© Algebra and Discrete Mathematics
Volume **33** (2022). Number 2, pp. 78–91
DOI:10.12958/adm1793 RESEARCH ARTICLE

Further combinatorial results for the symmetric inverse monoid*

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Communicated by V. Mazorchuk

Abstract. Let $\mathcal{I}_n$ be the set of partial one-to-one transformations on the chain $X_n = \{1, 2, \ldots, n\}$ and, for each α in $\mathcal{I}_n$, let $h(\alpha) = |\operatorname{Im}\alpha|$, $f(\alpha) = |\{x \in X_n : x\alpha = x\}|$ and $w(\alpha) = \max(\operatorname{Im}\alpha)$. In this note, we obtain formulae involving binomial coefficients of $F(n; p, m, k) = |\{\alpha \in \mathcal{I}_n : h(\alpha) = p \wedge f(\alpha) = m \wedge w(\alpha) = k\}|$ and $F(n; \cdot, m, k) = |\{\alpha \in \mathcal{I}_n : f(\alpha) = m \wedge w(\alpha) = k\}|$ and analogous results on the set of partial derangements of $\mathcal{I}_n$.

Introduction and preliminaries

As remarked by Gomes and Howie [10], inverse semigroups (see [11, Chapter V]) are of interest not only as a naturally occurring special case of semigroups but also for their role in describing partial symmetries. Mathematically this property is expressed by the Vagner-Preston Theorem [11, Theorem 5.1.7], by which every (finite) inverse semigroup is embedded in an appropriate (finite) symmetric inverse semigroup $\mathcal{I}_X$, consisting of all partial one-to-one maps (equivalently subpermutations) of X.

Let $X_n = \{1, 2, \ldots, n\}$. A (partial) transformation $\alpha : \operatorname{Dom}\alpha \subseteq X_n \to \operatorname{Im}\alpha \subseteq X_n$ is said to be *full* or *total* if $\operatorname{Dom}\alpha = X_n$; otherwise it is called *strictly* partial. Let $\mathcal{I}_n$ be the set of partial one-one transformations on X_n. Then $\mathcal{I}_n$ is the *(inverse) semigroup of partial one-one maps* more

*The authors would like to acknowledge support from King Fahd University of Petroleum & Minerals and Khalifa University of Science and Technology, respectively.
2020 MSC: 20M18, 20M20, 05A10, 05A15.
Key words and phrases: partial one-to-one transformation, symmetric inverse monoid, height of α, fix of α, (left) waist of α, permutation, (partial) derangement.

commonly known as the *symmetric inverse monoid*. Enumerative problems of an essentially combinatorial nature arise naturally in the study of semigroups of transformations. In particular, for the semigroup $\mathcal{I}_n$ and some of its subsemigroups many interesting and delightful combinatorial results were obtained in [1–3, 7–9, 12–18, 22, 24, 25]. See also the remark by Cameron in [5, § 4.7]. One of the authors in [23] attempted to give a unified and coherent account of these and other results and also proposed some combinatorial questions that need further investigation. Observe that in the special case of *permutations*, Cameron [4, 5], Comtet [6] and Riordan [19] are replete with many interesting results. This paper is concerned with investigating some of the questions proposed by Umar [23] on $\mathcal{I}_n$, where we provide some answers in this section (§ 1) while in § 2 we obtain analogous results about partial one-to-one *derangements* in $\mathcal{I}_n$. As in [17], recurrence relations and (exponential) generating functions play a pivotal role in the approach used in this paper.

On a partial one-one mapping of X_n the following parameters are defined: the *height* of α is $h(\alpha) = |\operatorname{Im}\alpha|$, the *fix* of α is $f(\alpha) = |F(\alpha)|$, where $F(\alpha)$ is the set $\{x \in X_n : x\alpha = x\}$ of fixed points of α, and the *right waist* (or *waist*, for brevity) of α is $w(\alpha) = \max(\operatorname{Im}\alpha)$ (the *left waist* is defined as $\min(\operatorname{Im}\alpha)$). We recall the dual of [15, Proposition 2.1], which says that $c(n, p)$, defined as the number of surjective partial derangements $\alpha : X_n \longrightarrow Y_p = \{y_1, y_2, \ldots, y_p\} \subseteq X_n$, satisfies

$$c(n,p) = p!\sum_{j=0}^{p}\binom{n-j}{p-j}\frac{(-1)^j}{j!}. \tag{1}$$

Lemma 1. *Let $c(n, p)$ be as defined in (1). Then*

$$\sum_{m=0}^{p}\binom{p}{m}c(n-m,p-m) = \binom{n}{p}p!\,.$$

Proof. Let $\alpha \in \mathcal{I}_n$ be such that $\operatorname{Im}\alpha = Y_p = \{y_1, y_2, \ldots, y_p\} \subseteq X_n$. Then the expression $\binom{p}{m}c(n-m, p-m)$ is the number of such α with exactly m fixed points. Thus, taking the sum over m from 0 to p gives the number of all such maps. However, the expression $\binom{n}{p}p!$ clearly represents the number of all such maps. Hence the result follows. □

Next, as in Umar [23], define equivalences on $\mathcal{I}_n$ by equalities of all or some of the three parameters defined above as

$$F(n;p,m,k) = |\{\alpha \in \mathcal{I}_n : h(\alpha) = p \wedge f(\alpha) = m \wedge w(\alpha) = k\}|, \quad (2)$$

$$F(n;p,m,\cdot) = |\{\alpha \in \mathcal{I}_n : h(\alpha) = p \wedge f(\alpha) = m\}|, \quad (3)$$

$$F(n;p,\cdot,k) = |\{\alpha \in \mathcal{I}_n : h(\alpha) = p \wedge w(\alpha) = k\}|, \quad (4)$$

$$F(n;\cdot,m,k) = |\{\alpha \in \mathcal{I}_n : f(\alpha) = m \wedge w(\alpha) = k\}|. \quad (5)$$

The following lemma will be useful in what follows.

Lemma 2. *[23, Lemma 2.1] Let $X_n = \{1, 2, \ldots, n\}$. For a given $\alpha \in \mathcal{I}_n$, we set $p = h(\alpha), m = f(\alpha)$ and $k = w(\alpha)$. We also define $F(n;p,\cdot,k) = 1$ if $k = p = 0$. Then we have the following:*

1) $n \geqslant k \geqslant p \geqslant m \geqslant 0$;
2) $k = 1 \implies p = 1$;
3) $p = 0 \Leftrightarrow k = 0$.

Observe that

$$F(n;p,m,\cdot) = \sum_{k=p}^{n} F(n;p,m,k), \ \ F(n;p,\cdot,k) = \sum_{m=0}^{p} F(n;p,m,k), \ldots$$

and any two-variable function can be expressed as a sum of appropriate three-variable functions and so on.

We now have the following theorem.

Theorem 1. *Let $c(n,p)$ and $F(n;p,m,k)$ be as defined in (1) and (2), respectively. Then*

$$F(n;p,m,k) = \binom{k-1}{p-1}\binom{p}{m} c(n-m, p-m).$$

Proof. Let $\alpha \in \mathcal{I}_n$ be such that $h(\alpha) = p$, $f(\alpha) = m$ and $w(\alpha) = k$. First, we fix our image set to be Y_p. The p images which must include k, can be chosen in $\binom{k-1}{p-1}$ ways, since k is the maximum element in $\operatorname{Im}\alpha$. Next, the m fixed points can be chosen from the already chosen p images in $\binom{p}{m}$ ways. Now from the remaining $n-m$ unused elements for the domain and $p-m$ elements for the images, there are $c(n-m, p-m)$ ways to map these elements surjectively and injectively without fixed points. Hence the result follows. □

From the above result we recover the following results which can be found in [23, Table 3, p. 119]:

Lemma 3. *Let $c(n,p)$ and $F(n;p,m,\cdot)$ be as defined in (1) and (3), respectively. Then*

$$F(n;p,m,\cdot) = \frac{n!}{m!(n-p)!}\sum_{j=0}^{p-m}\binom{n-m-j}{p-m-j}\frac{(-1)^j}{j!}.$$

Lemma 4. *Let $c(n,p)$ and $F(n;p,\cdot,k)$ be as defined in (1) and (4), respectively. Then*

$$F(n;p,\cdot,k) = \binom{n}{p}\binom{k-1}{p-1}p!.$$

Proof. This follows directly from Lemmas 1, 2 and Theorem 1. □

However, we also get the following new result, thereby completing [23, Table 3, p. 119].

Theorem 2. *Let $c(n,p)$ and $F(n;\cdot,m,k)$ be as defined in (1) and (5), respectively. Then*

$$F(n;\cdot,m,k) = \sum_{p=0}^{k}\binom{k-1}{p-1}\binom{p}{m}c(n-m,p-m).$$

An alternative expression for $F(n;\cdot,m,k)$ is given in the proposition below.

Proposition 1. *For $1 \leqslant k \leqslant n$,*

$$F(n;\cdot,m,k) = \binom{k-1}{m}F(n-m;\cdot,0,k-m) + \sum_{j=m-1}^{k-1}F(n-1;\cdot,m-1,j).$$

Proof. Let $\mathcal{F}_m(n,k)$ be the set of all partial one-one maps on X_n with waist k and m fixed points. Let x be the largest fixed point of a map α in $\mathcal{F}_1(n,k)$ and β be the restriction of α to $X_n\setminus\{x\}$. If $x<k$, then β can be considered as an element of $\mathcal{F}_0(n-m,k-m)$ (remove all fixed points and their images) and there are $\binom{k-1}{m}$ choices of x. Hence the number of maps in $\mathcal{F}_m(n,k)$ with all fixed points less than k is $\binom{k-1}{m}F(n-m;\cdot,0,k-m)$. If k is the largest fixed point of α, then $\max(\operatorname{Im}\beta)<k$ and β can be considered as an element of $\bigcup_{j=1}^{k-1}\mathcal{F}_{m-1}(n-m,j)$, so there are $\sum_{j=0}^{k-1}F(n-1;\cdot,m-1,j)$ maps in $\mathcal{F}_m(n,k)$ with fixed point k. □

Remark 1. The triangular array of numbers $F(n;\cdot,1,k)$, $F(n;\cdot,2,k)$ and sequences $\Sigma F(n;\cdot,1,k)$ and $\Sigma F(n;\cdot,2,k)$ are as at the time of submitting this paper not in Sloane [20]. For computed values of $F(n;\cdot,1,k)$ and $F(n;\cdot,2,k)$ (for $1 \leqslant k \leqslant n \leqslant 6$) see Tables 1 and 2, respectively.

$n\backslash k$	1	2	3	4	5	6		$\Sigma F(n;\cdot,1,k)$
0	0							0
1	1	0						1
2	1	1	0					2
3	1	3	8	0				12
4	1	5	18	48	0			72
5	1	7	34	126	372	0		540
6	1	9	56	270	1044	3300		4680

Table 1.

$n\backslash k$	1	2	3	4	5	6		$\Sigma F(n;\cdot,2,k)$
0	0							0
1	0	0						0
2	0	1	0					1
3	0	1	2	0				3
4	0	1	5	18	0			24
5	0	1	8	39	132	0		180
6	0	1	11	72	336	1410		1830

Table 2.

1. Partial derangements

The number of *derangements* (and its various generalizations) of an n-set have attracted the attention of mathematicians as far back as the 18th century [21]. Thus, we were very surprised that the number of *partial derangements* defined as $\alpha \in \mathcal{I}_n$ having no fixed points was not known and so we computed it in 2007, see [15] and [20, A144085]. In this section we will find formulas for partial derangements of fixed waist. Note that in this case the number of partial derangements of fixed left waist must be equal to those of corresponding fixed waist.

From Theorems 1, 2 and Lemma 3, respectively, we deduce the following results.

Proposition 2. *Let $c(n,p)$ and $F(n;p,m,k)$ be as defined in (1) and (2), respectively. Then the number of partial derangements (of an n-set) of* *height p and waist k is given by*

$$F(n;p,0,k) = \binom{k-1}{p-1} c(n,p).$$

Proposition 3. *Let $c(n,p)$ and $F(n;p,m,\cdot)$ be as defined in (1), and (3), respectively. Then the number of partial derangements (of an n-set) of height p is given by*

$$F(n;p,0,\cdot) = \frac{n!}{(n-p)!} \sum_{j=0}^{p} \binom{n-j}{p-j} \frac{(-1)^j}{j!} = \binom{n}{p} c(n,p).$$

Proposition 4. *Let $c(n,p)$ and $F(n;\cdot,m,k)$ be as defined in (1) and (5), respectively. Then the number of partial derangements (of an n-set) of right waist k is given by*

$$F(n;\cdot,0,k) = \sum_{p=0}^{n} \binom{k-1}{p-1} c(n,p).$$

Remark 2. The triangular array of numbers $F(n;\cdot,0,k)$ and sequence $\Sigma F(n;\cdot,0,k)$, are as at the time of submitting this paper not in Sloane [20]. For computed values of $F(n;\cdot,0,k)$ for $1 \leqslant k \leqslant n \leqslant 6$ see Table 3.

$n\backslash k$	0	1	2	3	4	5	6	$\Sigma F(n;\cdot,0,k)$
0	1							1
1	1	0						1
2	1	1	2					4
3	1	2	5	10				18
4	1	3	10	28	66			108
5	1	4	17	62	192	504		780
6	1	5	26	118	462	1548	4440	6600

Table 3.

It is clear that $F(n;\cdot,0,0) = 1$ and $F(n;\cdot,0,1) = n-1$. Moreover, we have

$F(n;\cdot,0,2) = n^2 - 2n + 2;$
$F(n;\cdot,0,3) = n^3 - 4n^2 + 9n - 8;$
$F(n;\cdot,0,4) = n^4 - 7n^3 + 26n^2 - 50n + 42;$
$F(n;\cdot,0,5) = n^5 - 11n^4 + 61n^3 - 193n^2 + 346n - 276;$

$$F(n;\cdot,0,6) = n^6 - 16n^5 + 125n^4 - 580n^3 + 1674n^2 - 2824n + 2160.$$

We next obtain a recurrence relation and a generating function for $F(n;\cdot,0,k)$. However, for notational brevity, we will denote from here on $F(n;\cdot,m,k)$ by $F_m(n,k)$.

Proposition 5. *Let $F_m(n,k) = F(n;\cdot,m,k)$ be as defined in (5). Then for $1 \leqslant k < n$, we have*

$$\begin{aligned} F_0(n,k) - F_0(n,k-1) &= (k-1)F_0(n-1,k-1) \\ &\quad -(k-2)F_0(n-1,k-2) + F_0(n-1,k). \end{aligned}$$

Proof. First observe that since $\sum_{j=2}^{i-k+2}\binom{k-j}{i-2} = \binom{k-1}{i-1}$, it follows that

$$\sum_{i=1}^{n}\binom{k-1}{i-1}c(n-1,i-1) = \sum_{i=1}^{k-1}F_0(n-1,i).$$

Next we show that

$$F_0(n,k) = kF_0(n-1,k-1) + \sum_{i=0}^{k-2}F_0(n-1,i) + F_0(n-1,k). \tag{6}$$

We have

$$\begin{aligned} F_0(n,k) &= \sum_{i=0}^{n}\binom{k-1}{i-1}c(n,i) \\ &= \sum_{i=0}^{n}\binom{k-1}{i-1}[ic(n-1,i-1) + c(n-1,i)] \\ &= \sum_{i=0}^{n}\binom{k-1}{i-1}[(i-1)c(n-1,i-1) + c(n-1,i-1) + c(n-1,i)] \\ &= (k-1)F_0(n-1,k-1) + \sum_{i=0}^{k-1}F_0(n-1,i) + F_0(n-1,k) \\ &= kF_0(n-1,k-1) + \sum_{i=0}^{k-2}F_0(n-1,i) + F_0(n-1,k). \end{aligned}$$

Finally, note that substituting (6) into $F_0(n,k) - F_0(n,k-1)$ and simplifying we get the required result. $\square$

Theorem 3. *Let $F_m(n,k) = F(n;\cdot,m,k)$ be as defined in (5). Then, for $n \geqslant 2$, we have*

$$\sum_{k\geqslant 0} \frac{F_0(n,k+1)}{k!} x^k = e^{\frac{x^2}{1+x}} (n(1+x)^{n-1} - (1+x)^{n-2}).$$

Proof. We have

$$F_0(n,k) = \sum_{j=0}^{k} (k-1)! \frac{(-1)^j}{j!} \sum_{r=0}^{k-j} \binom{n-j}{r} \frac{r+j}{(k-j-r)!}.$$

Let $a = k-j$ and let $g(x) = \sum_{a\geqslant 0} \sum_{r=0}^{a} \binom{n-j}{r} \frac{r+j}{(a-r)!} x^a$. Then

$$\begin{aligned} g(x) =& \sum_{a\geqslant 0} \sum_{r=0}^{a} \binom{n-j}{r} x^r (r+j) \frac{x^{a-r}}{(a-r)!} \\ =& \sum_{r\geqslant 0} \binom{n-j}{r} x^r (r+j) \sum_{a\geqslant r} \frac{x^{a-r}}{(a-r)!} = e^x \sum_{r\geqslant 0} \binom{n-j}{r} x^r (r+j) \\ =& e^x (j(1+x)^{n-j} + (n-j)(1+x)^{n-j-1}) = e^x (1+x)^{n-j-1} (j+nx), \end{aligned}$$

using the fact that $\binom{n-j}{r} = 0$, if $r > n-j$. We now obtain

$$\begin{aligned} \sum_{k\geqslant 0} \frac{kF_0(n,k)}{k!} x^k &= \sum_{k\geqslant 0} \sum_{j=0}^{k} \frac{(-x)^j}{j!} \sum_{r=0}^{k-j} \binom{n-j}{r} \frac{r+j}{(k-j-r)!} x^{k-j} \\ &= \sum_{j\geqslant 0} \frac{(-x)^j}{j!} \sum_{\alpha\geqslant 0} \sum_{r=0}^{\alpha} \binom{n-j}{r} \frac{r+j}{(a-r)!} x^a \\ &= e^x (1+x)^{n-1} \sum_{j\geqslant 0} \frac{(-x/(1+x))^j}{j!} (j+nx) \\ &= e^x (1+x)^{n-1} \left\{ nxe^{-x/(1+x)} + \sum_{j\geqslant 0} \frac{j(-x/(1+x))^j}{j!} (j+nx) \right\} \\ &= e^x (1+x)^{n-1} \left(nxe^{-x/(1+x)} - \frac{x}{1+x} e^{-x/(1+x)} \right) \\ &= xe^{x^2/(1+x)} (1+x)^{n-1} (n - 1/(1+x)). \end{aligned}$$

This implies $\sum_{k\geqslant 0} \frac{F_0(n,k+1)}{k!} x^k = e^{x^2/(1+x)} (1+x)^{n-1} (n - 1/(1+x))$, as required. $\square$

Remark 3. A straightforward expansion of the power series above gives

$$\sum_{k\geqslant 0}\frac{F_0(n,k+1)}{k!}x^k = (n-1)+x(n^2-2n+2)+\frac{x^2}{2!}(n^3-4n^2+9n-8)$$
$$+\frac{x^3}{3!}(n^4-7n^3+26n^2-50n+42)$$
$$+\frac{x^4}{4!}(n^5-11n^4+61n^3-193n^2+346n-276)$$
$$+\frac{x^5}{5!}(n^6-16n^5+125n^4-580n^3+1674n^2-2824n+2160)+O(x^6).$$

We thus recover the polynomials listed after Remark 2.4.

The following result relates the number $F_m(n,n)$ of partial one-one maps with m fixed points and waist n to the number of partial derangements.

Proposition 6. *Let a_n be the number of partial derangements of $\mathcal{I}_n$ and let $F_m(n,k) = F(n;\cdot,m,k)$ be as defined in (5). Then, for $n-m\geqslant 2$,*

$$F_m(n,n) = (n-m-1)\binom{n-1}{m}(a_{n-m-1}+a_{n-m-2})+\binom{n-1}{m-1}a_{n-m}.$$

In particular, for $n\geqslant 2$,

$$F_0(n,n) = (n-1)(a_{n-1}+a_{n-2}).$$

Proof. Let $n\geqslant 2$ and let $b_n = F_0(n,n)$ be the number of partial derangements of $\mathcal{I}_n$ with waist n, i.e., with $\max(\operatorname{Im}\alpha) = n$. We first prove that $b_n = (n-1)(a_{n-1}+a_{n-2})$. By [15, Proposition 3.1], the exponential generating function (e.g.f.) of a_n is $a(x) = e^{x^2/(1-x)}/(1-x)$. For computational convenience, let $b(x)$ be the e.g.f. of b_{n+1} rather than that of b_n. We have

$$\begin{aligned} b(x) &= \sum_{n\geqslant 0} b_{n+1}\frac{x^n}{n!}\\ &= \sum_{n\geqslant 0}\frac{1}{n!}\sum_{p=0}^{n} x^{n-p}\binom{n}{p}\sum_{j=0}^{p} x^{p-j}\frac{(-x)^j}{j!}(p+1)!\binom{n+1-j}{p+1-j}\frac{(-1)^j}{j!}\\ &= \sum_{n\geqslant 0}\sum_{p=0}^{n}\frac{x^{n-p}}{(n-p)!}(p+1)\sum_{j=0}^{p} x^{p-j}\frac{(-x)^j}{j!}\binom{n+1-j}{p+1-j}. \end{aligned}$$

Let $u = n - p$, $v = p - j$. Then

$$b(x) = \sum_{j \geqslant 0} \frac{(-x)^j}{j!} \sum_{u \geqslant 0} \frac{x^u}{u!} g(x, j, u)$$

where

$$\begin{aligned} g(x, j, u) &= \sum_{v \geqslant 0} (v + j + 1) \binom{u + v + 1}{v + 1} x^v \\ &= \frac{j}{x} \sum_{v \geqslant 0} x^{v+1} \binom{u + v + 1}{v + 1} + \sum_{v \geqslant 0} (v + 1) \binom{u + v + 1}{v + 1} x^v \\ &= \frac{j}{x} \left(\frac{1}{(1 - x)^{u+1}} - 1 \right) + \frac{d}{dx} \sum_{t \geqslant 1} t x^{t-1} \binom{u + t}{t} \\ &= \frac{j}{x} \left(\frac{1}{(1 - x)^{u+1}} - 1 \right) + \frac{u + 1}{(1 - x)^{u+2}}. \end{aligned}$$

Hence, by algebraic manipulations we see that

$$\begin{aligned} b(x) &= \sum_{j \geqslant 0} \frac{(-x)^j}{j!} \sum_{u \geqslant 0} \frac{x^u}{u!} \left(\frac{j}{x} \left(\frac{1}{(1 - x)^{u+1}} - 1 \right) + \frac{u + 1}{(1 - x)^{u+2}} \right) \\ &= \sum_{j \geqslant 0} \frac{(-x)^j}{j!} \left\{ \frac{j e^{x/(1-x)}}{x(1 - x)} - \frac{j e^x}{x} \right. \\ &\qquad \left. + \frac{x}{(1 - x)^3} \sum_{u \geqslant 0} \frac{x^{u-1}}{(u - 1)!(1 - x)^{u-1}} + \frac{e^{x/(1-x)}}{(1 - x)^2} \right\} \\ &= e^{x/(1-x)} \left\{ \frac{1}{x(1 - x)} \sum_{j \geqslant 0} \frac{j(-x)^j}{j!} \right. \\ &\qquad \left. + \left(\frac{x}{(1 - x)^3} + \frac{1}{(1 - x)^2} \right) \sum_{j \geqslant 0} \frac{(-x)^j}{j!} \right\} + 1 \\ &= \frac{2x - x^2}{(1 - x)^3} e^{x^2/(1-x)} + 1. \end{aligned}$$

Thus we obtain

$$\sum_{n \geqslant 0} b_{n+2} \frac{x^n}{n!} = b'(x) = \frac{2 + x^2 - 3x^3 + x^4}{(1 - x)^5} e^{x^2/(1-x)}.$$

On the other hand,

$$\begin{aligned}
&\sum_{n\geqslant 0}(n+1)(a_{n+1}+a_n)\frac{x^n}{n!}\\
&\quad= x\sum_{n\geqslant 1}\frac{a_{n+1}x^{n-1}}{(n-1)!}+\sum_{n\geqslant 0}\frac{a_{n+1}x^n}{n!}+x\sum_{n\geqslant 1}\frac{a_nx^{n-1}}{(n-1)!}+\sum_{n\geqslant 0}\frac{a_nx^n}{n!}\\
&\quad= xa''(x)+(1+x)a'(x)+a(x)\\
&\quad= \frac{2+x^2-3x^3+x^4}{(1-x)^5}e^{x^2/(1-x)}=\sum_{n\geqslant 0}b_{n+2}\frac{x^n}{n!}.
\end{aligned}$$

This completes the proof that for all $n \geqslant 0$, $b_{n+2} = (n+1)(a_{n+1}+a_n)$. We next turn to the case when $m \geqslant 1$. By Proposition 1.7 and the above part,

$$\begin{aligned}
F_m(n,n) &= \binom{n-1}{m}F_0(n-m,n-m)+\sum_{j=0}^{n-1}F_{m-1}(n-1,j)\\
&= (n-m-1)\binom{n-1}{m}(a_{n-m-1}+a_{n-m-2})+a_{n-1,m-1}
\end{aligned}$$

where $a_{r,t}$ is the number of partial one-one maps on $\{1,\ldots,r\}$ with exactly t fixed points. By [15, (2.5a)] $a_{r,t} = \binom{r}{t}a_{r-t}$, so we obtain

$$F_m(n,n) = (n-m-1)\binom{n-1}{m}(a_{n-m-1}+a_{n-m-2})+\binom{n-1}{m-1}a_{n-m}. \qquad \square$$

Remark 4. *From Proposition 1.7 and (6), we easily deduce that for* $1 \leqslant k < n$

$$F_1(n,k) = F_0(n,k) - F_0(n-1,k).$$

We conclude this article with the following divisibility properties. In particular, we obtain the curious fact that $\mathrm{lcm}(1,\ldots,k-1)$ divides $F_0(n,k)$ and $F_1(n,k)$ if $k \geqslant 3$.

Theorem 4. *(i) For* $k-m \geqslant 2$, $\mathrm{lcm}(1,\ldots,k-1)|m!F_m(n,k)$.
(ii) If $k-m \geqslant 4$, *then* $F_m(n,k)$ *is divisible by* 3.
(iii) $\mathrm{lcm}(1,2,\ldots,n-m-1)|F_m(n,n)$.

Proof. (i) We use induction on m. For each $r \in \mathbb{N}$, let $\mu_r = \mathrm{lcm}(1,\ldots,r)$. Since the sequence (a_k) of partial derangements of $\mathcal{I}_k$ has e.g.f.

$\sum_{k\geqslant 0}\frac{a_k}{k!}x^k = e^{x^2/(1-x)}/(1-x)$, Theorem 2.6 implies

$$\sum_{k\geqslant 0} F_0(n,k+1)\frac{x^k}{k!} = (1+x)^{n-1}(n(1+x)-1)\sum_{k\geqslant 0} a_k\frac{(-x)^k}{k!}.$$

This means $\frac{F_0(n,k+1)}{k!}$ is a $\mathbb{Z}$-linear combination of terms of the form $\frac{a_j}{j!}$ $(0\leqslant j\leqslant k)$ and so $F_0(n,k+1)=\sum_{j=0}^{k}\alpha_j a_j\frac{k!}{j!}$ for some $\alpha_j\in\mathbb{Z}$. By [15, Proposition 2.11], $\mu_j|a_j$ and hence each term $a_j\frac{k!}{j!}$ is divisible by μ_k. We thus obtain $\mu_{k-1}|F_0(n,k)$ for $k\geqslant 2$.
Now assume that for some nonnegative integer M, $\mu_{k-1}|M!F_M(n,k)$ whenever $k-M\geqslant 2$. By Proposition 1.7

$$F_{M+1}(n,k) = A_1 + A_2 - A_3$$

where

$$A_1 = \binom{k-1}{M+1}F_0(n-M-1,k-M-1)$$
$$A_2 = \binom{n-1}{M}a_{n-M-1}$$
$$A_3 = \sum_{j=k}^{n-1} F_M(n-1,j).$$

Hence, to prove that $\mu_{k-1}|(M+1)!F_{M+1}(n,k)$ for $k-M\geqslant 3$, it suffices to show that $\mu_{k-1}|(M+1)!A_i$ $(1\leqslant i\leqslant 3)$.
Clearly $\mu_{k-1}|(M+1)!\binom{k-1}{M+1}\mu_{k-M-2}$. Since $\mu_{k-M-2}|F_0(n-M-1,k-M-1)$, we deduce that μ_{k-1} divides $(M+1)!A_1$. Also, $(M+1)!a_{n-M-1}$ is divisible by $(n-1)(n-2)\cdots(n-M)\mu_{n-M-1}$, so $\mu_{k-1}|(M+1)!A_2$. Finally, by the induction hypothesis, $\mu_{k-1}|M!F_M(n,k)$, hence $\mu_{k-1}|(M+1)!F_M(n-1,j)$ for $j\geqslant k$ and therefore $\mu_{k-1}|(M+1)!A_3$.
(ii) This follows by an argument similar to the one above.
(iii) This follows from Proposition 2.8 and the fact that $\mu_n|a_n$. $\square$

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Received by the editors: 03.04.2021
and in final form 21.10.2021.