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RESEARCH ARTICLE

Groupoids: Direct products, semidirect products and solvability

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Abstract. We present some constructions of groupoids such as: direct product, semidirect product and give necessary and sufficient conditions for a groupoid to be embedded into a direct product of groupoids. Also, we establish necessary and sufficient conditions to determine when a semidirect product is direct. Finally the notion of solvable groupoid is introduced and studied, in particular it is shown that a finite groupoid $\mathcal{G}$ is solvable if and only if its isotropy groups are.

Introduction

A groupoid is a small category in which every morphism is invertible. An equivalent notion of groupoid from an axiomatic approach, such as that of a group, is presented in [11, p. 89]. Using this point of view, Paques and Tamusiunas gave necessary and sufficient conditions for a subgroupoid to be normal (see Definition 1.2) and constructed the quotient groupoid [17] which plays an important role in the study of Galois extensions for groupoid actions. In [4, 6] several equivalent characterizations of groupoids are presented. Applications of groupoids appear in several branches, for instance in [10] the author uses groupoids to simplify proofs of basic results in group theory. Groupoids are also used to study partial actions and partial representations of groups, for example in [12] the authors

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presented an alternative way to study globalizations through groupoids, in [1] it is shown that a partial action gives rise to a groupoid provided with a Haar system, whose C^*-algebra agrees with the crossed product by the partial action, while in [8] and [9] is constructed and studied a special groupoid algebra which controls the partial representations of an arbitrary group. On the other hand, Étale groupoids are topological groupoids where the domain and range are local homeomorphisms. In [18] it was defined a convolution algebra of the R-module (where R being a commutative unital ring) generated by characteristic functions of certain types of compact open subsetes of an ample étale groupoid, denomined étale groupoid algebras, and in [19] chain conditions were established on étale groupoid algebras and applications to Leavitt path algebras and inverse semigroup algebras were showed. In [13] orbifolds were described in terms of a certain kind of groupoids. In algebraic topology Brown in [7] used the van Kampen theorem for the fundamental groupoid on a set of base points to prove the Jordan Curve Theorem.

The main goal of this paper is to continue the works [3, 4] and [5] by presenting new constructions of groupoids and study some structural properties of them. For this, after the introduction in section 1 we introduce the necessary background on groupoids. In section 2 we define direct product of an arbitrary family of groupoids and give in Theorem 2.4 necessary and sufficient conditions for a groupoid to be embedded in a direct product of groupoids. At this point it is important to remark that in [16] the author deals with internal direct product of groupoids, but in his work groupoids are considered as binary system so his approach is different from ours. We also give in Theorem 2.9 necessary and sufficient conditions for a semidirect product to be direct. In section 3 we present the concept of solvable groupoid in terms of subnormal series, a characterization of solvable groupoid via derived series and normal subgroupoids are given in Theorem 3.4 and Theorem 3.6, respectively. Finally in Proposition 3.8 we show that under a mild restriction a groupoid is solvable provided that all its isotropy groups are.

1. Preliminaries

Recall that a *groupoid* is a small category in which every morphism is an isomorphism. The set of the objects of a groupoid $\mathcal{G}$ will be denoted by $\mathcal{G}_0$. If $g : e \to f$ is a morphism of $\mathcal{G}$ then $d(g) = e$ and $r(g) = f$ are called the *domain* and the *range* of g, respectively. We identify any object e of $\mathcal{G}$ with its identity morphism, that is, $e = \mathrm{id}_e$ and thus $\mathcal{G}_0 \subseteq \mathcal{G}$. The *isotropy* *group* associated to an object e of $\mathcal{G}$ is the group $\mathcal{G}_e = \{g \in \mathcal{G} \mid d(g) = r(g) = e\}$. The set $\mathrm{Iso}(\mathcal{G}) = \bigcup_{e\in\mathcal{G}_0} \mathcal{G}_e$ is called the *isotropy subgroupoid* of $\mathcal{G}$. The composition of morphisms of a groupoid $\mathcal{G}$ will be denoted via concatenation. Hence, for $g, h \in \mathcal{G}$, there exists gh, denoted $\exists gh$, if and only if $r(h) = d(g)$. Notice that, if $g \in \mathcal{G}$ then its inverse g^{-1} is unique, $d(g) = g^{-1}g$ and $r(g) = gg^{-1}$.

The following is well-known (see for instance [4, Proposition 2]).

Proposition 1.1. Let $\mathcal{G}$ be a groupoid, $g, h \in \mathcal{G}$ and $n \in \mathbb{N}$. The following assertions hold.

(i) If $\exists gh$, then $d(gh) = d(h)$ and $r(gh) = r(g)$.

(iii) $\exists gh$ if and only if $\exists h^{-1}g^{-1}$ and, $(gh)^{-1} = h^{-1}g^{-1}$.

(iii) For $g_1, g_2, \ldots, g_n \in \mathcal{G}$, $\exists g_1 g_2 \cdots g_n$ if and only if $r(g_{i+1}) = d(g_i)$, for $1 \leqslant i \leqslant n-1$.

Recall the notions of subgroupoid, wide and normal subgroupoid.

Definition 1.2. Let $\mathcal{G}$ be a groupoid and $\mathcal{H}$ a nonempty subset of $\mathcal{G}$.

(i) $\mathcal{H}$ is said to be a subgroupoid of $\mathcal{G}$ if for all $g, h \in \mathcal{H}$, $g^{-1} \in \mathcal{H}$ and $gh \in \mathcal{H}$ provided that $\exists gh$. In this case we denote $\mathcal{H} < \mathcal{G}$. If in addition $\mathcal{H}_0 = \mathcal{G}_0$ (or equivalently $\mathcal{G}_0 \subseteq \mathcal{H}$) then $\mathcal{H}$ is called a wide subgroupoid of $\mathcal{G}$.

(ii) A subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is said to be normal, denoted by $\mathcal{H} \lhd \mathcal{G}$, if $\mathcal{H}$ is wide and $g^{-1}\mathcal{H}g \subseteq \mathcal{H}$, where

$$g^{-1}\mathcal{H}g = \{g^{-1}hg \mid h \in \mathcal{H} \text{ and } r(h) = d(h) = r(g)\},$$

for all $g \in \mathcal{G}$.

Note that $\mathcal{G}_0$ is a wide subgroupoid of $\mathcal{G}$ and if $g \in \mathcal{G}$ then $g^{-1}\mathcal{G}_0 g = \{d(g)\} \subseteq \mathcal{G}_0$. That is $\mathcal{G}_0$ is a normal subgroupoid of $\mathcal{G}$.

Remark 1.3. Normality was defined in [6] as follows: A subgroupoid $\mathcal{H}$ is said to be normal if $\mathcal{H}_0 = \mathcal{G}_0$ and $g^{-1}\mathcal{H}_{r(g)}g = \mathcal{H}_{d(g)}$, for all $g \in \mathcal{G}$. The equivalence between this definition and the presented in Definition 1.2 appears in [17, Lemma 3.1].

Given a wide subgroupoid $\mathcal{H}$ of $\mathcal{G}$, in [17] was considered a relation on $\mathcal{G}$ as follows: for every $g, l \in \mathcal{G}$,

$$g \equiv_{\mathcal{H}} l \Longleftrightarrow \exists l^{-1}g \quad \text{and} \quad l^{-1}g \in \mathcal{H}.$$

Furthermore, this relation is a congruence, that is an equivalence relation which is compatible with products. The equivalence class of $\equiv_{\mathcal{H}}$ containing g is the set $g\mathcal{H} = \{gh \mid h \in \mathcal{H} \wedge r(h) = d(g)\}$. This set is called left coset of $\mathcal{H}$ in $\mathcal{G}$ containing g. Then we have the next.

Proposition 1.4. [17, Lemma 3.12]. If $\mathcal{H}$ is a normal subgroupoid of $\mathcal{G}$ and $\mathcal{G}/\mathcal{H}$ is the set of all left cosets of $\mathcal{H}$ in $\mathcal{G}$, then $\mathcal{G}/\mathcal{H}$ is a groupoid with the partial binary operation given by $(g\mathcal{H})(l\mathcal{H}) = gl\mathcal{H}$.

The groupoid $\mathcal{G}/\mathcal{H}$ in Proposition 1.4 is denomined the *quotient groupoid* of $\mathcal{G}$ by $\mathcal{H}$.

2. Direct and semidirect product of groupoids

We start this section by giving the definition of the direct product of an arbitrary family of groupoids, we also obtain a criteria to decide when a groupoid is embedded into a direct product of groupoids. Later semidirect products are also considered.

2.1. The direct product of a family of groupoids

Let $\{\mathcal{G}_i \mid i \in I\}$ be a family of groupoids and $\prod_{i\in I} \mathcal{G}_i$ the direct product of the family of sets $\{\mathcal{G}_i \mid i \in I\}$. We define a partially binary operation on $\{\mathcal{G}_i \mid i \in I\}$ as follows. Given $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} \mathcal{G}_i$,

$$\exists (x_i)_{i\in I}(y_i)_{i\in I} \iff \exists x_i y_i \ \forall i \in I.$$

In this case we set $(x_i)_{i\in I}(y_i)_{i\in I} = (x_i y_i)_{i\in I}$. It is clear that with this product the set $\prod_{i\in I} \mathcal{G}_i$ is a groupoid. Let n be a natural number, if I is a set with n elements and $\mathcal{G}_i = \mathcal{G}$ for all $i \in I$, the set $\prod_{i\in I} \mathcal{G}_i$ is denoted by $\mathcal{G}^n$.

Now consider a family $\{X_i\}_{1\leqslant i\leqslant n}$ consisting of non-empty subsets of $\mathcal{G}$. We set

$$(X_1 \times X_2 \times \cdots \times X_n)^{(n)} = \{(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n \mid \exists x_1 \cdots x_n\}$$

and

$$X_1 \cdots X_n = \{x_1 \cdots x_n \mid (x_1, x_2, \ldots, x_n) \in (X_1 \times X_2 \times \cdots \times X_n)^{(n)}\}.$$

In general it is not true that $(X_1 \times X_2 \times \cdots \times X_n)^{(n)}$ is a subgroupoid of $\mathcal{G}^n$, even though each X_i is a subgroupoid of $\mathcal{G}$, for all $1 \leqslant i \leqslant n$. In the next result we provide necessary and sufficient conditions for $(X_1 \times X_2 \times \cdots \times X_n)^{(n)}$ to be a groupoid.

Proposition 2.1. Let n be a natural number and $\mathcal{H}_i$ be a subgroupoid of $\mathcal{G}$, for $i \in \{1, \ldots, n\}$. Then $(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$ is a subgroupoid of $\mathcal{G}^n$ if and only if $r(h_i) = d(h_{i+1})$ for $1 \leqslant i \leqslant n-1$ with $(h_1, \ldots, h_n) \in (\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$.

Proof. ($\Rightarrow$) Let $(h_1, \ldots, h_n) \in (\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$. We need to show that $d(h_{i+1}) = r(h_i)$, for all $1 \leqslant i \leqslant n-1$. By assumption $d(h_i) = r(h_{i+1})$, for $1 \leqslant i \leqslant n-1$. Since $(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$ is a groupoid we have $(h_1^{-1}, \ldots, h_n^{-1}) \in (\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$, which gives $r(h_i) = d(h_{i+1})$ for $1 \leqslant i \leqslant n-1$.

($\Leftarrow$) Let $(h_1, \ldots, h_n) \in (\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$ and $1 \leqslant i \leqslant n-1$, then $r(h_{i+1}^{-1}) = d(h_{i+1}^{-1}) = r(h_i) = d(h_{i+1}^{-1})$ and thus $(h_1^{-1}, \ldots, h_n^{-1}) \in (\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$. Now if $(h_1', \ldots, h_n') \in (\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$ and $\exists (h_1, \ldots, h_n)(h_1', \ldots, h_n')$ then $(h_1 h_1', \ldots, h_n h_n') \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$. Now for $1 \leqslant i \leqslant n-1$

$$d(h_i') = r(h_{i+1}') = d(h_i) = r(h_{i+1})$$

and thus $\exists h_i' h_{i+1}$, which implies $(h_i h_1', \ldots, h_n h_n') \in (\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$, and we conclude that $(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$ is a subgroupoid of $\mathcal{G}^n$. $\square$

Corollary 2.2. Let n be a natural number and $\mathcal{H}_i$ be wide subgroupoids of $\mathcal{G}$, for $i \in \{1, \ldots, n\}$. Then $(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$ is a subgroupoid of $\mathcal{G}^n$ if and only if $\mathcal{H}_i = \operatorname{Iso} \mathcal{H}_i$ for $1 \leqslant i \leqslant n$.

Proof. ($\Rightarrow$) Take $h_i \in \mathcal{H}_i$ then $(r(h_i), \ldots, r(h_i), h_i, d(x_i), \ldots, d(x_i)) \in (\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$, by Proposition 2.1 we have that $r(r(h_i)) = d(h_i) = d(d(h_i))$, that is $r(h_i) = d(h_i)$ and $h_i \in \operatorname{Iso} \mathcal{H}_i$, that is $\mathcal{H}_i = \operatorname{Iso} \mathcal{H}_i$.

($\Leftarrow$) Let $(h_1, \ldots, h_n) \in (\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$ then $r(h_i) = d(h_i) = r(h_{i+1}) = d(h_{i+1})$, again by Proposition 2.1 we have that $(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$ is a subgroupoid of $\mathcal{G}^n$. $\square$

We have the following.

Proposition 2.3. Let $\mathcal{G}$ be a groupoid and $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be wide subgroupoids of $\mathcal{G}$. Consider the following assertions.

(i) $\mathcal{G} = \mathcal{H}_1 \cdots \mathcal{H}_n$.
(ii) $\mathcal{H}_i \lhd \mathcal{G}, \ \forall i = 1, \ldots, n$.
(iii) $\mathcal{H}_i \cap (\mathcal{H}_1 \cdots \mathcal{H}_{i-1} \mathcal{H}_{i+1} \cdots \mathcal{H}_n) = \mathcal{G}_0, \ \forall i = 1, \ldots, n$.
(iv) For each $g \in \mathcal{G}$, there exists unique elements $x_1 \in \mathcal{H}_1, \ldots, x_n \in \mathcal{H}_n$ such that $(x_1, \ldots, x_n) \in (\mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_n)^{(n)}$ and $g = x_1 \cdots x_n$.

(v) For each $i \neq j$, we have that $xy = yx$, $\forall x \in \mathcal{H}_i$ and $\forall y \in \mathcal{H}_j$ such that
$$r(y) = d(x) = r(x) = d(y).$$

Then (i),(ii) and (iii) hold if and only if (iv) and (v) hold.

Proof. Assume that the conditions (i), (ii), (iii) hold. We start by checking (v), for this take $x \in \mathcal{H}_i$ and $y \in \mathcal{H}_j$, with $i \neq j$, and $r(y) = d(x) = r(x) = d(y)$, then $\exists y^{-1}x^{-1}yx$ and $y^{-1}x^{-1}yx = (y^{-1}x^{-1}y)x \in \mathcal{H}_i$, because $\mathcal{H}_i \lhd \mathcal{G}$, and $y^{-1}x^{-1}yx = y^{-1}(x^{-1}yx) \in \mathcal{H}_j$, because $\mathcal{H}_j \lhd \mathcal{G}$. Thus, $y^{-1}x^{-1}yx \in \mathcal{H}_i \cap \mathcal{H}_j = \mathcal{G}_0$. that is,
$$y^{-1}x^{-1}yx = d(g), \tag{1}$$
for some $g \in \mathcal{G}$, which gives $d(x) = d(y^{-1}x^{-1}yx) = d(g)$ and $r(y) = d(y) = r(x) = d(x) = d(g)$. From this and (1) we get $xy = yx$ which is (v).

Now we prove (iv). Take $g \in \mathcal{G}$ then by (i) there are $x_i \in \mathcal{H}_i$, $1 \leqslant i \leqslant n$ such that $g = x_1 \cdots x_n$. If $g = y_1 \cdots y_n$, with $y_i \in \mathcal{H}_i$. Let $1 \leqslant i \leqslant n-1$ then $d(x_i) = r(x_{i+1})$ and $d(y_i) = r(y_{i+1})$, also $\exists y_1^{-1}(y_1 \cdots y_n)$ and $\exists x_1 \cdots x_n x_n^{-1} \cdots x_2^{-1}$, thus
$$y_1^{-1}x_1 = y_2y_3 \cdots y_{n-1}y_nx_n^{-1}x_{n-1}^{-1} \cdots x_2^{-1}.$$
By (v) we have
$$y_1^{-1}x_1 = y_2x_2^{-1}y_3x_3^{-1} \cdots y_nx_n^{-1},$$
where $y_i^{-1}x_i \in \mathcal{H}_i$ with $i = 1, \dots, n$. Thus, $y_1^{-1}x_1 \in \mathcal{H}_1 \cap (\mathcal{H}_2 \cdots \mathcal{H}_n) = \mathcal{G}_0$ and there exists $g \in \mathcal{G}$ such that $y_1^{-1}x_1 = d(g)$. Hence, $x_1 = y_1$.
Now as $x_1 \cdots x_n = y_1 \cdots y_n$ and $x_1 = y_1$ by the cancellation law for groupoids, we obtain $x_2 \cdots x_n = y_2 \cdots y_n$ and continuing this process we have $x_i = y_i$ for $i = 2, \dots, n$.

For the other implication, suppose that conditions (iv) and (v) are satisfied. First of all note that (i) follows from (iv). To show (ii) take $i \in \{1, \dots n\}$ since $\mathcal{H}_i$ is wide subgroupoid, then $\mathcal{G}_0 = (\mathcal{H}_i)_0$ and thus $g^{-1}\mathcal{H}_i g \neq \varnothing$ for all $i = 1, \dots n$ and $g \in \mathcal{G}$. We will see that given $g \in \mathcal{G}$ and $y \in \mathcal{H}_i$ such that $\exists g^{-1}yg$ then $g^{-1}yg \in \mathcal{H}_i$. Indeed, by (iv), there are $x_j \in \mathcal{H}_j, 1 \leqslant j \leqslant n$ such that $\exists x_1x_2 \cdots x_n$ and $g = x_1x_2 \cdots x_n$. Then
$$g^{-1}yg = x_n^{-1} \cdots x_i^{-1}x_{i-1}^{-1} \cdots x_1^{-1}yx_1 \cdots x_{i-1}x_ix_{i+1} \cdots x_n.$$
Furthermore, for $j = 1, \dots, i-1$, $yx_j = x_jy$ and $d(x_j) = r(y)$ thanks to (v). Thus,
$$x_j^{-1}yx_j = x_j^{-1}x_jy = d(x_j)y = r(y)y = y, \quad \text{for all} \quad j = 1, \dots, i-1.$$

Hence, $g^{-1}yg = x_n^{-1}\cdots x_i^{-1}yx_ix_{i+1}\cdots x_n$. Now, as for all $j = i+1,\ldots,n-1$, $x_j^{-1}x_{j+1}^{-1} = x_{j+1}^{-1}x_j^{-1}$ and $x_jx_{j+1} = x_{j+1}x_j$, we get

$$g^{-1}yg = x_i^{-1}x_{i+1}^{-1}\cdots x_n^{-1}yx_n\cdots x_{i+1}x_i.$$

Now, for $j = i,\ldots,n$, $yx_j = x_jy$ and $d(x_j) = r(y)$ by condition (v). It implies that

$$x_j^{-1}yx_j = x_j^{-1}x_jy = d(x_j)y = r(y)y = y, \quad \text{for all} \quad j = i+1,\ldots,n.$$

Thus, $g^{-1}yg = x_i^{-1}yx_i \in \mathcal{H}_i$ which is (ii).

To fininsh the proof, we show (iii). As $\mathcal{G}_0 = (\mathcal{H}_i)_0$ for all $i = 1,\ldots,n$, we have that $\mathcal{G}_0 \subseteq \mathcal{H}_i \cap (\mathcal{H}_1\cdots\mathcal{H}_{i-1}\mathcal{H}_{i+1}\cdots\mathcal{H}_n)$. Now take $g \in \mathcal{H}_i \cap (\mathcal{H}_1\cdots\mathcal{H}_{i-1}\mathcal{H}_{i+1}\cdots\mathcal{H}_n)$, then $g \in \mathcal{H}_i$, and

$$g = x_1\cdots x_{i-1}gx_{i+1}\cdots x_n$$

with $x_j = r(g)$ for $j = 1,\ldots,i-1$ and $x_k = d(g)$ for $k = i+1,\ldots,n$. On the other hand, as $g \in \mathcal{H}_1\cdots\mathcal{H}_{i-1}\mathcal{H}_{i+1}\cdots\mathcal{H}_n$, then

$$g = h_1\cdots h_{i-1}h_{i+1}\cdots h_n = x_1\cdots x_{i-1}h_ih_{i+1}\cdots x_n$$

with $h_i = d(x_{i-1})$, then $x_j, h_j \in \mathcal{H}_j$ for all $j = 1,\ldots,n$ and by (iv) we get $g \in \mathcal{G}_0$, as desired. $\square$

If $\mathcal{G}$ and $\mathcal{G}'$ are groupoids and $\mathbf{F} : \mathcal{G} \to \mathcal{G}'$ a groupoid homomorphism, it is not difficult to show that if $\mathcal{K}$ is a wide subgroupoid of $\mathcal{G}'$, then $\mathbf{F}^{-1}(\mathcal{K}) = \{x \in \mathcal{G} \mid \mathbf{F}(x) \in \mathcal{K}\}$ is a wide subgroupoid of $\mathcal{G}$. We say that $\mathbf{F}$ is a monomorphism if it is injective, in this case $\mathcal{G}$ is *embedded* in $\mathcal{G}'$.

Theorem 2.4. Let $\mathcal{G}$ be a groupoid, then $\mathcal{G}$ is embedded in a direct product of groupoids if and only if there are $\mathcal{H}_1,\ldots,\mathcal{H}_n$ subgroupoids of $\mathcal{G}$ such that $\mathcal{H}_1,\ldots,\mathcal{H}_n$ satisfy conditions (i), (ii) and (iii) of Proposition 2.3.

Proof. ($\Rightarrow$) Suppose that there exists a groupoid monomorphism $\mathbf{F} : \mathcal{G} \to \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$. Since

$$\mathcal{K}_i = (\mathcal{G}_1)_0 \times \cdots \times \mathcal{G}_i \times \cdots \times (\mathcal{G}_n)_0$$

is a normal subgroupoid of $\mathcal{G}_1 \times \cdots \times \mathcal{G}_n$, then the family of groupoids $\mathcal{H}_i = \mathbf{F}^{-1}(\mathcal{K}_i)$ for $i = 1,\ldots,n$ satisfies (i), (ii) and (iii) of Proposition 2.3.

($\Rightarrow$) Suppose that there exists subgroupoids $\mathcal{H}_i$ of $\mathcal{G}, 1 \leqslant i \leqslant n$ that satisfy conditions (i), (ii) and (iii). Consider the map

$$\mathbf{F} : \mathcal{G} = \mathcal{H}_1\cdots\mathcal{H}_n \ni (h_1\cdots h_n) \mapsto (h_1,\ldots,h_n) \in \mathcal{G}^n \tag{2}$$

Note that condition (iv) makes $\mathbf{F}$ well defined and injective. We prove that $\mathbf{F}$ is a homomorphism. Suppose that $\exists x_1x_2\cdots x_n, \exists y_1y_2\cdots y_n$ and $g = x_1x_2\cdots x_n$ and $g' = y_1y_2\cdots y_n$ in $\mathcal{G}$. If $\exists gg'$ we have

$$gg' = x_1x_2\cdots x_ny_1y_2\cdots y_n = x_1y_1x_2y_2\cdots x_ny_n.$$

Using condition (v) we get $\mathbf{F}(gg') = (x_1y_1, x_2y_2, \ldots, x_ny_n) = \mathbf{F}(g)\mathbf{F}(g')$. $\square$

Remark 2.5. The image of $\mathbf{F}$ defined by (2) is $(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)^{(n)}$, then under conditions of Theorem 2.4 $\mathcal{G}$ is isomorphic to a subgroupoid of a direct product, provided that the groupoids $\mathcal{H}_i$ satisfy the assumptions in Proposition 2.1.

Example 2.6. Consider the groupoid $\mathcal{G} = \{a, u, v, u^{-1}, v^{-1}, x, y\}$, where $\mathcal{G}_0 = \{x, y\}$, $d(a) = r(a) = x, d(v) = r(u) = y$ and $r(v) = d(u) = x$, and the following composition rules $vu = a$, $a^2 = u^{-1}u = vv^{-1} = x$ and $v^{-1}v = uu^{-1} = y$ hold in $\mathcal{G}$. The following diagram ilustrates the composition in $\mathcal{G}$

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[dr, "a"'] & y \arrow[d, "v"] \\ & x \end{tikzcd}$$

Set $\mathcal{H}_1 = \{u, u^{-1}, x, y\}$ and $\mathcal{H}_2 = \{v, v^{-1}, x, y\}$. It is not diffiult to check that $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfy conditions (i)-(iii) of Proposition 2.3. Then $\mathcal{G} = \mathcal{H}_1\mathcal{H}_2$ is embedded in $\mathcal{H}_1 \times \mathcal{H}_2$.

2.2. Semidirect product and groupoids

Let $\mathcal{G}$ be a groupoid, G be a group with identity 1_G and $\omega : G \to \mathrm{Aut}(\mathcal{G})$ be a homomorphism of groups. The homomorphism ω induce a action of G on $\mathcal{G}$ given by

$$\begin{aligned} \cdot : \mathcal{G} \times G &\to \mathcal{G} \\ (x, g) &\mapsto x \cdot g := \omega_{g^{-1}}(x). \end{aligned}$$

We use ω to define a groupoid structure on the set $\mathcal{G} \times G$. Define the partial product as follows:

$$\exists (x, g)(z, h) \text{ if and only if } d(x) = r(\omega_g(z)).$$

In this case we set

$$(x, g)(z, h) = (x\omega_g(z), gh).$$

Here the identities are given as follows. For each $(x,a) \in \mathcal{G} \times G$, we set $d(x,a) = (d(x)\cdot a, 1_G)$ and $r(x,a) = (r(x), 1_G)$. Further, for $(x,a) \in \mathcal{G} \times G$ we have

$$(x,a)^{-1} = (\omega_{a^{-1}}(x^{-1}), a^{-1}) \in \mathcal{G} \times G. \tag{3}$$

With this product the groupoid $\mathcal{G} \times G$ is denoted by $\mathcal{G} \times_\omega G$ and is called the *semidirect product of the groupoid $\mathcal{G}$ with the group G, via the homomorphism $\omega : G \to Aut(\mathcal{G})$.*

Before giving an example of the construction above we recall that a groupoid $\mathcal{G}$ is connected, if for all $e, f \in \mathcal{G}_0$ there is $g \in \mathcal{G}$ such that $d(g) = e$ and $r(g) = f$. The following result is well-known.

Proposition 2.7. There is a groupoid isomorphism $\mathcal{G} \simeq \mathcal{G}_0^2 \times \mathcal{G}_e$, where $e \in \mathcal{G}_0$ and $\mathcal{G}_0^2$ is the coarse groupoid associated to $\mathcal{G}_0$, that is

$$d(x,y) = (x,x) \quad \text{and} \quad r(x,y) = (y,y), \tag{4}$$

and the rule of composition in $\mathcal{G}_0^2$ is given by $(y,z)(x,y) = (x,z)$, for all $x,y,z \in \mathcal{G}_0$.

Example 2.8. Let $\mathcal{G}$ be a connected groupoid and take $e \in \mathcal{G}_0$, by Proposition 2.7 one may write $\mathcal{G} = \mathcal{G}_0^2 \times \mathcal{G}_e$. Consider an action $\cdot : \mathcal{G} \times \mathcal{G}_e \to \mathcal{G}_e$ then $\mathcal{G}_e$ acts on $\mathcal{G}$ via the homomorphism $\omega : \mathcal{G}_e \to \mathrm{Aut}(\mathcal{G})$, where for $a \in \mathcal{G}_e$, $\omega_a(x,y,g) = (x,y,a\cdot g)$, for all $((x,y),g) \in \mathcal{G}$. Now $(\mathcal{G}_0^2 \times \mathcal{G}_e)_0 = \{(x,x,e) \mid x \in \mathcal{G}_0\}$ and $r(\omega_a(x,y,g)) = (y,y,e)$. Then $\mathcal{G} \times_\omega \mathcal{G}_e$ is a groupoid with partial product

$$(y,z,g,a)(x,y,h,b) = ((y,z,g)(x,y,a\cdot h), ab) = (x,z,g(a\cdot h), ab).$$

for all $(y,z,g), (x,y,h) \in \mathcal{G}$ and $a, b \in \mathcal{G}_e$.

The following result characterizes when a semidirect product is direct.

Theorem 2.9. Let $\mathcal{G}$ be a groupoid, G be a group with identity 1_G and $\omega : G \to Aut(\mathcal{G})$ be a group homomorphism. Then the following assertions are equivalent.

(i) The identity map between $\mathcal{G} \times G$ and $\mathcal{G} \times_\omega G$ is a homomorphism, and thus an isomorphism.
(ii) ω is trivial.
(iii) $\mathcal{G}_0 \times G$ is normal in $\mathcal{G} \times_\omega G$.

Proof. (i) $\Rightarrow$ (ii) Take $g \in G$ we need to show that ω_g is the identity map on $\mathcal{G}$. Take $z \in \mathcal{G}$, then $\exists (r(z), g)(z,b) \in \mathcal{G} \times G$, for $b \in G$. Since the identity map is a homomorphism of groupoids we have that $\exists(r(z), g)(z, b)$ and $(z, gb) = (r(z)\omega_a(z), gb)$. In particular $z = \omega_g(z)$ and ω is trivial.

(ii) $\Rightarrow$ (i) Si ω is trivial, then ω_g is the identity on $\mathcal{G}$ for all $g \in G$. Hence $\mathcal{G} \times_\omega G = \mathcal{G} \times G$ and the identity map is an isomorphism.

(ii) $\Rightarrow$ (iii) It is not difficult to check that $\mathcal{G}_0 \times G$ is a subgroupoid of $\mathcal{G} \times_\omega G$, also $(\mathcal{G}_0 \times G)_0 = \mathcal{G}_0 \times \{1_G\} = (\mathcal{G} \times_\omega G)_0$, that is $\mathcal{G}_0 \times G$ is wide. Let $(x, g) \in \mathcal{G} \times_\omega G$ and $(e, h) \in \mathcal{G}_0 \times G$ such that $\exists(x, g)(e, h)(x, g)^{-1} \in \mathcal{G} \times_\omega G$, but ω is trivial, then

$$\begin{aligned}(x, g)(e, h)(x, g)^{-1} &\overset{(3)}{=} (xe, gh)(x^{-1}, g^{-1}) = (xex^{-1}, ghg^{-1})\\ &= (e, ghg^{-1}) \in \mathcal{G}_0 \times G.\end{aligned}$$

(iii) $\Rightarrow$ (ii) Suppose that $\mathcal{G}_0 \times G$ is normal in $\mathcal{G} \times_\omega G$. Take $l \in G$, $x \in \mathcal{G}$ and write $l = ghg^{-1}$ for some $g, h \in G$. Now $(\omega_{g^{-1}}(d(x)), h) \in \mathcal{G}_0 \times G$ and $\exists(x, g)(\omega_{g^{-1}}(d(x)), h) \in \mathcal{G} \times_\omega G$ which equals (x, gh). Moreover by (3) $(x, g)^{-1} = (\omega_{g^{-1}}(x^{-1}), g^{-1})$ and

$$r(\omega_g(\omega_{g^{-1}}(x^{-1}))) = r(x^{-1}) = d(x),$$

thus $\exists(x, g)(\omega_{g^{-1}}(d(x)), h)(x, g)^{-1}$ and

$$\begin{aligned}(x, g)(\omega_{g^{-1}}(d(x)), h)(x, g)^{-1} &\overset{(3)}{=} (x, gh)(\omega_{g^{-1}}(x^{-1}), g^{-1})\\ &= (x\omega_{ghg^{-1}}(x^{-1}), ghg^{-1}) \in \mathcal{G}_0 \times G,\end{aligned}$$

from this $x\omega_l(x^{-1}) = x\omega_{ghg^{-1}}(x^{-1}) \in \mathcal{G}_0$ then $x = \omega_l(x)$ and ω is trivial. $\square$

The following result tells us how to recognice semidirect products inside a groupoid.

Proposition 2.10. Let $\mathcal{G}$ be a groupoid, G be a subgroup of $\mathcal{G}$ and $\mathcal{H}$ a normal subgroupoid of $\mathcal{G}$. Suppose that $\mathcal{H} \cap G = \{1_G\}$ and G acts in $\mathcal{H}$ via conjugation, then $\mathcal{H}G \simeq \mathcal{H} \times_w G$.

Proof. First of all by [4, Proposition 11], $\mathcal{H}G$ is a subgroupoid of $\mathcal{G}$. Since $\mathcal{H} \cap G = \{1_G\}$ every element in $\mathcal{H}G$ has a unique expresion hg, for some $h \in \mathcal{H}, g \in G$ with $d(h) = r(g)$, and thus the map $\varphi : \mathcal{H}G \ni hg \mapsto (h, g) \in \mathcal{H} \times_w G$ is a bijection. To check that it is an isomorphism take $h, h' \in \mathcal{H}, g, g' \in G$ such that $\exists hg, \exists h'g'$ and $\exists(hg)(h'g')$, then

$\exists gh', \exists gh'g^{-1}, \exists hgh'g^{-1}$ and

$$\begin{aligned}\varphi((hg)(h'g')) &= \varphi((hgh'g^{-1})(gg')) \\ &= \varphi((h\omega_g(h'))(gg')) \\ &= (h\omega_g(h'), gg') \\ &= \varphi(hg)\varphi(h'g').\end{aligned}$$

Hence $\mathcal{H}G \simeq \mathcal{H} \times_w G$. □

3. Solvable groupoids

In this chapter we introduce the notion of solvable groupoid and show that being solvable is a hereditary property and it is preserved by taking quotients.

Recall that a groupoid $\mathcal{G}$ is called abelian if all its isotropy groups are abelian. Clearly $\mathcal{G}_0$ is an abelian subgroupoid of $\mathcal{G}$.

Definition 3.1. A groupoid $\mathcal{G}$ is called solvable if there exists a series:

$$\mathcal{G}_0 = \mathcal{H}_0 \lhd \mathcal{H}_1 \lhd \cdots \lhd \mathcal{H}_s = \mathcal{G}$$

such that each factor $\mathcal{H}_{i+1}/\mathcal{H}_i$ is abelian, for all $0 \leqslant i \leqslant s-1$.

Before giving a characterization of solvable groupoid we need to recall some notions and facts.

Proposition 3.2. [3, Proposition 2.6] Let $\mathcal{G}$ be a groupoid, $\{\mathcal{H}_i\}_{i\in I}$ a family of subgroupoids of $\mathcal{G}$, and $\varnothing \neq B \subseteq \mathcal{G}$. Then:

(i) If $\bigcap_{i\in I} \mathcal{H}_i \neq \varnothing$, then $\bigcap_{i\in I} \mathcal{H}_i$ is a subgroupoid of $\mathcal{G}$;

(ii) If $\mathcal{H}_i$ is wide for each $i \in I$, then $\bigcap_{i\in I} \mathcal{H}_i$ is a wide subgroupoid of $\mathcal{G}$;

(iii) There exists a smallest subgroupoid of $\mathcal{G}$ that contains B.

If $\mathcal{G}$ is a groupoid and $\varnothing \neq B \subseteq \mathcal{G}$, then the subgroupoid given in the Proposition 3.2 is called the *subgroupoid generated by* B and it will be denoted by $\langle B\rangle$. It can be proved that the set $\langle B\rangle$ is given by

$$\langle B\rangle = \{x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n} \mid \exists x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}, x_i \in B, \alpha_i \in \{1,-1\}\, \forall i, n \in \mathbb{N}\}.$$

Now let X, Y be non-empty subset of $\mathcal{G}$ we denote by $[X,Y]$ the subgroupoid generated by the set $\{xyx^{-1}y^{-1} \mid x \in X, y \in Y,\ r(x) = d(x) = r(y) = d(y)\}$, provided that this set is non-empty. When $X = Y = \mathcal{G}$ the groupoid $[\mathcal{G},\mathcal{G}]$ is called the *derived subgroupoid of* $\mathcal{G}$, by [3, Proposition 4.4] we know that $[\mathcal{G},\mathcal{G}]$ is a normal subgroupoid of $\mathcal{G}$. We proceed with the next.

Definition 3.3. Let $\mathcal{G}$ be a groupoid. Define the following sequence of subgroupoids:

$$\mathcal{G}^{(0)} = \mathcal{G}, \quad \mathcal{G}^{(1)} = [\mathcal{G}, \mathcal{G}] \quad \text{and} \quad \mathcal{G}^{(i+1)} = [\mathcal{G}^{(i)}, \mathcal{G}^{(i)}] \quad \text{for all } i \geqslant 1.$$

These series of subgroupoids is called the *derived* or *commutator* series of $\mathcal{G}$.

Theorem 3.4. Let $\mathcal{G}$ be a groupoid. Then, $\mathcal{G}$ is solvable if and only if $\mathcal{G}^{(n)} = \mathcal{G}_0$ for some $n \geqslant 0$.

Proof. Assume that $\mathcal{G}$ is solvable. Then there exists a series

$$\mathcal{G}_0 = \mathcal{H}_0 \lhd \mathcal{H}_1 \lhd \cdots \lhd \mathcal{H}_s = \mathcal{G}$$

such that each factor $\mathcal{H}_{i+1}/\mathcal{H}_i$ is abelian. We prove that $\mathcal{G}^{(i)} < \mathcal{H}_{s-i}$ for all $i \geqslant 0$. By induction on i. For $i = 0$ is evident. Suppose that $\mathcal{G}^{(i)} < \mathcal{H}_{s-i}$. Then

$$\mathcal{G}^{(i+1)} = [\mathcal{G}^{(i)}, \mathcal{G}^{(i)}] < [\mathcal{H}_{s-i}, \mathcal{H}_{s-i}].$$

As $\mathcal{H}_{s-1}/\mathcal{H}_{s-i-1}$ is abelian, by [3, Proposition 4.4 (5)], $[\mathcal{H}_{s-i}, \mathcal{H}_{s-i}] < \mathcal{H}_{s-i-1}$. Hence $\mathcal{G}^{(i+1)} < \mathcal{H}_{s-i-1}$. Since $\mathcal{H}_0 = \mathcal{G}_0$, then $\mathcal{G}^{(s)} = \mathcal{G}_0$. For the other implication, suppose that $\mathcal{G}^{(n)} = \mathcal{G}_0$ for some $n \geqslant 0$. By [3, Proposition 4.4 (5)] taking $\mathcal{H}_i = \mathcal{G}^{(n-i)}$ then $\mathcal{H}_i$ is a normal subgroupoid of $\mathcal{H}_{i+1}$ with abelian quotient, so the derived series satisfies the definition of solvability of $\mathcal{G}$. □

If $\mathcal{G}$ is solvable, the *solvable length* of $\mathcal{G}$ is the smallest nonnegative n such that $\mathcal{G}^{(n)} = \mathcal{G}_0$. As in the case group, the derived series of a groupoid is a series of shortest length whose successive quotients are abelian.

Proposition 3.5. Let $\mathcal{G}$ and $\mathcal{K}$ be groupoids, $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ and let $\mathbf{F} : \mathcal{G} \to \mathcal{K}$ be a groupoid epimorphism. Then

(i) $\mathcal{H}^{(i)} < \mathcal{G}^{(i)}$ for all $i \geqslant 0$. In particular, if $\mathcal{G}$ is solvable then $\mathcal{H}$ is solvable.

(ii) $\mathbf{F}(\mathcal{G}^{(i)}) = \mathcal{K}^{(i)}$. In particular, homomorphic images and quotient groupoids of solvable groupoids are solvable.

(iii) If $\mathcal{G}$ is solvable then every non-trivial normal subgroup of $\mathcal{G}$, that is different from $\mathcal{G}_0$, contains a non-trivial abelian subgroupoid.

Proof. (i). By induction on i. As $\mathcal{H} < \mathcal{G}$, then $[\mathcal{H}, \mathcal{H}] < [\mathcal{G}, \mathcal{G}]$ by definition of commutator subgroupoids, i.e., $\mathcal{H}^{(1)} < \mathcal{G}^{(1)}$. Now, assume that $\mathcal{H}^{(i)} < \mathcal{G}^{(i)}$. Then

$$\mathcal{H}^{(i+1)} = [\mathcal{H}^{(i)}, \mathcal{H}^{(i)}] < [\mathcal{G}^{(i)}, \mathcal{G}^{(i)}] = \mathcal{G}^{(i+1)}.$$

In particular, if $\mathcal{G}$ is solvable then $\mathcal{G}^{(n)} = \mathcal{G}_0$ for some n. Hence $\mathcal{H}^{(n)} < \mathcal{G}^{(n)} = \mathcal{G}_0$. Thus $\mathcal{H}$ is solvable.
(ii). Note that for $x, y \in \mathcal{G}^{(i-1)}$,

$$\mathbf{F}([x,y]) = \mathbf{F}(x^{-1}y^{-1}xy) = \mathbf{F}(x)^{-1}\mathbf{F}(y)^{-1}\mathbf{F}(x)\mathbf{F}(y) = [\mathbf{F}(x), \mathbf{F}(y)]. \quad (5)$$

Now, we prove by induction that $\mathbf{F}(\mathcal{G}^{(i)}) < \mathcal{K}^{(i)}$. Indeed, for $i = 0$ note that $\mathbf{F}(\mathcal{G}) < \mathcal{K}$. Now, suppose that $\mathbf{F}(\mathcal{G}^{(i)}) < \mathcal{K}^{(i)}$. Using (5) we have

$$\mathbf{F}(\mathcal{G}^{(i+1)}) = \mathbf{F}([\mathcal{G}^{(i)}, \mathcal{G}^{(i)}]) = [\mathbf{F}(\mathcal{G}^{(i)}), \mathbf{F}(\mathcal{G}^{(i)})] < [\mathcal{K}^{(i)}, \mathcal{K}^{(i)}] = \mathcal{K}^{(i+1)}.$$

Since $\mathbf{F}$ is epimorphism, then for each $[x', y'] \in \mathcal{K}$ there exist $[x.y] \in \mathcal{G}$ such that $\mathbf{F}([x,y]) = [x', y']$. Hence, we prove by induction that $\mathbf{F}(\mathcal{G}^{(i)}) = \mathcal{K}^{(i)}$. In fact, for $i = 0$ we have that $\mathbf{F}(\mathcal{G}) = \mathcal{K}$ since $\mathbf{F}$ is epimorphism. Suppose that $\mathbf{F}(\mathcal{G}^{(i)}) = \mathcal{K}^{(i)}$ and note that

$$\mathbf{F}(\mathcal{G}^{(i+1)}) = \mathbf{F}([\mathcal{G}^{(i)}, \mathcal{G}^{(i)}]) = [\mathbf{F}(\mathcal{G}^{(i)}), \mathbf{F}(\mathcal{G}^{(i)})] = [\mathcal{K}^{(i)}, \mathcal{K}^{(i)}] = \mathcal{K}^{(i+1)}.$$

Again, if $\mathcal{G}^{(n)} = \mathcal{G}_0$ for some n then $\mathcal{K}^{(n)} = \mathbf{F}(\mathcal{G}^{(n)}) = \mathbf{F}(\mathcal{G}_0) = \mathcal{K}_0$.

(iii) Suppose that $\mathcal{G}$ is solvable and let $\mathcal{N}$ be a normal subgroupoid of $\mathcal{G}$ with $\mathcal{N} \neq \mathcal{G}_0$ and let $\mathcal{G}_0 = \mathcal{H}_0 \lhd \mathcal{H}_1 \lhd \cdots \lhd \mathcal{H}_s = \mathcal{G}$ such that each factor $\frac{\mathcal{H}_{i+1}}{\mathcal{H}_i}$ is abelian. Let i be the maximal possible integer such that $\mathcal{N} \cap \mathcal{H}_i = \mathcal{G}_0$. Notice that $i < n$ since $\mathcal{N}$ is non-trivial. Then $\mathcal{K} = \mathcal{N} \cap \mathcal{H}_{i+1}$ is a nontrivial subgroupoid of $\mathcal{G}$ which is normal in $\mathcal{G}$. Consider the map $j : \mathcal{H}_{i+1} \to \frac{\mathcal{H}_{i+1}}{\mathcal{H}_i}$ then

$$j(\mathcal{K}) = \frac{\mathcal{K}\mathcal{H}_i}{\mathcal{H}_i} \simeq \frac{\mathcal{K}}{\mathcal{K} \cap \mathcal{H}_i} = \frac{\mathcal{K}}{\mathcal{G}_0} \simeq \mathcal{K},$$

where the isomorphism follows from [4, Theorem 2], then $\mathcal{K}$ is isomorphic to a subgroupoid of the abelian groupoid $\frac{\mathcal{H}_{i+1}}{\mathcal{H}_i}$. Hence it is abelian. □

We proceed with the next.

Theorem 3.6. A groupoid $\mathcal{G}$ is solvable if and only if for any normal subgroupoid $\mathcal{N}$ of $\mathcal{G}$, $\mathcal{N}$ and $\mathcal{G}/\mathcal{N}$ are solvable.

Proof. ($\Rightarrow$) Let $\mathcal{N}$ be a normal subgroupoid of $\mathcal{G}$, we shall prove that $\mathcal{N}$ is solvable. Let $\mathcal{G}_0 = \mathcal{H}_0 \lhd \mathcal{H}_1 \lhd \cdots \lhd \mathcal{H}_s = \mathcal{G}$ be a series such that each factor $\mathcal{H}_{i+1}/\mathcal{H}_i$ is abelian, for all $0 \leqslant i \leqslant s-1$. Since $\mathcal{H}_{i+1} \cap \mathcal{N} = (\mathcal{H}_i \cap \mathcal{N}) \cap \mathcal{H}_{i+1}$ is a normal subgroupoid of $\mathcal{H}_i \cap \mathcal{N}$ for all $0 \leqslant i \leqslant s-1$, we obtain a series

$$\mathcal{N}_0 = (\mathcal{H}_0 \cap \mathcal{N}) \lhd (\mathcal{H}_1 \cap \mathcal{N}) \lhd \cdots \lhd (\mathcal{H}_s \cap \mathcal{N}) = \mathcal{N}.$$

Now, from [4, Theorem 2] it follows that

$$\frac{\mathcal{H}_{i+1}\cap\mathcal{N}}{\mathcal{H}_i\cap\mathcal{N}}\simeq\frac{\mathcal{H}_i(\mathcal{H}_{i+1}\cap\mathcal{N})}{\mathcal{H}_i}.$$

As $\frac{\mathcal{H}_i(\mathcal{H}_{i+1}\cap\mathcal{N})}{\mathcal{H}_i}<\frac{\mathcal{H}_{i+1}}{\mathcal{H}_i}$ and $\frac{\mathcal{H}_{i+1}}{\mathcal{H}_i}$ is abelian, then $\frac{\mathcal{H}_{i+1}\cap\mathcal{N}}{\mathcal{H}_i\cap\mathcal{N}}$ is abelian. Hence $\mathcal{N}$ is solvable. The fact that $\frac{\mathcal{G}}{\mathcal{N}}$ is solvable follows from Theorem 3.5 (ii). For the converse, suppose that $\mathcal{N}$ is solvable of length m and $\frac{\mathcal{G}}{\mathcal{N}}$ is solvable of length n then by (ii) of Proposition 3.5 applied to the natural epimorphism $j:\mathcal{G}\to\frac{\mathcal{G}}{\mathcal{N}}$ we obtain $j(\mathcal{G}^{(n)})=\left(\frac{\mathcal{G}}{\mathcal{N}}\right)^{(n)}=\mathcal{G}_0\mathcal{N}\subseteq\mathcal{N}$, i.e., $\mathcal{G}^{(n)}<\mathcal{N}$. Thus $\mathcal{G}^{(n+m)}=(\mathcal{G}^{(n)})^{(m)}<\mathcal{N}^{(m)}=\mathcal{G}_0$. Hence by Theorem 3.4 we have that $\mathcal{G}$ is solvable. □

Corollary 3.7. The following assertions hold.

(i) Let $\mathcal{K}$ and $\mathcal{H}$ be groupoids, then $\mathcal{K}\times\mathcal{H}$ is solvable if and only if $\mathcal{K}$ and $\mathcal{H}$ are.

(ii) If $\mathcal{G}$ is a connected groupoid, then $\mathcal{G}$ is solvable if and only if the isotropy group $\mathcal{G}_e$ is solvable for all $e\in\mathcal{G}_0$.

Proof. (i) ($\Rightarrow$) This part follows from (ii) of Proposition 3.5 since $\mathcal{K}$ and $\mathcal{H}$ are epimorphic images of $\mathcal{K}\times\mathcal{H}$. For ($\Leftarrow$) We first check that $\mathcal{K}\times\mathcal{H}_0$ is solvable, it is not difficult to check that for all $i\geqslant 0$ one has that $(\mathcal{K}\times\mathcal{H}_0)^{(i)}=\mathcal{K}^{(i)}\times\mathcal{H}_0^{(i)}=\mathcal{K}^{(i)}\times\mathcal{H}_0$. Since $\mathcal{K}$ is solvable Theorem 3.4 implies that there exists $n\geqslant 0$ such that $\mathcal{K}^{(n)}=\mathcal{K}_0$ thus $(\mathcal{K}\times\mathcal{H}_0)^{(n)}=\mathcal{K}_0\times\mathcal{H}_0$ and $\mathcal{K}\times\mathcal{H}_0$ is solvable. Now by [4, Theorem 1], there is a groupoid isomorphism $\frac{\mathcal{K}\times\mathcal{H}}{\mathcal{K}\times\mathcal{H}_0}\simeq\mathcal{H}$ and thus $\frac{\mathcal{K}\times\mathcal{H}}{\mathcal{K}\times\mathcal{H}_0}$ is solvable, then $\mathcal{K}\times\mathcal{H}$ is solvable due to Theorem 3.6.
(ii) Take $e\in\mathcal{G}_0$, by Proposition 2.7 there is a groupoid isomorphism $\mathcal{G}\simeq\mathcal{G}_0^2\times\mathcal{G}_e$, but for $f\in\mathcal{G}_0^2$ we have by (4) that $(\mathcal{G}_0^2)_{(f,f)}=\{(f,f)\}$ and $\mathcal{G}_0^2$ is abelian and thus solvable, then the result follows from (i). □

Now we shall show that under suitable conditions the notion of solvability for a groupoid $\mathcal{G}$ can be reduced to the group case. Let $\mathcal{G}$ be a groupoid, it is well-known that $\mathcal{G}$ is a disjoint union of connected subgroupoids. In fact, consider the following equivalence relation on $\mathcal{G}_0$: $x,y\in\mathcal{G}_0$, $x\sim y$ if and only if there exists $g\in\mathcal{G}$ such that $s(g)=x$ and $t(g)=y$. For each equivalence class $X\in\mathcal{G}_0/\sim$ corresponds a full connected subgroupoid $\mathcal{G}_X$ of $\mathcal{G}$ such that $(\mathcal{G}_X)_0=X$. By construction, $\mathcal{G}$ is the disjoint union of the subgroupoids $\mathcal{G}_X$, that is,

$$\mathcal{G}=\dot{\bigcup}_{X\in\mathcal{G}_0/\sim}\mathcal{G}_X. \tag{6}$$

and (6) gives $[\mathcal{G},\mathcal{G}] = \bigcup_{X\in\mathcal{G}_0/\sim} [\mathcal{G}_X,\mathcal{G}_X]$, and $\mathcal{G}^{(n)} = \bigcup_{X\in\mathcal{G}_0/\sim} (\mathcal{G}_X)^{(n)}$.

Suppose that $\mathcal{G}$ is solvable, we know by (i) of Proposition 3.5 that $\mathcal{G}_e$ is solvable for all $e \in \mathcal{G}_0$. Conversely, suppose that $\mathcal{G}_e$ is solvable for all $e \in \mathcal{G}_0$. Take $X \in \mathcal{G}_0/\sim$ then $\mathcal{G}_X$ is solvable thanks to (ii) of Corollary 3.7, and there is $n_X \in \mathbb{N}$ such that $(\mathcal{G}_X)^{(n_X)} = X$. If there exists $N = \max\{n_X \mid X \in \mathcal{G}_0/\sim\}$, then $\mathcal{G}^{(N)} = \bigcup_{X\in\mathcal{G}_0/\sim} (\mathcal{G}_X)^{(N)} = \bigcup_{X\in\mathcal{G}_0/\sim} X = \mathcal{G}_0$, and $\mathcal{G}$ is solvable. Hence we have proved the following.

Proposition 3.8. Let $\mathcal{G}$ a groupoid, and write $\mathcal{G} = \dot{\bigcup}_{X\in\mathcal{G}_0/\sim}\mathcal{G}_X$ as the union of connected subgroupoids given by (6). Then the following assertions hold.

(i) If $\mathcal{G}$ is solvable, then $\mathcal{G}_e$ is solvable for all $e \in \mathcal{G}_0$.

(ii) If $\mathcal{G}_e$ is solvable for all $e \in \mathcal{G}_0$, then $\mathcal{G}_X$ is solvable for all $X \in \mathcal{G}_0/\sim$. In this case let n_X be the solvable length of $\mathcal{G}_X$, then $\mathcal{G}$ is solvable provided that the sequence $\{n_X\}_{X\in\mathcal{G}_0/\sim}$ is upper bounded.

It is clear that the condition on the sequence $\{n_X\}_{X\in\mathcal{G}_0/\sim}$ given in (ii) of Proposition 3.8 holds for any finite groupoid. Then we have the next.

Corollary 3.9. let $\mathcal{G}$ be a finite groupoid, then $\mathcal{G}$ is solvable if and only if $\mathcal{G}_e$ is solvable for all $e \in \mathcal{G}_0$.

Remark 3.10. It is possible to construct a non solvable groupoid $\mathcal{G}$ such that $\mathcal{G}_e$ is solvable for all $e \in \mathcal{G}_0$. Indeed let $n \in \mathbb{N}$ by [2, Theorem 2.2] there is a group G_n of solvable length n, let $\mathcal{G}$ be the disjoint union $\mathcal{G} = \dot{\bigcup}_{n\in\mathbb{N}}G_n$ then $\mathcal{G}$ is not solvable.

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