negative loops. If Σ° is degree paintable (degree choosable), then Σ is degree paintable (degree choosable)

Proof. Let Σ be connected, and suppose that e and f be parallel edges such that $\sigma(e) = \sigma(f)$. Let u, v be the endpoints of e, and let $\Sigma' = \Sigma - e$. Then by Proposition 5, with $U = \{u, v\}$, we see that Σ' is $(d_{\Sigma'} + \delta_{\{u,v\}})$-paintable. Since $d_\Sigma = d_{\Sigma'} + \delta_{\{u,v\}}$, it follows that Painter's winning strategy for Σ' is also a winning strategy for Σ.

Suppose that Σ is connected, and Σ° is degree paintable. We describe a winning strategy for Painter. Let S be the subset of vertices that contain a negative loop, and let A be the set of vertices v such that $0 \in L(v)$ after the 0th round. Then Painter follows the winning strategy for Σ° if Lister presented $A \setminus S$. For all subsequent rounds, Painter follows the winning strategy for Σ°. For $v \in S \cap A$, we see that $|L(v)| \leqslant 1 + d_{\Sigma^\circ}(v) = d_\Sigma(v)$, as we start with $0 \in L(v)$, and we know that Painter finishes painting before Lister can add $d_{\Sigma^\circ}(v)$ more colors. For $v \in S \setminus A$, we have $|L(v)| \leqslant d_{\Sigma^\circ}(v) = d_\Sigma(v) - 1$. Finally, for $v \in V \setminus S$, we have $|L(v)| \leqslant d^\circ_\Sigma(v) = d_\Sigma(v)$. Thus Σ is degree paintable.

Similarly, suppose that Σ° is degree choosable. Let L be a degree compatible list assignment for Σ. For each v that has a negative loop, if $0 \in L(v)$, delete 0. If $0 \notin L(v)$, choose $i \in L(v)$ and delete it. This gives us a new list assignment L' such that $L'(v) \subseteq L(v)$ for all v. Now delete all the negative loops. We now have Σ° with a degree compatible list assignment L'. Since Σ° is degree choosable, there is an L'-coloring f. Since $0 \notin L(v)$ whenever v has a negative loop, we see that f is an L-coloring of Σ. □

5. Special cases

In this section, we study paintability for special classes of signed graphs, such as cycles and cliques.

Lemma 10. Let Σ be a loopless signed graph such that $\underline{\Sigma}$ is a cycle. If Σ is not degree choosable if and only if Σ is a balanced odd cycle, unbalanced even cycle, or odd double cycle. Also Σ is not degree paintable if and only if Σ is a balanced odd cycle, unbalanced even cycle, odd double cycle, or an unbalanced odd cycle.

Proof. By Lemma 9, if Σ has parallel edges of the same sign, then the Σ is degree paintable. Thus there are at most two edges between any pair of distinct vertices u and v, and if there is a pair of edges, they have to have opposite signs. First, we study the case where Σ has a pair of vertices that have exactly one edge between them, and a pair of vertices that have two edges between them. In light of Lemma 6, when showing that Painter has a winning strategy we restrict ourselves to games where on the first nontrivial step i the list assignment is satisfied. Similarly, in light of Lemma 7, when showing that Σ is degree choosable we restrict ourselves to degree-compatible *satisfied* list assignments.

Suppose that we have labeled the vertices of Σ so that there is exactly one edge from v_n to v_0, exactly two edges from v_0 to v_1, and v_j is adjacent to v_{j+1} for all j. Suppose that $i = 0$. Then Painter should color v_1 with 0, and then color the vertices v_j greedily by increasing j, resulting in a maximum independent set I in $\Sigma \setminus \{v\}$ of vertices that are colored 0. If the uncolored vertices form an independent set, then Painter wins, as we see that Lister must add at least one more color to the lists for each uncolored vertex. Otherwise, the only possibility is that v_0 and v_n are uncolored.

If we let X be the set of uncolored vertices, and $U = \{v_n\}$, then we can apply Proposition 5 and conclude that Painter has a winning strategy for $\Sigma[X]$.

Suppose that $i > 0$. Since L is satisfied at the ith round, we see that $L(v_1) = L(v_0)$ and $L(v_1) = -L(v_0)$. This forces $L(v_1) = \{i, -i\}$, and thus $L(x) = \{-i, i\}$ for all $x \in V$. Now we start by coloring v_n with i, and start coloring v_j greedily by *decreasing* j. Note that we skip vertices that cannot be colored. If a vertex cannot be colored, then it is adjacent to a colored vertex by a double edge. After we finish, we see that the uncolored vertices of $\Sigma \setminus v_1$ from an independent set. Moreover, since uncolored vertex x was incident to a double edge, the degree of those vertices was at least three, so Lister still has to add more colors to $L(x)$. Thus, Painter will win if Painter can paint v_1. If v_1 is still uncolored, then it is because it is incident to two double edges, and has degree 4. If we let X be the set of uncolored vertices, and $U = \{v_1\}$, then Proposition 5 implies that Painter has a winning strategy for $\Sigma[X]$.

Thus, we are left in the case where Σ is a simple graph or is a double cycle. We show that balanced odd cycles, unbalanced even cycles, and odd double cycles are not degree choosable. If Σ is a balanced odd cycle, or an unbalanced even cycle, then it is not degree choosable. Setting $L(v) = \{-1, 1\}$ for all $v \in V$ gives a list assignment for which no proper coloring exists. Similarly, if Σ is an odd double cycle, then it is not degree choosable, and setting $L(v) = \{-2, -1, 1, 2\}$ for all $v \in V$ gives a list assignment where no proper coloring exists.

Next, we show that unbalanced odd cycles are degree choosable, but not degree paintable. Suppose that Σ is an unbalanced odd cycle. Label the vertices as v_0 through v_n with v_j adjacent to v_{j+1} for all j. Up to switch equivalence, we can assume that every edge of Σ is positive except the edge from v_n to v_0, which is negative. Let L be a satisfied list assignment. Thus $L(v_j) = L(v_{j+1})$ for $j < n$ and $L(v_0) = -L(v_n)$. Hence there exists an i such that $L(u) = \{i, -i\}$ for all u. We color v_0 with i. For $j > 0$, we color v_j with $(-1)^j i$. This yields a proper coloring.

However, Σ is not degree paintable: Lister starts by placing $0 \in L(v)$ for every v. This yields a satisfied list assignment for the first round. Since Σ is odd, Painter has to leave a signed edge uv^ϵ with uncolored vertices u and v. Then Lister puts $1 \in L(u)$, and $\epsilon 1 \in L(v)$, ensuring that Painter cannot properly color the graph.

In the remaining case Σ is either a balanced even cycle or even double cycle. We show the graph is paintable. Let i be the first round where Lister adds colors to the lists. If it is a balanced even cycle, then up to switching equivalence, we assume every edge is positive. If $i = 0$, or $i > 0$ and Σ is an even double cycle. Painter finds a maximum independent set I and colors it i. The remainder of the graph is an independent set which can be 1-painted, so Painter wins. So we are left in the case where $i > 0$, and Σ is an even cycle where every edge is positive. In this case, we have $L(x) = \{i\}$ for all x, or we have $L(x) = \{i, -i\}$ for all x. We take a maximum independent set and color it i. If $L(x) \neq \{i\}$, then we color the remaining vertices with $-i$. Otherwise, $|L(x)| = 1 < 2 = d(x)$ for all x, so Lister must add a color to the lists for each of the remaining vertices. Hence Painter wins. □

Lemma 11. Let Σ be a loopless signed graph such that $\underline{\Sigma}$ is a theta graph. Then Σ is degree paintable.

Proof. Let x, y be the unique vertices of degree 3 in $\underline{\Sigma}$. Let i be the first stage for which Lister adds colors to lists. By Lemma 6, we assume that L is a satisfied list assignment.

Suppose that $i > 0$, and that Σ is not balanced. Then $L(v) = \{i, -i\}$ for all v. Then we know that we can color $\Sigma - x$ with the colors $\pm i$. Since $d(x) \geqslant 3$, Lister must give at least one more color to $L(x)$, and hence Painter can complete the coloring.

Suppose that Σ is a balanced graph. We know that Σ contains an even cycle C. By Lemma 10, C is degree paintable. By Lemma 8, it follows that Σ is as well. □

Lemma 12. Let Σ be a signed graph such that $\underline{\Sigma}$ is a complete graph on at least 4 vertices. Then the following are equivalent:

1) Σ° is not a balanced complete graph or a double complete graph.
2) Σ is degree choosable.
3) Σ is degree paintable.

Proof. Suppose that Σ° is a simple graph. If it is unbalanced, then it contains an unbalanced triangle T. Let B be a subset on 4 vertices including T. By Lemma 8, it suffices to show that B is d_B-paintable.

Now we give a winning strategy for Painter on an unbalanced simple signed clique on four vertices a, b, c, d where a, b, c form the vertices of an unbalanced triangle T. Let i be the first stage for which Lister adds colors to lists. By Lemma 6, we assume that L is satisfied. Since Σ is unbalanced, this implies that $L(v) = \{i, -i\}$ for all v. Suppose that $i = 0$. Then color d with the color 0. What is left is an unbalanced odd cycle, so by Lemma 10, Painter has a winning strategy to d_T-paint T. Hence Painter has a winning strategy.

If $i > 0$, then Painter should color the triangle a, b, c. Then we see that Lister still has to add at least one more color to $L(d)$, and hence Painter will win.

On the other hand, if Σ° is a balanced simple complete graph, then Σ is not degree choosable: Let A, B be a balanced bipartition of Σ, and suppose Σ has degree k. Let $L(v) = \{1, \ldots, k\}$ for $v \in A$, and let $L(v) = \{-1, \ldots, -k\}$ for $v \in B$. Finally, add $0 \in L(v)$ if v contains a negative loop. Then clearly Σ has no proper L-coloring.

Now assume that Σ has parallel edges. By Lemma 9, we can assume that any pair of vertices that have two edges between them have *exactly* two edges between them, and that the edges are opposite in sign. Suppose that Σ' is not a double complete graph. Then Σ° must contain a triangle $T = \{a, b, c\}$ such that there is exactly one edge between a and b, and exactly two edges between b and c. By Lemma 8, it suffices to show that T is degree paintable. However $\underline{T}$ is a cycle, so the result follows from Lemma 10.

So assume that Σ° is a double clique on n vertices. Let $L(v) = \{-n+1, \ldots, -1, 1, \ldots, n-1\}$ for all $v \in \Sigma$, and put 0 into $L(v)$ whenever v contains a negative loop. Then Σ is not L-colorable, and hence is not degree choosable. $\square$

Recall that a *block* of Σ is a maximal biconnected subgraph. A vertex that belongs to more than one block of G is a *cut vertex*. A *leaf block* is a block with only one cut vertex. A *Gallai tree* is a connected graph where each block is an odd cycle or a clique. If each block of Σ° is an odd cycle, unbalanced even cycle, odd double cycle, balanced complete graph, or double complete graph, then we call Σ a *signed Gallai tree*. Note that the definition is meant to be a signed analogue of Gallai tree: since we allow unbalanced even cycles, it could be the case that Σ is a signed Gallai tree while $\underline{\Sigma}$ is not a Gallai tree.

Lemma 13. Let Σ be a signed Gallai tree such that no block of Σ is an unbalanced odd cycle. Then Σ is not degree choosable.

Proof. We prove that there is a degree-compatible list assignment L on Σ that involves only non-zero integers, and for which Σ is not L-colorable whenever Σ is a loopless and contains no unbalanced odd cycle blocks. The result is proven by induction on the number b of blocks.

Suppose that $b = 1$. Then Σ is a balanced complete graph, a double complete graph, a balanced odd cycle, an unbalanced even cycle, or an odd double cycle. By the proofs of Lemma 12 or Lemma 10, there is a degree compatible list assignment L for Σ involving only non-zero integers for which there is no L-coloring.

So let Σ be a loopless, signed Gallai tree with no unbalanced odd cycle blocks. Suppose Σ has b blocks, where $b > 0$. Let B be a leaf block, with cut vertex v. By the inductive hypothesis, $\Sigma' = \Sigma \setminus (B \setminus \{v\})$ has fewer blocks, and hence a degree compatible list assignment L' involving only non-zero integers for which there is no L'-coloring. Similarly, by induction, B has a degree compatible list assignment L'' involving non-zero integers for which there is no L''-coloring.

Let m be the maximum integer such that $\{m, -m\} \cap L'(u) \neq \varnothing$ for some $u \in \Sigma'$. For $u \in B$, let $L_m^-(u) = \{i - m : i < 0, i \in L''(u)\}$, and $L_m^+(u) = \{i + m : i > 0, i \in L''(u)\}$. Now we construct our list assignment. For $u \in \Sigma \setminus B$, let $L(u) = L'(u)$. For $u \in B \setminus \{v\}$, let $L(u) = L_m^-(u) \cup L_m^+(u)$. Finally, let $L(v) = L'(v) \cup L_m^-(u) \cup L_m^+(u)$.

Observe that L is degree compatible for Σ. Suppose that there is an L-coloring f. If $f(v) \in L'(v)$, then f restricts to an L'-coloring of Σ', which is impossible, by our inductive assumption. Thus $f(v) \in L_m^-(v) \cup L_m^+(v)$. Then we define a proper L''-coloring of B, to obtain a contradiction. For $u \in B$, we let $f'(u) = f(u) - m$ if $f(u) > 0$, and $f'(u) = f(u) + m$, if $f(u) < 0$. Then $f'(u) \in L''(u)$ for all $u \in B$, giving a proper L''-coloring.

Now suppose that Σ is a signed Gallai tree that contains negative loops, but no unbalanced odd cycle blocks. Then we have constructed a degree-compatible list assignment L' for Σ° that contains only non-zero numbers and for which Σ° is not L'-colorable. If we add 0 to $L'(v)$ for any v that is part of a negative loop, then we obtain a new list assignment L that is degree-compatible for Σ. We see that Σ is not L-colorable. $\square$

Lemma 14. Let Σ be a Gallai tree that does not have a pair of intersecting unbalanced odd cycle blocks. Suppose that, for every unbalanced cycle block B and every $v \in B$, v does not have a negative loop. Then Σ is not degree paintable.

Proof. We prove by induction on b, the number of blocks, that there is a winning strategy for Lister where Lister puts 0 into $L(v)$ if and only if v has a negative loop or is contained in an unbalanced odd cycle block.

First, suppose that $b = 1$. Suppose that Σ is an unbalanced odd cycle block. We describe a winning strategy for Lister by induction on b, the number of blocks. Moreover, the strategy puts 0 into $L(v)$ if and only if v has a negative loop, or v is contained in an unbalanced odd cycle block. Then Lister follows the winning strategy for Lister detailed in the proof of Lemma 10 for unbalanced odd cycles. We observe that the strategy given in that proof also starts by putting $0 \in L(v)$ for every $v \in B$. Hence the Lister wins.

Now suppose that Σ is not an unbalanced odd cycle block. Then Σ° is not degree choosable. For any $v \in \Sigma$ that contains a negative loop, we add 0 to $L(v)$. We know from the proof of Lemma 10 or Lemma 12 that there is a degree-compatible List assignment L' for Σ° such that Σ° is not L'-colorable and $0 \notin L'(v)$ for any v. Lister adapts the following strategy: in the i-th round, Lister adds ϵi to $L(v)$ if and only if $\epsilon i \in L'(v)$, for all $v \in \Sigma$ and $\epsilon \in \{+, -\}$. This results in a winning strategy for Lister, so Σ is not degree paintable.

Now suppose that $b > 1$. Let B be a leaf block and let x be a cut vertex with $x \in B$. We let $\Sigma' = \Sigma \setminus (B \setminus \{x\})$. First, Lister puts $0 \in L(v)$ for all v such that v has a negative loop or v is part of an unbalanced cycle block.

By induction, Lister has a winning strategy for B where the 0th step matches the 0th step we just described. We let the Lister follow that strategy until there is an uncolored vertex v with $d_B(v) = |L(v)|$. If $v \neq x$, then the Lister wins. So suppose that $v = x$. Then by induction, the Lister has a winning strategy for Σ' where the 0th step involves adding $0 \in L(v)$ if and only if v contains a negative loop or is contained in an unbalanced odd cycle block. We implement that strategy for several rounds, and we must end with an uncolored vertex. If we find a vertex $u \neq x$ with with $d_{\Sigma'}(u) = |L(u)|$, then Lister wins. Otherwise, every vertex of the graph is colored but x, and $|L(x)| = d_{\Sigma'}(x) + d_B(x) = d_\Sigma(x)$. In that case, Lister wins. □

6. Proof of the main theorems

Proof of Theorem 2. First, suppose that $\underline{\Sigma^\circ}$ is not a Gallai tree. Then Σ° contains a block B such that $\underline{B}$ is not a cycle or clique. By Lemma 8 and Lemma 9, it suffices to show that B is degree paintable. Since $\underline{B}$ is 2-connected and is not a cycle or clique, B contains an induced subgraph T such that $\underline{T}$ is a theta graph. By Lemma 11, T is degree paintable. By Lemma 8, B is degree paintable. Thus Σ is degree paintable.

We assume that $\underline{\Sigma^\circ}$ is a Gallai tree. Suppose that Σ° has a block B that is degree choosable. Then by Lemma 8, Σ° is degree choosable, and by Lemma 9, we see that Σ is degree choosable. So assume that every block of Σ° is not degree choosable. By Lemma 10 and Lemma 12, we see that every block of Σ° must be a balanced odd cycle, an unbalanced even cycle, an odd double cycle, a balanced clique, or a double clique. Then Σ

is the type of Gallai tree described in Lemma 13, and hence is not degree choosable.

Similarly, if Σ is a Gallai tree that contains a block B that is degree paintable, then Σ is degree paintable. Conversely, if Σ meets the conditions of Lemma 14, then it is not degree paintable. Suppose that Σ° is a signed Gallai tree that contains two intersecting unbalanced odd cycle blocks. Let A be the union of these two cycles. We show that A is degree paintable. Then Σ is degree paintable by Lemma 8. We describe a winning strategy for Painter. As usual, let i be the first stage for which Lister adds colors to the lists, and assume that L is satisfied.

Suppose first that $i > 0$. Then $L(u) = \{i, -i\}$ for all u, and A can be colored. Suppose that $i = 0$. Suppose that A' is still the intersection of two unbalanced odd cycles. Let x be the cut vertex of A. We find a maximum independent I set of $A \setminus \{x\}$, and color the vertices in I with 0. Then $A \setminus I$ consists of an independent set, and a path on three vertices u, x and v, with x as the internal node. However, Lister still needs to add three more colors to $L(x)$, and one more color to the remaining vertices' lists. Hence Painter will be able to win by choosing to color u and v before x whenever such a choice is available. Hence A° is degree paintable, and so A is degree paintable.

We are left in the case that Σ has an unbalanced odd cycle block A and a vertex $v \in A$ that contains a negative loop. We show that A is degree paintable, and hence so is Σ. We label the vertices of A as $v_0, \ldots, v_n$ where v_j is adjacent to v_{j+1} for all j. We also have $L(v) = \{0\}$ for every v. Suppose that there exists a vertex v that does not have a negative loop. Without loss of generality, we assume that v_0 does not have a negative loop. Let $I = \{v_0\}$, and consider each v_j, as j increases, greedily adding v_j to I if v_j does not have a negative loop and adding v_j to I results in an independent set. Then we color the vertices in I with 0. Then $A \setminus I$ consists of paths. Moreover, given a path P on $v_j, \ldots, v_{j+k}$ with $k > 0$, we see that $v_{j+1}, \ldots, v_{j+k}$ all have negative loops. Hence if we set $U = v_{j+k}$, then P° is $(d_P + \delta_U)$-paintable. Hence Painter wins.

Hence every vertex of A must have a negative loop. Then Painter leaves every vertex uncolored. However, Painter is able to paint A° on round $i > 0$ as we have shown above. Thus Painter will win. □

Now we discuss deriving Brooks' Theorem for signed graphs.

Proof of Theorem 1. Let Σ be a loopless connected signed graph. Suppose that there is a $v \in V$ with $d(v) < \Delta(\Sigma)$. Let C be a component of $\Sigma - v$.

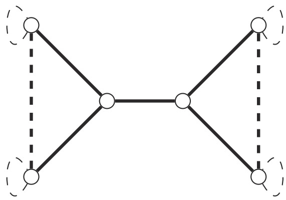

Figure 2. A regular signed Gallai tree.

Let $U = N(v) \cap C$. By Proposition 5, we see that C is $(d_C + \delta_U)$-paintable. Since $(d_C + \delta_U)(x) \leqslant \Delta(x)$ for all $x \in C$, it follows that C is Δ_Σ-paintable.

Taking the union over all the components of $\Sigma - v$, we see that $\Sigma - v$ is Δ_Σ-paintable. By Lemma 4, applied to $A = V \setminus \{v\}$, we see that Σ is Δ-paintable.

Suppose that Σ is Δ-regular and is not Δ-paintable. In light of Theorem 2. We see that Σ is a regular signed Gallai tree. Suppose that Σ has at least two blocks. Let B be a block, and let v be a cut vertex with $v \in B$. We observe that every block of a loopless signed Gallai tree is regular, since each block is a simple cycle, a double cycle, a simple clique, or a double clique. Thus $d(v) > d(x)$ for any $x \in B \setminus \{v\}$, contradicting the fact that Σ is regular. Hence Σ only has one block, and is biconnected. Then Σ is an odd cycle, an unbalanced even cycle, an odd double cycle, a balanced clique, or a double clique. Similarly, if we assume that Σ is Δ-regular, but not Δ-choosable, we see that Σ is biconnected, and is a balanced odd cycle, an unbalanced even cycle, an odd double cycle, a balanced clique, or a double clique. □

Note that if we allow loops, then there are more examples, such as in Figure 2. The best we can say is that Σ is a regular signed Gallai tree subject to the conditions of Theorem 2.

7. Two choosability and paintability

In this section we classify the 2-choosable signed graphs: those Σ where $\chi_\ell(\Sigma) \leqslant 2$. We see that $\chi(\Sigma) \leqslant 2$ if and only $-\Sigma$ is balanced. This is because Σ is balanced if and only if Σ has a balanced bipartition A, B. If $-\Sigma$ has a balanced bipartition A, B, then coloring all vertices in A with the color 1 and all vertices in B with the color -1 results in a signed 2-coloring. Similarly, given a signed 2-coloring f of Σ, we see that $f^{-1}(-1), f^{-1}(1)$ is a balanced bipartition of $-\Sigma$.

Recall that the core of a graph is obtained by removing vertices of degree 1 until no such vertices remain.

Theorem 3 ([2])**.** Let G be connected graph. Then G is 2-choosable if and only if its core is a vertex, and even cycle, or a theta graph $\theta_{2,2,2m}$ where $m \geqslant 1$.

We prove a version of this result for signed graphs.

Theorem 15. A connected signed graph Σ is 2-choosable if and only if its core is one of the following graphs:
1) a vertex,
2) a balanced even cycle,
3) an unbalanced odd cycle,
4) a balanced $\theta_{2,2,2m}$, with $m \geqslant 1$.

The proof mimics the original proof due to Rubin in the classical graph case. We first discuss some preliminary lemmas.

Lemma 16. Let Σ be a signed graph. Let $u, v, w \in \Sigma$ such that u and w are not adjacent, uv and vw are positive, and $d(v) = 2$. If $\Sigma/\{u, v, w\}$ is not 2-choosable, then Σ is not 2-choosable.

Proof. If Σ/uvw is not 2-choosable, then there exists a list assignment L' for Σ/uvw such that $|L'(x)| = 2$ for all $x \in V(\Sigma/uvw)$ such that Σ/uvw has no L'-coloring. Define L on Σ by

$$L(x) = \begin{cases} L'(x) & x \notin \{u, v, w\} \\ L'(x_{uvw}) & x \in \{u, v, w\} \end{cases}$$

where x_{uvw} is the contracted vertex. If Σ is L-colorable with coloring f, then defining f' by

$$f'(x) = \begin{cases} f(x) & x \neq x_{uvw} \\ f(u) & x = x_{uvw} \end{cases}$$

yields an L'-coloring of Σ/uvw, which is a contradiction. □

Now we define the notion of weight of a cycle. The weight of C is $(-1)^{e_+(C)}$, where $e_+(C)$ is the number of positive edges in C.

Lemma 17. Let Σ be a connected signed graph such that $\delta(\Sigma) \geqslant 2$. If Σ is neither K_1, nor a positive weight cycle, nor a balanced $\theta_{2,2,2m}$ with $m \geqslant 1$, then Σ contains one of the following subgraphs:

(i) a cycle of negative weight,
(ii) two vertex disjoint cycles of positive weight joined by a path,
(iii) two cycles of positive weight that share exactly one vertex,
(iv) a theta graph whose underlying graph is $\theta_{a,b,c}$, $a \neq 2, b \neq 2$,
(v) an unbalanced theta graph whose underlying graph is $\theta_{2,2,m}$,
(vi) a balanced theta graph whose underlying graph is $\theta_{2,2,2,2m}$.

Proof. If Σ is balanced, then we perform switching operations to obtain an ordinary graph. In this case, Erdös, Rubin, and Taylor showed that this graph has one of the following as a subgraph (not necessarily induced): (a) an odd cycle, (b) two vertex disjoint even cycles joined by a path, (c) two even cycles meeting in exactly one vertex, (d) a theta graph $\theta_{a,b,c}$ with $a \neq 2$ and $b \neq 2$, or (e) a $\theta_{2,2,2,2m}$ with $m \geqslant 1$. So suppose that Σ is unbalanced.

If Σ has an induced negative weight cycle, then we are done, since (i) above holds. So we assume that every cycle has positive weight; that is, the number of positive edges on each cycle is even. This implies that $-\Sigma$ is balanced. Suppose that Σ contains two vertex disjoint cycles. Since Σ is connected, it contains a path that connects these two cycles; so Σ contains (ii) above. Similarly, if two cycles share a single vertex, then we are done, since Σ contains (iii) above.

Let C be a minimum unbalanced cycle. By assumption, Σ is not a positive weight cycle. If $C = \Sigma$, then C also has negative weight, and Σ is of type (i). Thus $C \neq \Sigma$. Since $\delta(\Sigma) \geqslant 2$, there are other cycles of Σ. We may assume that there exists a cycle which intersects C in more than one vertex. Thus, Σ must contain a path R whose ends are on C, but whose internal vertices are not. Together $C \cup R$ forms a theta graph $\theta_{a,b,c}$. If $a \neq 2$ and $b \neq 2$, then Σ contains (iv) above. So suppose $\underline{\Sigma}$ contains $\theta_{2,2,c}$. Then this theta graph is unbalanced, and thus Σ contains a graph of the form (v) above. □

Lemma 18. Let Σ be a signed graph of negative weight. Suppose that Σ is one of the following signed graphs:

(i) a cycle,
(ii) two vertex disjoint cycles of positive weight joined by a path,
(iii) two cycles of positive weight that share exactly one vertex,
(iv) a theta graph whose underlying graph is $\theta_{a,b,c}$, $a \neq 2, b \neq 2$,
(v) an unbalanced theta graph whose underlying graph is $\theta_{2,2,m}$,

(vi) a balanced theta graph whose underlying graph is $\theta_{2,2,2,2m}$.
Then Σ is not 2-choosable.

Proof. Let Σ be a cycle of negative weight. Then Σ is not 2-colorable, and hence not 2-choosable.

Suppose that there exists a signed graph Σ that is of type (ii)-(vi) that is 2-choosable. We prove the theorem by induction on $|\Sigma|$. We see that the base case is $|\Sigma| = 2$, where Σ consists of two vertices connected by an edge, where each vertex has a negative loop. The resulting graph appears in Figure 3, along with a list assignment for which it is not L-colorable.

So we assume that $|\Sigma| > 2$. Suppose that Σ is of type (ii), and that the connecting path P has length at least two. By switching equivalence, we can assume that there are three vertices $x, y, z \in P$ with $xy, yz \in E(\Sigma)$ and $\sigma(xy) = \sigma(yz) = +$. We let $\Sigma' = \Sigma/x, y, z$. We see that Σ' is of type (ii) or (iii), so by induction Σ' is not 2-choosable. By Lemma 16, we see that Σ is also not 2-choosable.

Now suppose that Σ is of type (iii) or type (ii) where the path connecting the two cycles consists of one edge. Suppose that at least one cycle C of Σ has length at least five. Then, up to switch equivalence, we may assume that there are vertices $x, y, z \in C$ with $xy, yz \in E(\Sigma)$, $xz \notin E(\Sigma)$, and $\sigma(xy) = \sigma(yz) = +$. Moreover, Σ/xyz is also of type (ii) or (iii). By induction, Σ/xyz is not 2-choosable, and by Lemma 16, Σ is not 2-choosable.

For graphs of type (ii) or (iii), we are left with the case where both cycles have length at most four, and the connecting path, if it exists, is an edge. If Σ is balanced, then it is switch equivalent to an ordinary graph, which is not 2-choosable by Theorem 3. Since every cycle of Σ is positive, $-\Sigma$ is balanced, and thus Σ is switch-equivalent to a signed graph with only negative edges. Thus we only need to consider unbalanced signed graphs of type (ii) or (iii) where each cycle has length at most four, each edge is negative, and the connecting path, if it exists is an edge. We list all such graphs in Figure 3, along with list assignments for which the graphs are not L-colorable.

If Σ is a balanced graph of type (iv) or (vi), then up to switch equivalence Σ is an ordinary graph. Then we already know that Σ is not 2-choosable by Theorem 3. So suppose that Σ is an unbalanced graph of type (iv) or (v). Let d be the maximum of a, b, and c. Suppose that $d \geqslant 3$. Then there exists vertices x, y, z on the path of d vertices such that $xy, yz \in E(\Sigma)$, $xz \notin E(\Sigma)$, and $\sigma(xy) = \sigma(yz) = +$. If $d \geqslant 4$, then $\Sigma/\{x, y, z\}$ is still of type (iv) or (vi), so by induction is not 2-choosable. By Lemma 16, neither is Σ. Thus we have $d \leqslant 3$. If Σ contains a cycle

C of negative weight, then C is not 2-choosable, and hence neither is Σ. If every cycle of Σ has positive weight, then $-\Sigma$ is balanced, and Σ is switch-equivalent to a signed graph with only negative edges. Thus we are reduced to the case of an unbalanced θ graph with only negative edges and $d \leqslant 3$.

Hence we are in the case where $a = 1, b = 2$ and $c = 3$, or $a \leqslant b \leqslant c = 2$. We may assume that Σ has no cycle of negative weight. This implies that $-\Sigma$ is balanced, and hence Σ is switching equivalent to a signed graph with only negative edges. There are only two

We see in Figure 3 various such examples where the lengths of the two cycles are at most four. $\square$

Proof of Theorem 15. Let Σ' denote the core of Σ. If Σ' is one of those graphs listed in Theorem 15, then it is 2 choosable. The first three families are regular graphs with max degree two, that are already known to be degree choosable, by Lemma 10. On the other hand, since choosability is invariant under switching, a balanced $\theta_{2,2,2m}$ is 2-choosable if and only if $\theta_{2,2,2m}$ is 2-choosable, by just switching vertices until there are only positive edges. Erdős, Rubin and Taylor already showed that $\theta_{2,2,2m}$ is 2-choosable. We see that we can use induction on $|\Sigma \setminus \Sigma'|$ to show that Σ is also 2-choosable, since we obtain Σ' from Σ by recursively removing vertices of degree one.

Suppose that Σ' is not one of the graphs listed in Theorem 15. Then by Lemma 17 and Lemma 18, Σ' contains a subgraph B that is not 2-choosable. Let L' be a list assignment for B such that $|L'(u)| = 2$ for all $u \in B$ and such that B has no L'-coloring. Define a list assignment L for Σ by

$$L(x) = \begin{cases} L'(x) & x \in B \\ \{1, -1\} & x \notin B \end{cases}$$

If f is an L-coloring of Σ, then f restricts to an L'-coloring of B, a contradiction. Hence Σ is not 2-choosable. $\square$

We know unbalanced odd cycles are not 2-paintable, and it was shown in [7] that $\theta_{2,2,2m}$ is only 2-paintable when $m = 1$. Thus, we obtain the following:

Proposition 19. A connected signed graph Σ is 2-paintable if and only if its core is a vertex, a balanced even cycle, or a balanced $K_{2,3}$.

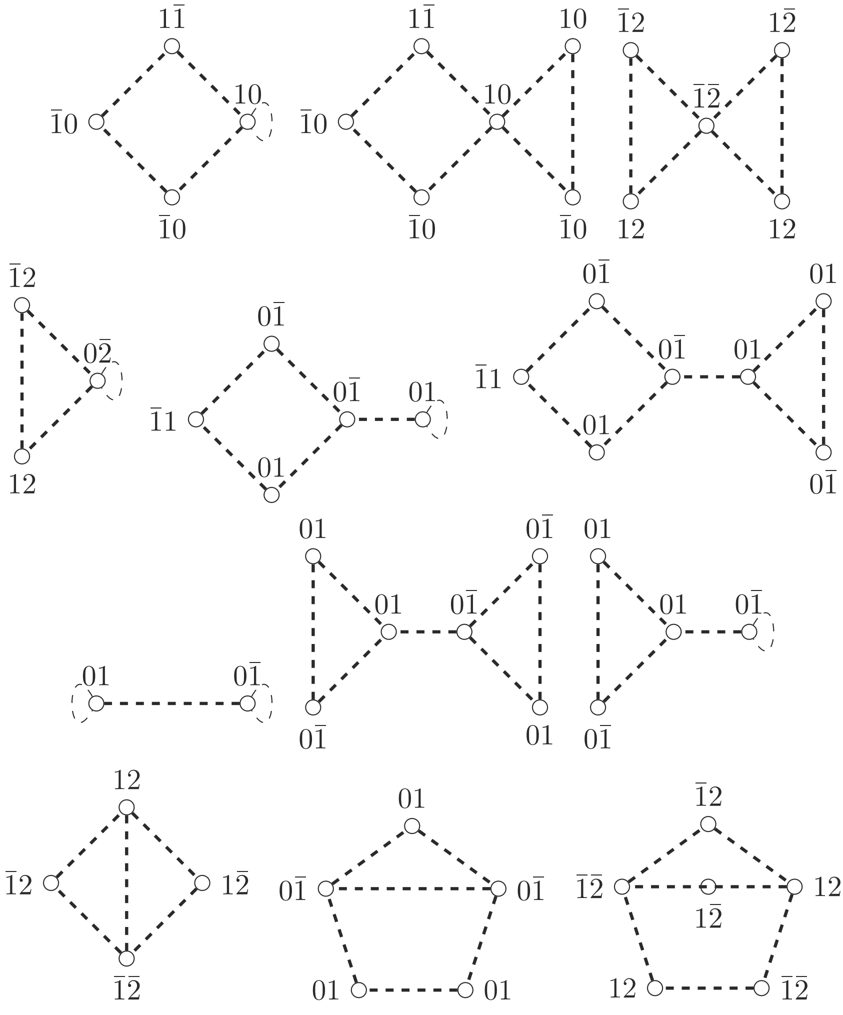

FIGURE 3. Signed graphs that are not 2-choosable. We use $\bar{1}$ and $\bar{2}$ for -1 and -2.

8. Open questions

Finally, we end with some open questions regarding possible future work. One problem to consider for signed graphs would be to study an analogue of Reed's conjecture [4]. For ordinary graphs, Reed's conjecture is that $\chi(G) \leqslant \lfloor \frac{\omega(G)+\Delta(G)+1}{2} \rfloor$. In our case, we must define $\omega(\Sigma)$ for a signed graph. One definition is that $\omega(\Sigma)$ is the largest k such that Σ

contains a $(k-1)$-regular signed Gallai tree. With that in mind, one can ask if $\chi(\Sigma) \leqslant \lfloor \frac{\omega(\Sigma)+\Delta(\Sigma)+1}{2} \rfloor$.

Reed's conjecture is known in certain cases, particularly when we replace $\chi(G)$ with the fractional chromatic number. What is the right definition of fractional chromatic number of a signed graph? Can we prove an analogue of Reed's conjecture in that case?

There is also work on studying hypergraph coloring. When is a signed hypergraph degree paintable? Also, what is the definition of paintability, or choosability, for arbitrary gain graphs? Recall that a gain graph is an oriented graph where the edges are labeled with elements of a fixed group G. In the case of signed graphs, the group is $\{-1, 1\}$, and the orientation does not matter.

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