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On dual Rickart modules and weak dual Rickart modules

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Abstract. Let R be a ring. A right R-module M is called d-*Rickart* if for every endomorphism φ of M, $\varphi(M)$ is a direct summand of M and it is called wd-*Rickart* if for every nonzero endomorphism φ of M, $\varphi(M)$ contains a nonzero direct summand of M. We begin with some basic properties of (w)d-Rickart modules. Then we study direct sums of (w)d-Rickart modules and the class of rings for which every finitely generated module is (w)d-Rickart. We conclude by some structure results.

1. Introduction

In [10], Lee, Rizvi and Roman introduced and studied a notion called d-Rickart modules. A module M is said to be d-*Rickart* (or *dual Rickart*) if for every $\varphi \in \mathrm{End}_R(M)$, $\mathrm{Im}\,\varphi$ is a direct summand of M. Actually, this notion is dual to the notion of Rickart modules introduced by Lee, Rizvi and Roman in [9]. A module M is called a *Rickart module* if for every endomorphism φ of M, $\mathrm{Ker}\,\varphi$ is a direct summand of M. Later in [13], Tribak introduced and investigated the notion called wd-Rickart modules, which is a generalization of the concept of d-Rickart modules. A module M is said to be wd-*Rickart* (or *weak dual Rickart*) if for every

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nonzero endomorphism φ of M, $\operatorname{Im}\varphi$ contains a nonzero direct summand of M. Let M and N be two modules. Then M is called N-wd-*Rickart* if for every nonzero homomorphism $\varphi : M \to N$, $\operatorname{Im}\varphi$ contains a nonzero direct summand of N.

In Section 2, we investigate some basic properties of (w)d-Rickart modules.

In Section 3, we study direct sums of (w)d-Rickart modules. We provide a characterization for a direct sum of two d-Rickart modules to be d-Rickart. We also show that if $M_1, \ldots, M_n$ are modules such that M_i is M_j-projective for all $j > i$ in $\{1, \ldots, n\}$. Then $\oplus_{i=1}^{n} M_i$ is a wd-Rickart module if and only if M_i is M_j-wd-Rickart for all $i, j \in \{1, \ldots, n\}$.

Section 4 is devoted to the study of the class of rings over which finitely generated modules are (w)d-Rickart. Among other results, the class of commutative rings R for which every finitely generated R-module is d-Rickart is shown to be precisely that of semisimple rings.

We conclude this paper by a short section in which we present some structure results.

Throughout this paper, R is an associative ring with identity and all the modules are unital right R-modules. Let M be a module. The notation $N \leqslant M$ means that N is a submodule of M. By $\operatorname{Soc}(M)$ and $\operatorname{End}_R(M)$, we denote the socle of M and the endomorphism ring of M, respectively. By $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ we denote the set of rational, integer and natural numbers, respectively.

2. Some properties of d-Rickart modules and wd-Rickart modules

Let M and N be two modules. Following [10, Definition 2.14], the module M is called N-d-*Rickart* (or *relatively* d-*Rickart to* N) if for every homomorphism $\varphi : M \to N$, $\operatorname{Im}\varphi$ is a direct summand of N. Therefore M is a d-Rickart module if and only if M is M-d-Rickart.

Recall that a module M is called a (C_3)-*module* if whenever A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M. Note that every injective module is a (C_3)-module.

Example 2.1. *Let M_1 be a semisimple module and let M_2 be a module such that the module $M = M_1 \oplus M_2$ is a (C_3)-module. Then M_1 and M_2 are relatively* d-*Rickart to each other by [2, Proposition 2.3].*

If M is a d-Rickart (wd-Rickart) module, then a factor module of M may not be d-Rickart (wd-Rickart) as we see in the following example.

Example 2.2. Let R be a von Neumann regular ring which is not a right V-ring (see [8, Example 3.74A]). By [10, Remark 2.2], R_R is a d-Rickart module. Then by [10, Proposition 2.25], every finitely generated free R-module is a d-Rickart module. Since R is not a right V-ring, there exists a finitely generated R-module M such that M is not a wd-Rickart module (Proposition 4.1). It is well known that every finitely generated R-module is a homomorphic image of a finitely generated free R-module. Therefore there exists a positive integer n such that $M \cong R^{(n)}/K$ for some submodule K of $R^{(n)}$. Hence $R^{(n)}/K$ is not a wd-Rickart (so $R^{(n)}/K$ is not a d-Rickart) module while $R^{(n)}$ is a d-Rickart module.

The following proposition provides a sufficient condition under which some factor modules of a d-Rickart module are d-Rickart.

Proposition 2.3. *Let M be a* d-*Rickart module and let N be a fully invariant submodule of M. If every endomorphism of M/N can be lifted to an endomorphism of M, then M/N is also a* d-*Rickart module.*

Proof. Let φ be a nonzero endomorphism of M/N. By assumption, there exists an endomorphism ψ of M such that $\pi\psi = \varphi\pi$, where $\pi : M \to M/N$ is the canonical projection. It is clear that $\psi \neq 0$. As M is d-Rickart, $\operatorname{Im}\psi$ is a direct summand of M. Note that $\operatorname{Im}\varphi = \varphi\pi(M) = \pi\psi(M) = (\psi(M)+N)/N$. Since N is fully invariant in M, $\operatorname{Im}\varphi$ is a direct summand of M/N. □

Corollary 2.4. *Let M be a quasi-projective* d-*Rickart module. If N is a fully invariant submodule of M, then M/N is a* d-*Rickart module.*

Proof. By Proposition 2.3. □

Next, we investigate connections between a wd-Rickart module and its endomorphism ring.

A ring R is called *left w-Rickart* if for every nonzero element $x \in R$, $l_R(x) = \{r \in R \mid rx = 0\}$ is contained in a proper direct summand of the left R-module $_RR$.

Proposition 2.5. *If M is a* wd-*Rickart module, then $S = \operatorname{End}_R(M)$ is a left w-Rickart ring.*

Proof. Let φ be a nozero endomorphism of M. Since M is wd-Rickart, there exists a nonzero idempotent $e \in S$ with $e(M) \subseteq \varphi(M)$. Then clearly $l_S(\varphi) \subseteq S(1-e)$ and $S(1-e) \neq S$. This proves the proposition. □

The following example shows that the converse of the above proposition is not true, in general.

Example 2.6. The $\mathbb{Z}$-module $\mathbb{Z}$ is not wd-Rickart, but $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z}$ is a left w-Rickart ring.

Corollary 2.7. *If R is a right* wd-*Rickart ring, then eRe is a left w-Rickart ring for any idempotent e in R.*

Proof. This follows from [13, Corollary 2.5] and Proposition 2.5. □

Let M be an R-module and let $S = \mathrm{End}_R(M)$. We denote $r_M(I) = \{m \in M \mid Im = 0\}$ for $\varnothing \neq I \subseteq S$ and $l_S(N) = \{\varphi \in S \mid \varphi(N) = 0\}$ for a submodule N of M. In [1, Corollary 4.2], it is presented some examples of submodules K of a module M for which $r_M(l_S(K)) = K$. Moreover, it is shown in [10, Corollary 3.7] that a module M is a d-Rickart module if and only if $r_M l_S(\varphi(M)) = \varphi(M)$ and $r_M l_S(\varphi(M))$ is a direct summand of M for all $\varphi \in S = \mathrm{End}_R(M)$.

It is natural to ask when the converse of Proposition 2.5 holds. In this vein we give the next theorem. But first we need the following lemma.

Lemma 2.8. *Let M be a module with $S = \mathrm{End}_R(M)$. Then S is a left w-Rickart ring if and only if $r_M l_S(\varphi(M))$ contains a nonzero direct summand of M for all nonzero endomorphisms φ of M.*

Proof. ($\Rightarrow$) Let $\varphi : M \to M$ be a nonzero endomorphism of M. Since S is left w-Rickart, there exists an idempotent f of S such that $l_S(\varphi) \subseteq Sf$ and $Sf \neq S$. Then $r_M(Sf) \subseteq r_M l_S(\varphi(M))$. This implies that the nonzero direct summand $(1 - f)(M)$ of M is contained in $r_M l_S(\varphi(M))$.

($\Leftarrow$) Let $0 \neq \varphi \in S$. By hypothesis, there exists $0 \neq e = e^2 \in S$ such that $e(M) \subseteq r_M l_S(\varphi(M))$. Thus $l_S r_M l_S(\varphi(M)) \subseteq l_S(e(M))$. Hence $l_S(\varphi(M)) \subseteq l_S(e(M))$. So $l_S(\varphi) \subseteq l_S(e) = S(1 - e) \neq R$. This completes the proof. □

Theorem 2.9. *Let M be a module with the property that $r_M l_S(\varphi(M)) = \varphi(M)$ for every nonzero endomorphism φ of M. Then M is a* wd-*Rickart module if and only if $S = \mathrm{End}_R(M)$ is a left w-Rickart ring.*

Proof. ($\Rightarrow$) By Proposition 2.5.

($\Leftarrow$) This follows from Lemma 2.8. □

Recall that a module M is called *retractable* if for every nonzero submodule $N \leqslant M$, there exists a nonzero endomorphism φ of M such that $\mathrm{Im}\,\varphi \subseteq N$. It was shown in [10, Proposition 4.10] that if M is a retractable d-Rickart module, then every nonzero submodule of M contains a nonzero direct summand of M. Now we give the following.

Proposition 2.10. *Let M be a* wd-*Rickart module. Then M is retractable if and only if every nonzero submodule of M contains a nonzero direct summand of M.*

Proof. ($\Rightarrow$) By [13, Proposition 2.13].

($\Leftarrow$) This is clear. □

Let M and N be two modules. The module M is called N-wd-*Rickart* (or *relatively* wd-*Rickart to N*) if for every nonzero homomorphism $\varphi : M \to N$, $\operatorname{Im}\varphi$ contains a nonzero direct summand of N. Therefore M is a wd-Rickart module if and only if M is M-wd-Rickart (see [13, Definition 2.1]).

Lemma 2.11. *Let M and N be modules. Then M is N-*wd-*Rickart (N-*d-*Rickart) if and only if M/X is N-*wd-*Rickart (N-*d-*Rickart) for any submodule $X \leqslant M$.*

Proof. ($\Rightarrow$) Assume that M is N-wd-Rickart (N-d-Rickart). Let $\varphi : M/X \to N$ be a nonzero homomorphism. Consider the nonzero homomorphism $\varphi\pi : M \to M/X \to N$, where $\pi : M \to M/X$ is the natural epimorphism. By the assumption, there exists a nonzero direct summand T of N such that $T \subseteq \operatorname{Im}\varphi\pi = \operatorname{Im}\varphi$ ($\operatorname{Im}\varphi\pi = \operatorname{Im}\varphi$ is a direct summand of N).

($\Leftarrow$) The result follows by taking $X = 0$. □

Theorem 2.12. *The following conditions are equivalent for a module M:*

(a) *M is a* wd-*Rickart module;*

(b) *For any submodule N of M and every direct summand K of M, M/N is K-*wd-*Rickart;*

(c) *For every pair of direct summands K and N of M, N is K-*wd-*Rickart.*

Proof. $(a) \Rightarrow (b)$ This is clear by Lemma 2.11 and [13, Proposition 2.4].

$(b) \Rightarrow (c)$ Clear.

$(c) \Rightarrow (a)$ Take $N = K = M$. □

Definition 2.13. A module M is called w-C_2 if for every nonzero submodule N of M and every direct summand K of M, $N \cong K$ implies that N contains a nonzero direct summand of M.

Proposition 2.14. *A module M is* wd-*Rickart if and only if M has w-C_2 condition and for every nonzero $\varphi \in \operatorname{End}_R(M)$, there exists a nonzero submodule A of M such that A is isomorphic to a nonzero direct summand of M and $A \subseteq \operatorname{Im}\varphi$.*

Proof. This follows from [13, Proposition 2.3] and the definition of a wd-Rickart module. □

Theorem 2.15. *The following are equivalent for a module M:*

(a) *M is a* wd-*Rickart module;*

(b) *For every nonzero finitely generated right ideal I of $S = \operatorname{End}_R(M)$, $\sum_{\varphi \in I} \varphi(M)$ contains a nonzero direct summand of M.*

Proof. $(a) \Rightarrow (b)$ Let $I =< \varphi_1, \ldots, \varphi_n >$ be a finitely generated right ideal of S, where each φ_i is a nonzero endomorphism of M. Note that $\sum_{\varphi \in I} \varphi(M) = \varphi_1(M) + \cdots + \varphi_n(M)$. Since M is wd-Rickart, there exists a nonzero direct summand T of M such that $T \subseteq \varphi_1(M) \subseteq \sum_{\varphi \in I} \varphi(M)$.

$(b) \Rightarrow (a)$ This is clear. □

3. Direct sums of d-Rickart (wd-Rickart) modules

We begin with the following theorem which gives a characterization for a direct sum of two d-Rickart modules to be d-Rickart.

Theorem 3.1. *Let $M = M_1 \oplus M_2$ be a module. The following conditions are equivalent:*

(a) *M is a* d-*Rickart module;*

(b) (i) *M_i and M_j are relatively* d-*Rickart for $i, j \in \{1, 2\}$, and*
(ii) *for every $\varphi \in \operatorname{End}_R(M)$ such that $\operatorname{Im}\varphi + M_1$ is a direct summand of M, $\operatorname{Im}\varphi$ is a direct summand of M.*

(c) (i) *M_i and M_j are relatively* d-*Rickart for $i, j \in \{1, 2\}$, and*
(ii) *for every $\varphi \in \operatorname{End}_R(M)$ with $(\operatorname{Im}\varphi + M_1) \oplus N = M$ for some submodule $N \leqslant M_2$, $\operatorname{Im}\varphi$ is a direct summand of M.*

Proof. (a) ⇒ (b) By [10, Theorem 2.19] and the definition of a d-Rickart module.

(b) ⇒ (c) This is clear.

(c) ⇒ (a) Let $\varphi : M \to M$ be a nonzero homomorphism. Let $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$ be the natural epimorphisms. Consider the homomorphisms $\varphi_1 = \pi_1\varphi : M \to M_1$ and $\varphi_2 = \pi_2\varphi : M \to M_2$. Note that M is M_1-d-Rickart and M is M_2-d-Rickart by [10, Corollary 5.4]. Then there exists a direct summand M_1' of M_1 and a direct summand M_2' of M_2 such that $M_1 = \varphi_1(M) \oplus M_1'$ and $M_2 = \varphi_2(M) \oplus M_2'$. It is easy to check that $\varphi(M) + M_1 = \varphi_1(M) \oplus \varphi_2(M) \oplus M_1' = M_1 \oplus \varphi_2(M)$. So $(\varphi(M) + M_1) \oplus M_2' = M$. By assumption, $\varphi(M)$ is a direct summand of M. Hence M is a d-Rickart module. □

Recall that an element c of a ring R is called *regular* if $cr \neq 0$ and $rc \neq 0$ for all nonzero $r \in R$. Following [5, p. 104], an R-module X is called *divisible* in case $X = Xc$ for every regular element c of R. An R-module Y is called *torsion* if for any $y \in Y$, there exists a regular element c in R such that $yc = 0$. On the other hand, an R-module Z is called *torsion-free* if whenever $z \in Z$ satisfies $zd = 0$ for some regular element d of R then $z = 0$. The ring R is called a right *Goldie ring* if R_R has finite rank and R has the acc on right annihilators. The following theorem provides many examples of d-Rickart modules.

Theorem 3.2. *Let R be a prime right Goldie ring such that R is not right primitive and let an R-module M be a direct sum of a torsion-free divisible submodule X and a torsion semisimple submodule Y. Then M is a* d-*Rickart module.*

Proof. By [5, Propositions 6.12 and 6.13], X is a nonsingular injective module. Hence X is d-Rickart since $\text{End}_R(X)$ is von Neumann regular. Moreover, in the proof of [7, Corollary 2.16] it is shown that $\text{Hom}_R(X, Y) = 0$ and $\text{Hom}_R(Y, X) = 0$. Therefore X and Y are fully invariant submodules of M. Then M is a d-Rickart module by [10, Proposition 5.14]. □

Corollary 3.3. *Let R be a prime PI-ring which is not artinian and let an R-module M be a direct sum of a torsion-free divisible submodule X and a torsion semisimple submodule Y. Then M is a* d-*Rickart module.*

Proof. By [7, Corollary 2.17] and [11, Corollary 13.6.6 and Theorem 13.3.8], R is a right Goldie ring and R is not right primitive. The result follows from Theorem 3.2. □

The following proposition is inspired by [10, Proposition 5.2]. This result provides a rich source of examples showing that the wd-Rickart property does not go to direct sums of wd-Rickart modules. It extends [13, Example 2.6] to arbitrary modules.

Proposition 3.4. *Let M be an indecomposable module with a nonzero proper socle. Then $M \oplus \text{Soc}(M)$ is not a* wd-*Rickart module.*

Proof. Assume that $M \oplus \text{Soc}(M)$ is wd-Rickart. By Theorem 2.12, $\text{Soc}(M)$ is M-wd-Rickart. Let $\mu : \text{Soc}(M) \to M$ be the inclusion map. Then there exists a nonzero direct summand T of M such that $T \subseteq \mu(\text{Soc}(M)) = \text{Soc}(M)$. Since M is indecomposable, we have $T = M = \text{Soc}(M)$, which is a contradiction. □

In [13, Proposition 2.7], it is studied when a direct sum $\oplus_{i\in I}M_i$ of modules M_i ($i \in I$) is N-wd-Rickart for some module N. Next, we provide a sufficient condition under which N is $(\oplus_{i\in I}M_i)$-wd-Rickart for some finite index set I.

Proposition 3.5. *Let $M = M_1 \oplus M_2$ such that M_2 is M_1-projective and let N be a module. Then N is M-*wd*-Rickart if and only if N is M_i-*wd*-Rickart for all $i = 1, 2$.*

Proof. ($\Rightarrow$) By Theorem 2.12.

($\Leftarrow$) Let $\varphi : N \to M$ be a nonzero homomorphism. Let $\pi_2 : M \to M_2$ be the projection on M_2 along M_1. Let $\varphi_2 = \pi_2\varphi : N \to M_2$.

Case 1: Assume that φ_2 is nonzero. Since N is M_2-wd-Rickart, there exists a nonzero direct summand K_2 of M_2 such that $K_2 \subseteq \operatorname{Im}\varphi_2 = (\operatorname{Im}\varphi + M_1) \cap M_2$. Then $K_2 = (\operatorname{Im}\varphi + M_1) \cap K_2$. Let L_2 be a submodule of M_2 such that $M_2 = L_2 \oplus K_2$. Note that K_2 is M_1-projective by [15, 18.1]. On the other hand, $K_2 \oplus M_1 = [\operatorname{Im}\varphi \cap (K_2 \oplus M_1)] + M_1$. Then by [15, 41. 14], $K_2 \oplus M_1 = C \oplus M_1$ for some submodule $C \leqslant \operatorname{Im}\varphi \cap (K_2 \oplus M_1)$. Clearly, C is a nonzero direct summand of M which is contained in $\operatorname{Im}\varphi$.

Case 2: Assume that $\varphi_2 = 0$. Then $(\operatorname{Im}\varphi + M_1) \cap M_2 = 0$. This implies that $\operatorname{Im}\varphi + M_1 = M_1$ and hence $\operatorname{Im}\varphi \subseteq M_1$. Since N is M_1-wd-Rickart, $\operatorname{Im}\varphi$ contains a nonzero direct summand of M. □

Theorem 3.6. *Let $M = \oplus_{i=1}^{n} M_i$ such that M_j is M_i-projective for all $j > i$ in $\{1, \dots, n\}$, and let N be a module. Then N is M-*wd*-Rickart if and only if N is M_i-*wd*-Rickart for all $i = 1, \dots, n$.*

Proof. The proof is by induction on n and using Proposition 3.5, Theorem 2.12 and [15, 18.2(2)]. □

Corollary 3.7. *Assume that $M_1, \dots, M_n$ are R-modules such that M_i is M_j-projective for all $j > i$ in $\{1, \dots, n\}$. Then $\oplus_{i=1}^{n} M_i$ is a *wd*-Rickart module if and only if M_i is M_j-*wd*-Rickart for all $i, j \in \{1, \dots, n\}$.*

Proof. ($\Rightarrow$) Clear by Theorem 2.12.

($\Leftarrow$) By [13, Proposition 2.7], $\oplus_{i=1}^{n} M_i$ is M_j-wd-Rickart for all $j \in \{1, \dots, n\}$. Therefore $\oplus_{i=1}^{n} M_i$ is a wd-Rickart module by Theorem 3.6. □

4. Rings whose finitely generated modules are d-Rickart (wd-Rickart)

We begin with a result which gives some information about the class of rings over which every finitely generated module is wd-Rickart.

Proposition 4.1. *Let R be a ring such that every finitely generated R-module is a* wd-*Rickart module. Then*

(i) *R is a right V-ring.*

(ii) *Every indecomposable finitely generated R-module is a simple injective module.*

(iii) *Every uniform module is a simple injective module.*

Proof. (i) Assume that there is a simple R-module S with $E(S) \neq S$. Take a nonzero element $x \in E(S)$ which is not in S. Clearly, we have $\mathrm{Soc}(xR) = S$. By hypothesis, the finitely generated right R-module $xR \oplus \mathrm{Soc}(xR) = xR \oplus S$ is wd-Rickart. This is impossible (see Proposition 3.4).

(ii) Let M be an indecomposable finitely generated R-module. Let $0 \neq x \in M$. Since $xR \oplus M$ is wd-Rickart, xR is M-wd-Rickart by [13, Corollary 2.8(ii)]. Therefore xR contains a nonzero direct summand of M. As M is indecomposable, $xR = M$. Hence M is a simple module.

(iii) Let U be a uniform R-module and let $0 \neq x \in U$. So xR is indecomposable. Thus xR is simple by (ii). It follows that U is a semisimple module. But U is indecomposable. Then U is a simple module. □

The following example shows that, in general, a right V-ring may have a finitely generated module which is not wd-Rickart. Note that there exist right noetherian right V-rings which are not von Neumann regular (see [4]).

Example 4.2. Let R be a right noetherian right V-ring which is not von Neumann regular. Then R_R is not a d-Rickart module by [10, Remark 2.2]. Therefore R_R is not a wd-Rickart module by [13, Corollary 3.5].

Next, we focus on the class of rings over which every finitely generated module is d-Rickart.

A module M is said to be *regular* if every cyclic submodule of M is a direct summand of M. Equivalently, every finitely generated submodule of M is a direct summand of M (see [14, Remark 6.1]).

Lemma 4.3. (i) *If M is an R-module such that $R \oplus M$ is a* d-*Rickart R-module, then M is a von Neumann regular module and R is a von Neumann regular ring.*

(ii) *If N is a finitely generated R-module and M is a regular R-module, then N is M-*d-*Rickart.*

Proof. (i) Let $a \in M$ and consider the R-homomorphism $\varphi_a : R \to M$ defined by $\varphi_a(x) = ax$ for all $x \in R$. By (i) and [10, Theorem 2.19], R is M-d-Rickart. Therefore $\mathrm{Im}\,\varphi_a = aR$ is a direct summand of M. So M

is a von Neumann regular module. Similarly, we can see that R is a von Neumann regular ring.

(ii) Let $\varphi : N \to M$ be an R-homomorphism. Then $\operatorname{Im}\varphi$ is finitely generated. Hence $\operatorname{Im}\varphi$ is a direct summand of M since M is a regular module. It follows that N is M-d-Rickart. □

Proposition 4.4. *The following conditions are equivalent for a finitely generated R-module M:*

(i) *$R \oplus M$ is a* d-*Rickart module;*

(ii) *M is a von Neumann regular module and R is a von Neumann regular ring.*

Proof. (i) ⇒ (ii) By Lemma 4.3(i).

(ii) ⇒ (i) Applying Lemma 4.3(ii), we conclude that M is d-Rickart, R_R is M-d-Rickart, M is R_R-d-Rickart and R_R is d-Rickart. By [10, Corollary 5.6], it follows that $R \oplus M$ is a d-Rickart module. □

Corollary 4.5. *The following are equivalent for a ring R:*

(i) *Every finitely generated R-module is a* d-*Rickart module;*

(ii) *For any finitely generated R-module M, $R \oplus M$ is a* d-*Rickart module;*

(iii) *Every finitely generated R-module is a regular module.*

Proof. By Lemma 4.3 and Proposition 4.4. □

A ring R is called a *right FGC-ring* if every finitely generated right R-module is a direct sum of cyclic submodules.

Proposition 4.6. *Let R be a ring such that every finitely generated R-module is* d-*Rickart. Then the following hold:*

(i) *R is a von Neumann regular ring,*

(ii) *R is a right V-ring,*

(iii) *R is an FGC-ring,*

(iv) *Every indecomposable finitely generated R-module is a simple injective module, and*

(v) *For any right ideal I of R and any $x \in R$, there exists a right ideal I' of R such that $I \subseteq I'$, $xR \cap I' \subseteq I$ and $xR + I' = R$.*

Proof. (i) By Corollary 4.5 (see also [10, Remark 2.2]).

(ii) By Proposition 4.1.

(iii) By Corollary 4.5 and [14, Remark 6.2(2)].

(iv) By Proposition 4.1.

(v) Let I be a right ideal of R and let $x \in R$. By Corollary 4.5, R/I is a regular R-module. So $(xR + I)/I$ is a direct summand of R/I. Let I' be a right ideal of R which contains I such that $((xR+I)/I)\oplus(I'/I)=R/I$. Then $xR+I'=R$ and $xR\cap I'\subseteq I$. This completes the proof. □

Proposition 4.7. *Let R be a right noetherian ring. Then the following are equivalent:*

(i) *Every finitely generated R-module is a* d-*Rickart module;*
(ii) *R is a semisimple ring.*

Proof. (i) ⇒ (ii) Let I be a right ideal of R. Since R is right noetherian, I is finitely generated. Then by Corollary 4.5, I is a direct summand of R_R. Thus R is a semisimple ring.

(ii) ⇒ (i) This is clear. □

Note that there exists a commutative noetherian local ring R that may have an R-module which is not wd-Rickart, and hence not d-Rickart.

Example 4.8. Let F be a field. Consider $F[[x]]$, the formal power series ring over F. It is not hard to see that $F[[x]]$ is a commutative local noetherian ring (it is also a domain). Let $F((x))$ be the quotient field of $F[[x]]$. Take the cyclic $F[[x]]$-module $K=\{q\in F((x))\mid xq\in F[[x]]\}$. Note that $F[[x]]\subseteq K$. Consider the nonzero $F[[x]]$-monomorphism $\alpha:K\to K$ defined by $q\mapsto xq$. Clearly, $\operatorname{Im}\alpha\subseteq F[[x]]$. If $\operatorname{Im}\alpha$ contains a nonzero direct summand of K, then $\operatorname{Im}\alpha=F[[x]]$, which is a contradiction. This means that K is not a wd-Rickart $F[[x]]$-module.

Now we characterize commutative semisimple rings in terms of finitely generated d-Rickart modules.

Proposition 4.9. *The following are equivalent for a commutative ring R:*

(i) *Every finitely generated R-module is a* d-*Rickart module;*
(ii) *R is a semisimple ring.*

Proof. (i) ⇒ (ii) By Proposition 4.6, R is an FGC-ring which is von Neumann regular. Thus R is a direct sum of indecomposable rings by [3, Theorem 9.1]. Since R is von Neumann regular, it follows that R is a semisimple ring.

(ii) ⇒ (i) This is clear. □

Note that there exists a non-commutative artinian local ring R that may have a finitely generated injective R-module which is not wd-Rickart, and hence not d-Rickart.

Example 4.10. Let R be a local artinian ring with radical W such that $W^2 = 0$, $Q = R/W$ is commutative, $\dim({}_QW) = 2$ and $\dim(W_Q) = 1$. Then the indecomposable injective 2-generated right R-module $U = [(R \oplus R)/D]_R$ with $D = \{(ur, -vr) \mid r \in R\}$ and $W = Ru + Rv$ is not regular. For, let N be a cyclic submodule of U with length 2. Then $N \neq U$ since U has length 3. Therefore N cannot be a direct summand of U. On the other hand, note that U/N is simple and let $\pi : U \to U/N$ denote the canonical epimorphism. Since R is an artinian ring, we have $\mathrm{Soc}(U) \neq 0$. Let S be a simple submodule of U. Therefore there exists an isomorphism $\alpha : U/N \to S$ as R is a local ring. Let $\mu : S \to U$ be the inclusion map. It follows that $f = \mu\alpha\pi : U \to U$ is an endomorphism of U such that $\mathrm{Im}\, f = S$ is not a direct summand of U. This implies that U is not a d-Rickart module. Since U is indecomposable, U is not wd-Rickart, either.

5. Some structure results

Recall that a module M is said to be *dual Baer* if for every submodule $N \leqslant M$, there exists an idempotent $e \in S = \mathrm{End}_R(M)$ such that $D(N) = eS$, where $D(N) = \{\varphi \in S \mid \mathrm{Im}\,\varphi \subseteq N\}$. This notion was introduced by Keskin Tütüncü-Tribak in 2010 [6].

In this section, we present some structure results for some subclasses of wd-Rickart modules.

Since the properties of d-Rickart and wd-Rickart coincide for every noetherian module by [13, Corollary 3.5], the following three results can be obtained immediately from [10, Propositions 4.12 and 4.13 and Theorem 4.14], respectively.

Proposition 5.1. *Let M be a noetherian* wd-*Rickart module. Then there exists a decomposition $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ where for each i, M_i is an indecomposable noetherian* wd-*Rickart module with* $\mathrm{End}_R(M_i)$ *a division ring. Moreover, $n \in \mathbb{N}$ is uniquely determined, and the sequence of isomorphism types of $M_1, M_2, \ldots, M_n$ is uniquely determined up to permutation.*

Proposition 5.2. *Let M be a noetherian module over a commutative ring R. Then the following are equivalent for M:*

(a) *M is a* d-*Rickart module;*
(b) *M is a* wd-*Rickart module;*
(c) *M is a dual Baer module;*
(d) *M is a semisimple module.*

Theorem 5.3. *Let M be an n-generated module over a commutative noetherian ring R for $n \in \mathbb{N}$. Then the following are equivalent for M:*

(a) *M is a* d-*Rickart module;*
(b) *M is a* wd-*Rickart module;*
(c) *M is a dual Baer module;*
(d) *$M \cong R/\mathfrak{m}_1 \oplus R/\mathfrak{m}_2 \oplus \cdots \oplus R/\mathfrak{m}_n$, where $\mathfrak{m}_i$ are maximal ideals of R with $1 \leqslant i \leqslant n$.*

Let R be a Dedekind domain which is not a field. Then for each nonzero prime ideal P of R, $R(P^\infty)$ will denote the P-primary component of the torsion R-module K/R, where K is the quotient field of R.

Theorem 5.4. *Let R be a Dedekind domain which is not a field. Let K be the quotient field of R. The following are equivalent for an R-module $M = \oplus_{i \in I} M_i$, where M_i is indecomposable for each $i \in I$:*

(i) *M is a dual Baer module;*
(ii) *M is a* d-*Rickart module;*
(iii) *M is a* wd-*Rickart module;*
(iv) *M is a direct sum of copies of K, $(R(P_i^\infty))_{i \in I}$ and $(R/Q_j)_{j \in J}$, where $(P_i)_{i \in I}$ and $(Q_j)_{j \in J}$ are nonzero prime ideals of R with $P_i \neq Q_j$ for every couple $(i, j) \in I \times J$.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear by definitions.

(iii) $\Rightarrow$ (iv) By [13, Corollaries 2.5 and 3.4], each M_i $(i \in I)$ is an indecomposable dual Baer module. Applying [6, Theorem 3.4], we see that each M_i is either isomorphic to K or $R(P_i^\infty)$ or R/Q_i for some nonzero prime ideals P_i and Q_i of R. Moreover, by [13, Example 2.6], it follows that for every nonzero prime ideal P of R, the R-module $R(P^\infty) \oplus R/P$ is not a wd-Rickart module. The result follows.

(iv) $\Rightarrow$ (i) By [6, Theorem 3.4]. □

Corollary 5.5. *For a $\mathbb{Z}$-module $M = \oplus_{i \in I} M_i$, where M_i is indecomposable for each $i \in I$, the following are equivalent:*

(i) *M is a dual Baer module;*
(ii) *M is a* d-*Rickart module;*
(iii) *M is a* wd-*Rickart module;*
(iv) *M is isomorphic to a direct sum of arbitrarily many copies of $\mathbb{Q}$ and $(\mathbb{Z}(p_i^\infty))_{i \in I}$ and $(\mathbb{Z}/q_j\mathbb{Z})_{j \in J}$, where $p_i (i \in I)$ and $q_j (j \in J)$ are primes with $p_i \neq q_j$ for every couple $(i, j) \in I \times J$.*

Recall that a module M is called *lifting* if for every submodule N of M, there exists a direct summand K of M such that $K \leqslant N$ and N/K is small in M/K.

Theorem 5.6. *Let R be a non-local Dedekind domain. The following are equivalent for an R-module $M = \oplus_{i\in I} M_i$, where M_i is indecomposable for each $i \in I$:*

(i) *M is a dual Baer lifting module;*

(ii) *M is a* d-*Rickart lifting module;*

(iii) *M is a* wd-*Rickart lifting module;*

(iv) *M is torsion and every P-primary component of M is isomorphic either to $[R(P^\infty)]^{n_P}$ or $[R/P]^{(I_P)}$ for some natural number n_P and index set I_P.*

Proof. By Theorem 5.4 and [12, Propositions A.7 and A.8]. □

Corollary 5.7. *For a $\mathbb{Z}$-module $M = \oplus_{i\in I} M_i$, where M_i is indecomposable for each $i \in I$, the following are equivalent:*

(i) *M is dual Baer lifting;*

(ii) *M is* d-*Rickart lifting;*

(iii) *M is* wd-*Rickart lifting;*

(iv) *M is torsion and each p-primary component M_p is isomorphic either to $[\mathbb{Z}(p^\infty)]^{n_P}$ or $[\mathbb{Z}/p\mathbb{Z}]^{(I_P)}$ for some natural number n_P and index set I_P.*

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