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On products of 3-paths in finite full transformation semigroups

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Abstract. Let Sing_n denotes the semigroup of all singular self-maps of a finite set $X_n = \{1, 2, \ldots, n\}$. A map $\alpha \in \mathrm{Sing}_n$ is called a 3-path if there are $i, j, k \in X_n$ such that $i\alpha = j$, $j\alpha = k$ and $x\alpha = x$ for all $x \in X_n \setminus \{i, j\}$. In this paper, we described a procedure to factorise each $\alpha \in \mathrm{Sing}_n$ into a product of 3-paths. The length of each factorisation, that is the number of factors in each factorisation, is obtained to be equal to $\lceil \frac{1}{2}(g(\alpha)+m(\alpha))\rceil$, where $g(\alpha)$ is known as the gravity of α and $m(\alpha)$ is a parameter introduced in this work and referred to as the measure of α. Moreover, we showed that $\mathrm{Sing}_n \subseteq P^{[n-1]}$, where P denotes the set of all 3-paths in Sing_n and $P^{[k]} = P \cup P^2 \cup \cdots \cup P^k$.

1. Introduction

Let $X_n = \{1, 2, \ldots, n\}$. The full transformation semigroup $\mathcal{T}_n$ on X_n, that is the semigroup of all self-maps of X_n under composition of mappings, have been much studied. One of the outstanding contribution is given by Howie [5], where it was shown that the subsemigroup Sing_n, of all singular maps in $\mathcal{T}_n$, is generated by its set E_1 of all idempotents of defect one (that is element $e \in \mathcal{T}_n$ satisfying $e^2 = e$ and $|X_n \setminus \mathrm{im}(e)| = 1$). Later Howie [6] and Iwahori [7] independently computed the minimum number of factors in E_1 required to expressed each $\alpha \in \mathrm{Sing}_n$ to be $g(\alpha)$, the

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gravity of $\alpha \in \mathrm{Sing}_n$ (see [6] for details). The maximum possible value of this number was also obtained in [6] to be equal to $\lfloor \frac{3}{2}(n-1) \rfloor$, where $\lfloor x \rfloor$ is the floor of x (that is the unique integer m for which $x - 1 < m \leqslant x$). If E denote the set of all idempotents in Sing_n, the minimum number of factors in E required to expressed each $\alpha \in \mathrm{Sing}_n$ was found, by Saito [8], to be equal to $\lceil \frac{g(\alpha)}{d(\alpha)} \rceil$ or $\lceil \frac{g(\alpha)}{d(\alpha)} \rceil + 1$, where $d(\alpha) = |X_n \setminus \mathrm{im}(\alpha)|$ denotes the defect of α, and $\lceil x \rceil$ is the ceiling of x (that is the unique integer m for which $x \leqslant m < x + 1$).

Related lengths problems where addressed, for product of idempotents in semigroups of order-preserving maps in both full and partial cases, by Schein [9], Higgins [4] and Yang [10]. Garba [2] solved similar problems in the semigroup $\mathcal{P}_n$, of all partial transformations of X_n. Recently, Garba and Imam [3] also studied similar lengths problems in the symmetric inverse semigroup $\mathcal{I}_n$, of all partial one-to-one maps of X_n.

Ayik, *et. al.* [1] showed that the semigroup Sing_n can also be generated by certain primitive elements called path-cycles. Special class of path-cycles called m-paths can be regarded as generalisations of idempotents of defect one in the sense that all idempotents of defect one are 2-paths and vice-versa. In general, Ayik *et. al.* [1] proved that the semigroup Sing_n is generated by its set of m-paths for each m in $\{2, 3, \ldots, n\}$. In this paper, we describe a procedure to factorise each singular map α in Sing_n into a product of 3-paths. We then obtained the length of each factorisation, that is the number of 3-paths in the factorisation.

2. Preliminaries

Let $X_n = \{1, \ldots, n\}$ and let $\mathcal{T}_n$ be the full transformation semigroup on X_n. If $\{x_1, \ldots, x_m\} \subseteq X_n$ and $\alpha \in \mathcal{T}_n$ is defined by

$$x_i\alpha = x_{i+1},\ x_m\alpha = x_r\ (1 \leqslant r \leqslant m) \text{ and } x\alpha = x\ (x \in X_n \setminus \{x_1, \ldots, x_m\}),$$

then α is called a *path−cycle* of length m and period r, or simply, an (m, r)-*path-cycle*, and is denoted (in a linear notation) by $\alpha = [x_1, \ldots, x_m | x_r]$. If $r = m$, α is called an *m-path* to x_m or simply an *m-path*; if $m \geqslant 2$ and $r = 1$, α is called an *m-cycle*; if $m = r = 1$, α is called a *loop*; if $m = r = 2$, α is an *idempotent of defect one*; if $m \geqslant 2$ and $r \neq 1$, α is said to be a *proper path-cycle*.

Let $\xi = [x_1, x_2, x_3 | x_3]$ be an arbitrary 3-path in Sing_n, then ξ maps x_1 to x_2, x_2 to x_3 and all other elements of X_n identically. Instead of using the linear notation for ξ, we shall throughout this paper extend the array notation, used for idempotents of defect one (that is 2-paths) used in [6], and write ξ as

$$\xi = \begin{pmatrix} x_2 & x_1 \\ x_3 & x_2 \end{pmatrix}.$$

This will enable us a proper adoption of the methods of [6] in proving our results. In the array notation we shall refer to x_1 as the *upper entry* of ξ; to x_2 as the *middle entry* of ξ; and to x_3 as the *lower entry* of ξ.

Let α be in Sing_n. The equivalence relation ω on X_n, defined by

$$\omega = \{(x,y) \in X_n \times X_n : (\exists u, v \geqslant 0) x\alpha^u = y\alpha^v\},$$

partitioned X_n into orbits $\Omega_1, \dots, \Omega_k$. These orbits correspond to the connected components of the digraph associated to α with vertex set X_n in which there is a directed edge (x,y) if and only if $x\alpha = y$. Each orbit Ω has a kernel defined by

$$K(\Omega) = \{x \in \Omega : (\exists r > 0) x\alpha^r = x\}.$$

An orbit Ω is said to be:

standard	if and only if	$2 \leqslant \lvert K(\Omega)\rvert < \lvert\Omega\rvert$;
acyclic	if and only if	$1 = \lvert K(\Omega)\rvert < \lvert\Omega\rvert$;
cyclic	if and only if	$2 \leqslant \lvert K(\Omega)\rvert = \lvert\Omega\rvert$;
trivial	if and only if	$1 = \lvert K(\Omega)\rvert = \lvert\Omega\rvert$.

Every orbit of α falls into exactly one of these four categories and all four cases can arise. Let $c(\alpha)$ be the number of cyclic orbits of α and $f(\alpha)$ be the number of fixed points of α, this equals the sum of the number of trivial and the number of acyclic orbits of α. The *gravity* of α is defined as

$$g(\alpha) = n + c(\alpha) - f(\alpha).$$

For each standard or acyclic orbit Ω of $\alpha \in \mathrm{Sing}_n$ and each $x \in \Omega \backslash \mathrm{im}(\alpha)$, the sequence

$$x, x\alpha, x\alpha^2, \dots$$

eventually arrives in $K(\Omega)$, the kernel of Ω, and remains there for all subsequent iterations. Denote the set of all distinct elements in this sequence by $Z(x)$. Suppose that $\alpha \in \mathrm{Sing}_n$ has s standard orbits $\Omega_1, \Omega_2, \dots, \Omega_s$.

For each $j = 1, 2, \dots, s$, let $\Omega_j \setminus \mathrm{im}(\alpha) = \{x_{1j}, x_{2j}, \dots, x_{k_j j}\}$, where x_{1j} is such that

$$|Z(x_{1j})| = \begin{cases} \max\limits_{1 \leqslant i \leqslant k_j} \{|Z(x_{ij})| : \\ \qquad\qquad |Z(x_{ij})| \text{ is even}\} & \text{if } |Z(x_{ij})| \text{ is even for some } i, \\ \max_{1 \leqslant i \leqslant k_j} \{|Z(x_{ij})|\} & \text{if } |Z(x_{ij})| \text{ is odd for all } i. \end{cases}$$

Then there exist $m_j \geqslant 1$ and $r_j \geqslant 2$ (see [6]) such that

$$K(\Omega_j) = \{x_{1j}\alpha^{m_j}, \dots, x_{1j}\alpha^{m_j + r_j - 1}\},$$

where $x_{1j}\alpha^{m_j + r_j} = x_{1j}\alpha^{m_j}$. Note that this definition of $K(\Omega_j)$ is still valid for every x_{ij}, not only for x_{1j}, and moreover, they are all the same. We then define

$$Z_1(\Omega_j) = Z(x_{1j}) = \{x_{1j}, x_{1j}\alpha, \dots, x_{1j}\alpha^{m_j}, \dots, x_{1j}\alpha^{m_j + r_j - 1}\} \tag{1}$$

and

$$Z_i(\Omega_j) = \{x_{ij}, x_{ij}\alpha, \dots, x_{ij}\alpha^{p_{ij} - 1}\} \quad (2 \leqslant i \leqslant k_j) \tag{2}$$

where $x_{ij}\alpha^{p_{ij}} \in (Z_1(\Omega_j) \cup Z_2(\Omega_j) \cup \dots \cup Z_{i-1}(\Omega_j))$. Thus, $\{Z_i(\Omega_j) : 1 \leqslant i \leqslant k_j\}$ is a partition of Ω_j. Also, suppose that $\alpha \in \mathrm{Sing}_n$ has acyclic orbits; let Φ be the union of all its acyclic orbits and denote the set $\{x \in \Phi : x\alpha = x\}$ by $\mathrm{Fix}(\Phi)$. Let $\Phi \setminus \mathrm{im}(\alpha) = \{x_1, x_2, \dots, x_l\}$ where x_1 is such that

$$|Z(x_1)| \\ = \begin{cases} \max_{1 \leqslant u \leqslant l}\{|Z(x_u)| : |Z(x_u)| \text{ is odd}\} & \text{if } |Z(x_u)| \text{ is odd for some } u, \\ \max_{1 \leqslant u \leqslant l}\{|Z(x_u)|\} & \text{if } |Z(x_u)| \text{ is even for all } u. \end{cases}$$

Then, for $u = 1, 2, \dots, l$, define

$$Y_u(\Phi) = \{x_u, x_u\alpha, \dots, x_u\alpha^{q_u - 1}\}, \tag{3}$$

where $x_1\alpha^{q_1} \in \mathrm{Fix}(\Phi)$ and $x_u\alpha^{q_u} \in (Y_1(\Phi) \cup \dots \cup Y_{u-1}(\Phi) \cup \mathrm{Fix}(\Phi))$ $(u = 2, 3, \dots, l)$. Thus, $\{Y_u(\Phi) : 1 \leqslant u \leqslant l\}$ is a partition of Φ. We will be interested in the cardinalities of $Z_i(\Omega_j)$ and $Y_u(\Phi)$ being odd or even. For this, we define indicator functions z_{ij} and y_u by

$$z_{ij} = \begin{cases} 0 & \text{if } |Z_i(\Omega_j)| \text{ is even}, \\ 1 & \text{if } |Z_i(\Omega_j)| \text{ is odd}, \end{cases} \quad \text{and} \quad y_u = \begin{cases} 0 & \text{if } |Y_u(\Phi)| \text{ is even}, \\ 1 & \text{if } |Y_u(\Phi)| \text{ is odd}. \end{cases}$$

Finally, for each $\alpha \in \mathrm{Sing}_n$ we define the *measure* of α by

$$m(\alpha) = \begin{cases} l(\alpha) - e(\alpha) & \text{if } l(\alpha) > e(\alpha), \\ \\ 0 & \text{if } l(\alpha) \leqslant e(\alpha), \end{cases} \tag{4}$$

where $l(\alpha) = \sum_{j=1}^{s} \sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u$ and $e(\alpha)$ denote the number of cyclic orbits of α of even cardinality.

Before closing this section, we illustrate the above definitions and notations in an example.

Example 1. Consider the map

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 2 & 3 & 4 & 2 & 4 & 7 & 8 & 7 & 7 & 11 & 11 & 13 & 14 & 14 & 16 & 17 & 15 & 19 & 18 & 21 & 20 & 22 \end{pmatrix}$$

in Sing_{22} with orbits

$$\begin{array}{rr} \textit{standard}: & \Omega_1 = \{1,2,3,4,5\}, \Omega_2 = \{6,7,8,9\}; \\ \textit{acyclic}: & \Phi_1 = \{10,11\}, \Phi_2 = \{12,13,14\}; \\ \textit{cyclic}: & \Theta_1 = \{15,16,17\}, \Theta_2 = \{18,19\}, \Theta_3 = \{20,21\}; \\ \textit{trivial}: & \Psi_1 = \{22\}, \end{array}$$

as shown in Figure 1.

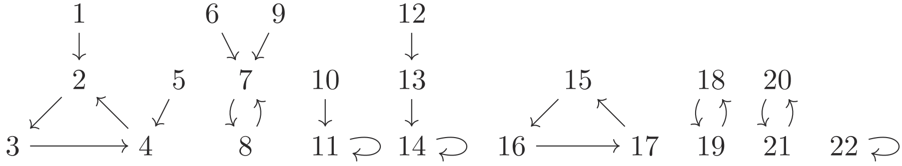

FIGURE 1. Orbits of $\alpha \in \mathrm{Sing}_{22}$.

For this α, we have $\Phi = \{10, 11, 12, 13, 14\}$ and so, $Z_1(\Omega_1) = \{1, 2, 3, 4\}$, $Z_2(\Omega_1) = \{5\}$ (note that, according to the concerning definitions, it is also possible that $Z_1(\Omega_1) = \{2, 3, 4, 5\}$ and $Z_2(\Omega_1) = \{1\}$), $Z_1(\Omega_2) = \{6, 7, 8\}$, $Z_2(\Omega_2) = \{9\}$, $Y_1(\Phi) = \{12, 13\}$, $Y_2(\Phi) = \{10\}$. Thus, $z_{11} = 0$, $z_{21} = 1$, $z_{12} = 1$, $z_{22} = 1$, $y_1 = 0$, $y_2 = 1$ and so, $l(\alpha) = z_{11}+z_{21}+z_{12}+z_{22}+y_1+y_2 = 4$, also $e(\alpha) = 2$. Therefore the measure of α is $m(\alpha) = 2$.

3. Products of 3-paths

In [1], it was proved that for a fixed m in $\{2, \ldots, n\}$, every element of Sing_n is a product of m-paths. The result was obtain via decomposing each 2-path in Sing_n as a product of 2 m-paths while each element of Sing_n is decomposable as a product of 2-paths. In this section, we consider the case when $m = 3$ and obtain a direct decomposition of each $\alpha \in \mathrm{Sing}_n$ as a product of 3-paths.

Let E be the set of all idempotents in Sing_n and E_1 be the set of all idempotents of defect 1 in E. First, we note that, in the notation of [6], each idempotent in E_1 is of the form $\binom{i}{j}$, with $i, j \in X_n$ and $i \neq j$. Thus, since $n \geqslant 3$, there is a $k \in X_n$, with $k \neq i$ and $k \neq j$, such that,

$$\begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} j & k \\ i & j \end{pmatrix} \begin{pmatrix} j & i \\ k & j \end{pmatrix}. \tag{5}$$

Theorem 1. *For $n \geqslant 3$, each $\alpha \in E \setminus E_1$ is expressible as a product of $g(\alpha)$ 3-path in* Sing_n.

Proof. Let $\alpha \in E \setminus E_1$ and let $A_1, A_2, \ldots, A_r$ be its non-singleton blocks. Then, each of the blocks A_i $(1 \leqslant i \leqslant r)$ is stationary. If $|A_i| \geqslant 3$ for some i, we can assume without loss of generality that $|A_1| \geqslant 3$. Let

$$A_i \setminus \{A_i\alpha\} = \{x_{i1}, x_{i2}, \ldots, x_{ia_i}\} \quad (1 \leqslant i \leqslant r)$$

and define products ξ_i $(1 \leqslant i \leqslant r)$ of 3-paths by

$$\xi_1 = \begin{pmatrix} x_{12} & x_{11} \\ A_1\alpha & x_{12} \end{pmatrix} \begin{pmatrix} x_{13} & x_{11} \\ A_1\alpha & x_{13} \end{pmatrix} \cdots \begin{pmatrix} x_{1a_1} & x_{11} \\ A_1\alpha & x_{1a_1} \end{pmatrix} \begin{pmatrix} x_{12} & x_{11} \\ A_1\alpha & x_{12} \end{pmatrix}$$

and

$$\xi_i = \begin{pmatrix} x_{i1} & x_{11} \\ A_i\alpha & x_{i1} \end{pmatrix} \begin{pmatrix} x_{i2} & x_{11} \\ A_i\alpha & x_{i2} \end{pmatrix} \cdots \begin{pmatrix} x_{ia_i} & x_{11} \\ A_i\alpha & x_{ia_i} \end{pmatrix} \qquad (2 \leqslant i \leqslant r).$$

Then, it is easy to verify that

$$\alpha = \xi_1\xi_2 \cdots \xi_r.$$

Also, observe that each point in $A_i \setminus \{A_i\alpha\}$ $(2 \leqslant i \leqslant r)$ appeared exactly once as a middle entry of a 3-path in ξ_i and each point in $A_1 \setminus \{A_1\alpha, x_{11}, x_{12}\}$ appeared exactly once as a middle entry of a 3-path in ξ_1. The point x_{12} appeared exactly twice as a middle entry of

3-paths in ξ_1 while the point x_{11} did not appear anywhere as a middle entry. Thus, the number of 3-paths used in the product $\xi_1\xi_2\cdots\xi_r$ is

$$\sum_{i=1}^{r} |A_i \setminus \{A_i\alpha\}| = n - f(\alpha) = g(\alpha).$$

If $|A_i| = 2$ for all i, let $A_i = \{x_i, x_i\alpha\}$ $(1 \leqslant i \leqslant r)$. Then,

$$\alpha = \begin{pmatrix} x_1 & x_r \\ x_1\alpha & x_1 \end{pmatrix} \begin{pmatrix} x_2 & x_r \\ x_2\alpha & x_2 \end{pmatrix} \cdots \begin{pmatrix} x_{r-1} & x_r \\ x_{r-1}\alpha & x_{r-1} \end{pmatrix} \begin{pmatrix} x_1 & x_r \\ x_r\alpha & x_1 \end{pmatrix},$$

and again, the number of 3-paths used is $n - f(\alpha) = g(\alpha)$. □

Example 2. Consider the idempotent

$$e = \begin{pmatrix} \{1,2,7,5\} & \{3,8,10,12\} & \{4,6,9,11\} \\ 2 & 8 & 11 \end{pmatrix} = \xi_1\xi_2\xi_3$$

where

$$\xi_1 = \begin{pmatrix} 5 & 1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 2 & 5 \end{pmatrix},$$

$$\xi_2 = \begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix} \begin{pmatrix} 10 & 1 \\ 8 & 10 \end{pmatrix} \begin{pmatrix} 12 & 1 \\ 8 & 12 \end{pmatrix},$$

$$\xi_3 = \begin{pmatrix} 4 & 1 \\ 11 & 4 \end{pmatrix} \begin{pmatrix} 6 & 1 \\ 11 & 6 \end{pmatrix} \begin{pmatrix} 9 & 1 \\ 11 & 9 \end{pmatrix}.$$

Theorem 2. *For $n \geqslant 3$, each $\alpha \in \mathrm{Sing}_n \setminus E$ is expressible as a product of $\lceil \frac{1}{2}(g(\alpha) + m(\alpha)) \rceil$ 3-paths in Sing_n.*

Proof. Suppose that $\alpha \in \mathrm{Sing}_n \setminus E$ has orbits as follows:

standard:	$\Omega_1, \Omega_2, \ldots, \Omega_s$;
acyclic:	$\Phi_1, \Phi_2, \ldots, \Phi_a$;
cyclic:	$\Theta_1, \Theta_2, \ldots, \Theta_c$;
trivial:	$\Psi_1, \Psi_2, \ldots, \Psi_t$.

For each standard orbit Ω_j let $\Omega_j \setminus \mathrm{im}(\alpha) = \{x_{1j}, x_{2j}, \ldots, x_{k_j j}\}$;

$$K(\Omega_j) = \{x_{1j}\alpha^{m_j}, x_{1j}\alpha^{m_j+1}, \ldots, x_{1j}\alpha^{m_j+r_j-1}\};$$

and define $Z_1(\Omega_j)$ and $Z_i(\Omega_j)$ $(i = 2, \ldots, k_j)$ as in Equations (1) and (2), respectively. Also, let

$$\Phi = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_a;$$

$$\Phi \setminus \mathrm{im}(\alpha) = \{x_1, x_2, \ldots, x_l\}$$

and define $Y_u(\Phi)$ $(u = 1, 2, \ldots, l)$ as in Equation (3). Moreover, let

$$\Theta_v = \{y_v, y_v\alpha, \ldots, y_v\alpha^{p_v-1}\}$$

(where $y_v\alpha^{p_v} = y_v$). Then we consider six possible cases that may arise.

Case 1. $0 = e(\alpha) = l(\alpha)$. In this case each $Z_i(\Omega_j)$ $(i = 1, 2, \ldots, k_j)$ and each $Y_u(\Phi)$ $(u = 1, 2, \ldots, l)$ is of even size; also, each Θ_v is of odd size. Thus, corresponding to each $Z_1(\Omega_j)$, $Z_i(\Omega_j)$ $(i = 2, 3, \ldots, k_j)$, $Y_u(\Phi)$ $(u = 1, 2, \ldots, l)$ and Θ_v $(v = 1, 2, \ldots, c)$ we define, respectively, products ξ_{1j}, ξ_{ij} $(i = 2, 3, \ldots, k_j)$, τ_u $(u = 1, 2, \ldots, l)$ and η_v $(v = 1, 2, \ldots, c)$ of 3-paths by

$$\xi_{1j} = \begin{pmatrix} x_{1j}\alpha^{m_j+r_j-1} & x_{1j}\alpha^{m_j+r_j-2} \\ x_{1j}\alpha^{m_j-1} & x_{1j}\alpha^{m_j+r_j-1} \end{pmatrix} \begin{pmatrix} x_{1j}\alpha^{m_j+r_j-3} & x_{1j}\alpha^{m_j+r_j-4} \\ x_{1j}\alpha^{m_j+r_j-2} & x_{1j}\alpha^{m_j+r_j-3} \end{pmatrix} \cdots \begin{pmatrix} x_{1j}\alpha^{3} & x_{1j}\alpha^{2} \\ x_{1j}\alpha^{4} & x_{1j}\alpha^{3} \end{pmatrix} \begin{pmatrix} x_{1j}\alpha & x_{1j} \\ x_{1j}\alpha^{2} & x_{1j}\alpha \end{pmatrix},$$

$$\xi_{ij} = \begin{pmatrix} x_{ij}\alpha^{p_{ij}-1} & x_{ij}\alpha^{p_{ij}-2} \\ x_{ij}\alpha^{p_{ij}} & x_{ij}\alpha^{p_{ij}-1} \end{pmatrix} \begin{pmatrix} x_{ij}\alpha^{p_{ij}-3} & x_{ij}\alpha^{p_{ij}-4} \\ x_{ij}\alpha^{p_{ij}-2} & x_{ij}\alpha^{p_{ij}-3} \end{pmatrix} \cdots \begin{pmatrix} x_{ij}\alpha & x_{ij} \\ x_{ij}\alpha^{2} & x_{ij}\alpha \end{pmatrix},$$

$$\tau_u = \begin{pmatrix} x_u\alpha^{q_u-1} & x_u\alpha^{q_u-2} \\ x_u\alpha^{q_u} & x_i\alpha^{q_u-1} \end{pmatrix} \begin{pmatrix} x_u\alpha^{q_u-3} & x_u\alpha^{q_u-4} \\ x_u\alpha^{q_u-2} & x_u\alpha^{q_u-3} \end{pmatrix} \cdots \begin{pmatrix} x_u\alpha & x_u \\ x_u\alpha^{2} & x_u\alpha \end{pmatrix}$$

and

$$\eta_v = \begin{pmatrix} y_v\alpha^{p_v-1} & y_v\alpha^{p_v-2} \\ z & y_v\alpha^{p_v-1} \end{pmatrix} \begin{pmatrix} y_v\alpha^{p_v-3} & y_v\alpha^{p_v-4} \\ y_v\alpha^{p_{v-2}} & y_v\alpha^{p_v-3} \end{pmatrix} \cdots \begin{pmatrix} y_v & z \\ x_v\alpha & y_v \end{pmatrix},$$

where z is any point in $X_n \setminus \mathrm{im}(\alpha)$.

For each $j = 1, 2, \ldots, s$, let

$$\beta_j = \xi_{1j}\xi_{2j} \cdots \xi_{k_jj},$$

then each element $x \in \Omega_j$ appears exactly once either as an upper entry or as a middle entry of a 3-path in the product β_j. Moreover, with the sole exception of $x = x_{1j}\alpha^{m_j-1}$, an element $x \in \Omega_j$ appearing as a lower entry or a middle entry never subsequently reappears as an upper or middle entry. Hence each $x \neq x_{1j}\alpha^{m_j+r_j-1}$ in Ω_j is moved by exactly one of the 3-paths appearing in the product β_j and moreover, it is moved to $x\alpha$. The exceptional element $x_{1j}\alpha^{m_j+r_j-1}$ is moved to $x_{1j}\alpha^{m_j-1}$ by the first 3-path in the product ξ_{1j} and then is moved, by either

$$\begin{pmatrix} x_{1j}\alpha^{m_j-1} & x_{1j}\alpha^{m_j-2} \\ x_{1j}\alpha^{m_j} & x_{1j}\alpha^{m_j-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x_{1j}\alpha^{m_j} & x_{1j}\alpha^{m_j-1} \\ x_{1j}\alpha^{m_j+1} & x_{1j}\alpha^{m_j} \end{pmatrix}$$

to $x_{1j}\alpha^{m_j}$ $(= x_{1j}\alpha^{m_j+r_j})$. Thus, $x\beta_j = x\alpha$ for every $x \in \Omega_j$, while $x\beta_j = x$ for every $x \in X_n \setminus \Omega_j$. Since the orbits Ω_j $(1 \leqslant j \leqslant s)$ are pairwise disjoint, we have a product $\beta_1\beta_2 \cdots \beta_s$ of 3-paths such that

$$x\beta_1\beta_2\cdots\beta_s = \begin{cases} x\alpha & \text{if} \quad x \in \cup_{j=1}^{s}\Omega_j, \\ \\ x & \text{if} \quad x \in X_n \setminus \cup_{j=1}^{s}\Omega_j. \end{cases}$$

Similarly, if

$$\gamma = \tau_1\tau_2\cdots\tau_l,$$

then each point $x \in \Phi$ appears either as an upper entry or a middle entry of a 3-path in the product γ. Moreover, each $x \in \Phi$ that appears as a lower entry or a middle entry never subsequently reappears as an upper or middle entry. Hence each $x \in \Phi$ is moved to $x\alpha$ by exactly one of the 3-paths appearing in the product γ. Thus, $x\gamma = x\alpha$ for each $x \in \Phi$ while $x\gamma = x$ for each $x \in X_n \setminus \Phi$.

Also, if

$$\delta = \eta_1\eta_2\cdots\eta_c,$$

then, again, we can observe that the product δ is such that $x\delta = x\alpha$ for each $x \in \cup_{v=1}^{c}\Theta_v$ and $x\delta = x$ for each $x \in X_n \setminus \cup_{v=1}^{c}\Theta_v$. Hence, it follows that

$$\alpha = \beta_1\beta_2\cdots\beta_s\gamma\delta,$$

a product of 3-paths in Sing_n.

Let us denote the number of 3-paths in the products ξ_{ij}, τ_u and η_v by $\#(\xi_{ij})$, $\#(\tau_u)$ and $\#(\eta_v)$, respectively (we shall also use similar notation in the sequel). Then, counting the number of points appearing at the top of each product ξ_{ij}, τ_i and η_j, we have $\#(\xi_{ij}) = \frac{1}{2}|Z_i(\Omega_j)|$, $\#(\tau_u) = \frac{1}{2}|Y_u(\Phi)|$ and $\#(\eta_v) = \frac{1}{2}(|\Theta_v| + 1)$. And so,

$$\#(\beta_j) = \frac{1}{2}\sum_{i=1}^{k_j}|Z_i(\Omega_j)| = \frac{1}{2}|\Omega_j|,$$

so that,

$$\#(\beta_1\beta_2\cdots\beta_s) = \frac{1}{2}\sum_{j=1}^{s}|\Omega_j|, \qquad \#(\gamma) = \frac{1}{2}\sum_{u=1}^{l}|Y_u(\Phi)|$$

and

$$\#(\delta) = \frac{1}{2}\sum_{v=1}^{c}(|\Theta_v| + 1) = \frac{1}{2}(\sum_{v=1}^{c}|\Theta_v| + c).$$

Using these, while noting that

$$\sum_{j=1}^{s} |\Omega_j| + \sum_{v=1}^{c} |\Theta_v| + \sum_{u=1}^{l} |Y_u(\Phi)| = n - (a+t),$$

we have

$$\#(\alpha) = \frac{1}{2}(n + c - (a+t)) = \frac{1}{2}(n + c(\alpha) - f(\alpha)) = \frac{g(\alpha)}{2}.$$

Case 2. $0 = l(\alpha) < e(\alpha)$. As in Case 1, each $Z_i(\Omega_j)$ $(i = 1, 2, \ldots, k_j)$ and each $Y_u(\Phi)$ $(u = 1, 2, \ldots, l)$ is of even size. Let $e(\alpha) = e$ and arrange the cyclic orbits such that

$$\Theta_1, \Theta_2, \ldots, \Theta_e$$

are of even sizes and

$$\Theta_{e+1}, \Theta_{e+2}, \ldots, \Theta_c$$

are of odd sizes. Then, corresponding to each $Z_i(\Omega_j)$ $(i = 1, 2, \ldots, k_j)$, $Y_u(\Phi)$ $(u = 1, 2, \ldots, l)$ and Θ_v $(v = e+1, e+2, \ldots, c)$, we define, respectively, products ξ_{ij}, τ_u and η_v of 3-paths as in Case 1. While if e is even, then corresponding to the even size cyclic orbits Θ_v $(v = 1, 2, \ldots, e)$, we define a product $\eta_v \eta_{v+1}$ $(v = 1, 2, \ldots, e-1)$ of 3-paths by

$$\begin{aligned} &\eta_v \eta_{v+1} \\ &= \begin{pmatrix} y_v \alpha^{p_v - 1} & y_v \alpha^{p_v - 2} \\ z & y_v \alpha^{p_v - 1} \end{pmatrix} \begin{pmatrix} y_v \alpha^{p_v - 3} & y_v \alpha^{p_v - 4} \\ y_v \alpha^{p_v - 2} & y_v \alpha^{p_v - 3} \end{pmatrix} \cdots \begin{pmatrix} z & y_{v+1} \alpha^{p_{v+1} - 1} \\ y_v & z \end{pmatrix} \\ &\begin{pmatrix} y_{v+1} \alpha^{p_{v+1} - 2} & y_{v+1} \alpha^{p_{v+1} - 3} \\ y_{v+1} \alpha^{p_{v+1} - 1} & y_{v+1} \alpha^{p_{v+1} - 2} \end{pmatrix} \cdots \begin{pmatrix} y_{v+1} \alpha^2 & y_{v+1} \alpha \\ y_{v+1} \alpha^3 & y_{v+1} \alpha^2 \end{pmatrix} \begin{pmatrix} y_{v+1} & z \\ x_{v+1} \alpha & y_{v+1} \end{pmatrix}, \end{aligned}$$

where z is any point in $X_n \setminus \mathrm{im}(\alpha)$.

If e is odd, then for each $v = 1, 2, \ldots, e-2$, we define the product $\eta_v \eta_{v+1}$ of 3-paths as above and for $v = e$ we define the product η_e of 3-paths by

$$\begin{aligned} \eta_e = &\begin{pmatrix} y_e \alpha^{p_e - 1} & y_e \alpha^{p_e - 2} \\ z & y_e \alpha^{p_e - 1} \end{pmatrix} \begin{pmatrix} y_e \alpha^{p_e - 3} & y_e \alpha^{p_e - 4} \\ y_e \alpha^{p_e - 2} & y_e \alpha^{p_e - 3} \end{pmatrix} \\ &\cdots \begin{pmatrix} y_e \alpha & y_e \\ y_e \alpha^2 & y_e \alpha \end{pmatrix} \begin{pmatrix} y_e & z \\ y_e \alpha & y_e \end{pmatrix}, \end{aligned}$$

where again z is any chosen point in $X_n \setminus \mathrm{im}(\alpha)$. It is then not difficult to observe that

$$\alpha = \beta_1 \beta_2 \cdots \beta_s \gamma \delta,$$

where

$$\beta_j = \xi_{1j}\xi_{2j}\cdots\xi_{k_j j}, \qquad \gamma = \tau_1\tau_2\cdots\tau_l, \quad \text{and} \quad \delta = \eta_1\eta_2\cdots\eta_c.$$

Also,

$$\#(\eta_v\eta_{v+1}) = \frac{1}{2}(|\Theta_v| + |\Theta_{v+1}| + 2)$$

and

$$\#(\eta_e) = \frac{1}{2}(|\Theta_e| + 2).$$

Thus, we have

$$\#(\delta) = \begin{cases} \frac{1}{2}(\sum_{v=1}^{c} |\Theta_v| + c) & \text{if} \quad e \text{ is even,} \\ \\ \frac{1}{2}(\sum_{v=1}^{c} |\Theta_v| + c + 1) & \text{if} \quad e \text{ is odd.} \end{cases}$$

And therefore

$$\#(\alpha) = \begin{cases} \frac{1}{2}(n + c - (a + t)) & \text{if} \quad e \text{ is even,} \\ \\ \frac{1}{2}(n + c - (a + t) + 1) & \text{if} \quad e \text{ is odd.} \end{cases}$$

That is

$$\#(\alpha) = \lceil \frac{1}{2}(n + c(\alpha) - f(\alpha)) \rceil = \lceil \frac{g(\alpha)}{2} \rceil.$$

Case 3. $0 = e(\alpha) < l(\alpha)$. Here, corresponding to each odd size subset $Z_i(\Omega_j)$ and $Y_u(\Phi)$, we define, respectively, products ξ_{ij} (while noting that $|Z_1(\Omega_j)| > 1$) and τ_u of 3-paths by

$$\xi_{ij} = \begin{cases} \begin{pmatrix} x_{1j}\alpha^{m_j+r_j-1} & x_{1j}\alpha^{m_j+r_j-2} \\ x_{1j}\alpha^{m_j-1} & x_{1j}\alpha^{m_j+r_j-1} \end{pmatrix} & \text{if } i = 1, \\ \quad \cdots \begin{pmatrix} x_{1j}\alpha^2 & x_{1j}\alpha \\ x_{1j}\alpha^3 & x_{1j}\alpha^2 \end{pmatrix} \begin{pmatrix} x_{1j}\alpha & x_{1j} \\ x_{1j}\alpha^2 & x_{1j}\alpha \end{pmatrix} & \\ & \\ \begin{pmatrix} x_{ij}\alpha^{p_{ij}-1} & x_{ij}\alpha^{p_{ij}-2} \\ x_{ij}\alpha^{p_{ij}} & x_{ij}\alpha^{p_{ij}-1} \end{pmatrix} & \text{if } i \neq 1 \\ \quad \cdots \begin{pmatrix} x_{ij}\alpha^2 & x_{ij}\alpha \\ x_{ij}\alpha^3 & x_{ij}\alpha^2 \end{pmatrix} \begin{pmatrix} x_{ij} & x_{1j} \\ x_{ij}\alpha & x_{ij} \end{pmatrix} & \end{cases}$$

and

$$\tau_u = \begin{pmatrix} x_u\alpha^{q_u-1} & x_u\alpha^{q_u-2} \\ x_u\alpha^{q_u} & x_u\alpha^{q_u-1} \end{pmatrix} \cdots \begin{pmatrix} x_u\alpha^2 & x_u\alpha \\ x_u\alpha^3 & x_u\alpha^2 \end{pmatrix} \begin{pmatrix} x_u\alpha & x_u \\ x_u\alpha^2 & x_u\alpha \end{pmatrix}$$

if $|Y_u(\Phi)| > 1$, otherwise, if $|Y_u(\Phi)| = 1$, define τ_u by

$$\tau_u = \begin{cases} \begin{pmatrix} x_u & x_1 \\ x_u\alpha & x_u \end{pmatrix} & \text{if } 1 < u \leqslant l, \\ \\ \begin{pmatrix} x_u & z \\ x_u\alpha & x_u \end{pmatrix} & \text{if } 1 = u \leqslant l, \end{cases}$$

where z is chosen to be any point of $X_n \setminus \mathrm{im}(\alpha)$ distinct from x_i which appeared in a standard orbit of α. Note that this choice of z is possible sine $\alpha \notin E$.

For the subsets $Z_i(\Omega_j)$ and $Y_u(\Phi)$ of even sizes and the cyclic orbits Θ_v, we define respectively, the products ξ_{ij}, τ_u and η_v as in Case 1. Then, here too, we can observe that, if

$$\beta_j = \xi_{1j}\xi_{2j}\cdots\xi_{k_j j}, \qquad \gamma = \tau_1\tau_2\cdots\tau_l, \quad \text{and} \quad \delta = \eta_1\eta_2\cdots\eta_c,$$

then

$$\alpha = \beta_1\beta_2\cdots\beta_s\gamma\delta$$

and that

$$\#(\beta_j) = \frac{1}{2}\sum_{i=1}^{k_j}(|Z_i(\Omega_j)| + z_{ij}) = \frac{1}{2}(|\Omega_j| + \sum_{i=1}^{k_j} z_{ij}),$$

$$\#(\gamma) = \frac{1}{2}\sum_{u=1}^{l}(|Y_u(\Phi)| + y_u)$$

and

$$\#(\delta) = \frac{1}{2}\sum_{v=1}^{c}(|\Theta_v| + c).$$

These give

$$\begin{aligned}
\#(\alpha) &= \frac{1}{2}\left(\sum_{j=1}^{s}|\Omega_j| + \sum_{u=1}^{l}|Y_u(\Phi)| + \sum_{v=1}^{c}|\Theta_v| + \sum_{j=1}^{s}\sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u + c\right)\\
&= \frac{1}{2}\left(n + c - (a+t) + \sum_{j=1}^{s}\sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u\right)\\
&= \frac{1}{2}\left(n + c(\alpha) - f(\alpha) + l(\alpha)\right)\\
&= \frac{1}{2}\left(g(\alpha) + m(\alpha)\right).
\end{aligned}$$

Case 4. $0 < l(\alpha) = e(\alpha)$. If $Z_1(\Omega_j)$ is of odd size, we define a product ξ_{1j} of 3-paths by

$$\begin{aligned}
\xi_{1j} = &\begin{pmatrix} x_{1j}\alpha^{m_j+r_j-1} & x_{1j}\alpha^{m_j+r_j-2} \\ x_{1j}\alpha^{m_j-1} & x_{1j}\alpha^{m_j+r_j-1} \end{pmatrix}\begin{pmatrix} x_{1j}\alpha^{m_j+r_j-3} & x_{1j}\alpha^{m_j+r_j-4} \\ x_{1j}\alpha^{m_j+r_j-2} & x_{1j}\alpha^{m_j+r_j-3} \end{pmatrix}\\
&\cdots\begin{pmatrix} x_{1j}\alpha^{2} & x_{1j}\alpha \\ x_{1j}\alpha^{3} & x_{1j}\alpha_2 \end{pmatrix}\begin{pmatrix} x_{1j} & y_v \\ x_{1j}\alpha & x_{1j} \end{pmatrix},
\end{aligned}$$

otherwise if $Z_1(\Omega_j)$ is of even size, we define a product ξ_{1j} of 3-paths as in Case 1. Corresponding to each $Z_i(\Omega_j)$ $(i \neq 1)$ and $Y_u(\Phi)$ of odd sizes, define products ξ_{ij} and τ_u of 3-paths, respectively, by

$$\xi_{ij} = \begin{pmatrix} x_{ij}\alpha^{p_{ij}-1} & x_{ij}\alpha^{p_{ij}-2} \\ x_{ij}\alpha^{p_{ij}} & x_{ij}\alpha^{p_{ij}-1} \end{pmatrix}\begin{pmatrix} x_{ij}\alpha^{p_{ij}-3} & x_{ij}\alpha^{p_{ij}-4} \\ x_{ij}\alpha^{p_{ij}-2} & x_{ij}\alpha^{p_{ij}-3} \end{pmatrix}\cdots\begin{pmatrix} x_{ij} & y_v \\ x_{ij}\alpha & x_{ij} \end{pmatrix}$$

and

$$\tau_u = \begin{pmatrix} x_u\alpha^{q_u-1} & x_u\alpha^{q_u-2} \\ x_u\alpha^{q_u} & x_u\alpha^{q_u-1} \end{pmatrix}\begin{pmatrix} x_u\alpha^{q_u-3} & x_u\alpha^{q_u-4} \\ x_u\alpha^{q_u-2} & x_u\alpha^{q_u-3} \end{pmatrix}\cdots\begin{pmatrix} x_u & y_v \\ x_u\alpha & x_u \end{pmatrix},$$

where the points y_v, appearing as upper entries of the last 3-paths in these products, ranges (distinctively) from the even cyclic orbits Θ_v $(v = 1, 2, \ldots, e(\alpha))$.

Now corresponding to each $Z_i(\Omega_j)$ $(i \neq 1)$ and $Y_u(\Phi)$ of even sizes as well as each cyclic orbit Θ_v of odd size, the products ξ_{ij}, τ_u and Θ_v of 3-paths are respectively defined as in Case 1. For the cyclic orbits Θ_v $(v = 1, 2, \ldots, e(\alpha))$ of even sizes, we define products η_v of 3-paths by

$$\eta_v = \begin{pmatrix} y_v\alpha^{p_j-1} & y_v\alpha^{p_v-2} \\ y_v & y_v\alpha^{p_v-1} \end{pmatrix}\begin{pmatrix} y_v\alpha^{p_v-3} & y_v\alpha^{p_v-4} \\ y_v\alpha^{p_{v-2}} & y_v\alpha^{p_v-3} \end{pmatrix}\cdots\begin{pmatrix} y_v\alpha & z \\ y_v\alpha^2 & y_j\alpha \end{pmatrix},$$

where z is the middle entry of the last 3-path, in the (already defined) product corresponding to the odd subset $Z_i(\Omega_j)$ or $Y_u(\Phi)$, to which y_v is an upper entry. As in the earlier cases, it can be observed that

$$\alpha = \beta_1\beta_2\cdots\beta_s\gamma\delta,$$

where

$$\beta_j = \xi_{1j}\xi_{2j}\cdots\xi_{k_j j}, \qquad \gamma = \tau_1\tau_2\cdots\tau_l \quad \text{and} \quad \delta = \eta_1\eta_2\cdots\eta_c.$$

Also, observing the points appearing at the top of the products ξ_{ij}, τ_i and η_i, we have

$$\#(\xi_{ij}) = \begin{cases} \frac{1}{2}|Z_i(\Omega_j)| & \text{if} \quad |Z_i(\Omega_j)| \text{ is even,} \\ \\ \frac{1}{2}(|Z_i(\Omega_j)|+1) & \text{if} \quad |Z_i(\Omega_j)| \text{ is odd,} \end{cases}$$

$$\#(\tau_u) = \begin{cases} \frac{1}{2}|Y_u(\Phi)| & \text{if} \quad |Y_u(\Phi)| \text{ is even,} \\ \\ \frac{1}{2}(|Y_u(\Phi)|+1) & \text{if} \quad |Y_u(\Phi)| \text{ is odd,} \end{cases}$$

and

$$\#(\eta_v) = \begin{cases} \frac{1}{2}|\Theta_v| & \text{if} \quad |\Theta_v| \text{ is even,} \\ \\ \frac{1}{2}(|\Theta_v|+1) & \text{if} \quad |\Theta_v| \text{ is odd.} \end{cases}$$

Thus, $\#(\beta_v) = \frac{1}{2}\sum_{i=1}^{k_j}(|Z_i(\Omega_j)| + z_{ij}) = \frac{1}{2}(|\Omega_j| + \sum_{i=1}^{k_j} z_{ij})$, $\#(\gamma) = \frac{1}{2}\sum_{u=1}^{l}(|Y_u(\Phi)| + y_u)$ and $\#(\delta) = \frac{1}{2}(\sum_{v=1}^{c}|\Theta_v| + c - e)$. Hence,

$$\begin{aligned} \#(\alpha) &= \frac{1}{2}\left(\sum_{j=1}^{s}|\Omega_j| + \sum_{u=1}^{l}|Y_u(\Phi)| + \sum_{v=1}^{c}|\Theta_v| + \sum_{j=1}^{s}\sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u + c - e\right) \\ &= \frac{1}{2}\left(n - (a+t) + \sum_{j=1}^{s}\sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u + c - e\right) \\ &= \frac{1}{2}\left(n - f(\alpha) + l(\alpha) + c(\alpha) - e(\alpha)\right) \\ &= \frac{1}{2}\left(n + c(\alpha) - f(\alpha)\right) \qquad (\text{since } l(\alpha) = e(\alpha)) \\ &= \frac{g(\alpha)}{2}. \end{aligned}$$

Case 5. $0 < l(\alpha) < e(\alpha)$. Here, corresponding to each $Z_i(\Omega_j)$ and each $Y_u(\Phi)$ of odd sizes and exactly $l(\alpha)$ cyclic orbits Θ_v of even sizes, we define, respectively, products ξ_{ij}, τ_u and η_v as described in Case 4. Corresponding to the even sizes subsets $Z_i(\Omega_j)$ and $Y_u(\Phi)$, as well as the odd sizes cyclic orbits Θ_v, we define , respectively, the products ξ_{ij}, τ_u and η_v as described in Case 1. For the remaining $e(\alpha) - l(\alpha)$ cyclic orbits Θ_v of even sizes, we define the products η_v as described in Case 2. It is then easily seen that, if

$$\beta_j = \xi_{1j}\xi_{2j}\cdots\xi_{k_j j}, \qquad \gamma = \tau_1\tau_2\cdots\tau_l, \quad \text{and} \quad \delta = \eta_1\eta_2\cdots\eta_c,$$

then

$$\alpha = \beta_1\beta_2\cdots\beta_s\gamma\delta$$

and that

$$\#(\beta_j) = \frac{1}{2}\sum_{i=1}^{k_j}(|Z_i(\Omega_j)| + z_{ij}) = \frac{1}{2}(|\Omega_j| + \sum_{i=1}^{k_j} z_{ij}),$$

$$\#(\gamma) = \frac{1}{2}\sum_{u=1}^{l}(|Y_u(\Phi)| + y_u)$$

and

$$\#(\delta) = \begin{cases} \frac{1}{2}\Big(\sum_{v=1}^{c}|\Theta_v| + c & \text{if } e(\alpha) - l(\alpha) \text{ is even,} \\ \qquad - \Big(\sum_{j=1}^{s}\sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u\Big)\Big) & \\ \frac{1}{2}\Big(\sum_{v=1}^{c}|\Theta_v| + c & \text{if } e(\alpha) - l(\alpha) \text{ is odd.} \\ \qquad - \Big(\sum_{j=1}^{s}\sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u\Big) + 1\Big) & \end{cases}$$

Thus, in this case we have

$$\#(\alpha) = \begin{cases} \frac{1}{2}(n + c - (a + t)) & \text{if} \quad e(\alpha) - l(\alpha) \text{ is even,} \\ \frac{1}{2}(n + c - (a + t) + 1) & \text{if} \quad e(\alpha) - l(\alpha) \text{ is odd.} \end{cases}$$

That is $\#(\alpha) = \lceil \frac{1}{2}(n + c(\alpha) - f(\alpha)) \rceil = \lceil \frac{g(\alpha)}{2} \rceil$.
Case 6. $0 < e(\alpha) < l(\alpha)$. Here, corresponding to each cyclic orbits Θ_v of even size and exactly $e(\alpha)$ subsets $Z_i(\Omega_j)$ and $Y_u(\Phi)$ of odd sizes, we define,

respectively, products η_v, ξ_{ij} and τ_u as described in Case 4. Corresponding to the even sizes subsets $Z_i(\Omega_j)$ and $Y_u(\Phi)$, as well as the odd sizes cyclic orbits Θ_v, we define, respectively, the products ξ_{ij}, τ_u and η_v as described in Case 1. For the remaining $l(\alpha) - e(\alpha)$ subsets $Z_i(\Omega_j)$ and $Y_u(\Phi)$ of odd sizes, we define, respectively, the products ξ_{ij} and τ_u as described in Case 3. Then, it is easily seen that, if

$$\beta_j = \xi_{1j}\xi_{2j}\cdots\xi_{k_j j}, \qquad \gamma = \tau_1\tau_2\cdots\tau_l \quad \text{and} \quad \delta = \eta_1\eta_2\cdots\eta_c,$$

then

$$\alpha = \beta_1\beta_2\cdots\beta_s\gamma\delta$$

and that

$$\#(\beta_j) = \frac{1}{2}\sum_{i=1}^{k_j}(|Z_i(\Omega_j)| + z_{ij}) = \frac{1}{2}(|\Omega_j| + \sum_{i=1}^{k_j} z_{ij}),$$

$$\#(\gamma) = \frac{1}{2}\sum_{u=1}^{l}(|Y_u(\Phi)| + y_i) \quad \text{and} \quad \#(\delta) = \frac{1}{2}(\sum_{v=1}^{c}|\Theta_v| + c - e).$$

Thus here,

$$\begin{aligned}\#(\alpha) &= \frac{1}{2}\left(n - (a+t) + \sum_{j=1}^{s}\sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u + c - e\right)\\ &= \frac{1}{2}\left(n + c(\alpha) - f(\alpha) + l(\alpha) - e(\alpha)\right) = \frac{1}{2}\left(g(\alpha) + m(\alpha)\right).\end{aligned}$$

Hence in all cases we have expressed $\alpha \in \mathrm{Sing}_n$ as a product of

$$\lceil \frac{1}{2}\left(g(\alpha) + m(\alpha)\right)\rceil$$

3-paths and so the proof of the theorem is now complete. □

Example 3. Let $\alpha \in \mathrm{Sing}_{22}$ be the map given in Example 1. Then $g(\alpha) = 22 + 3 - 3 = 22$ and $m(\alpha) = 2$, so that α can be expressed as a product of $k(\alpha) = \lceil \frac{1}{2}(22+2)\rceil = 12$ 3-paths in Sing_{22}. The processes of decomposition described in the proof of Theorem 2 give $\alpha = \xi_{11}\xi_{12}\xi_{21}\xi_{22}\tau_1\tau_2\eta_1\eta_2\eta_3$ where $\xi_{11} = \left(\begin{smallmatrix}4 & 3\\ 1 & 4\end{smallmatrix}\right)\left(\begin{smallmatrix}2 & 1\\ 3 & 2\end{smallmatrix}\right)$, $\xi_{12} = \left(\begin{smallmatrix}5 & 18\\ 4 & 5\end{smallmatrix}\right)$, $\xi_{21} = \left(\begin{smallmatrix}8 & 7\\ 6 & 8\end{smallmatrix}\right)\left(\begin{smallmatrix}6 & 1\\ 7 & 6\end{smallmatrix}\right)$, $\xi_{22} = \left(\begin{smallmatrix}9 & 20\\ 7 & 9\end{smallmatrix}\right)$, $\tau_1 = \left(\begin{smallmatrix}13 & 12\\ 14 & 13\end{smallmatrix}\right)$, $\tau_2 = \left(\begin{smallmatrix}10 & 12\\ 11 & 10\end{smallmatrix}\right)$, $\eta_1 = \left(\begin{smallmatrix}15 & 17\\ 1 & 15\end{smallmatrix}\right)\left(\begin{smallmatrix}16 & 1\\ 17 & 16\end{smallmatrix}\right)$, $\eta_2 = \left(\begin{smallmatrix}19 & 5\\ 18 & 19\end{smallmatrix}\right)$ and $\eta_3 = \left(\begin{smallmatrix}21 & 9\\ 20 & 21\end{smallmatrix}\right)$.

In the next lemma we obtain the maximum value of $g(\alpha) + m(\alpha)$ in Sing_n.

Lemma 1. *Let $n \geqslant 3$ and Sing_n be the semigroup of all singular self-maps of X_n. Then* $\max\{g(\alpha) + m(\alpha) : \alpha \in \mathrm{Sing}_n\} = 2(n-1)$.

Proof. From [6, Lemma 2.5], we have

$$\max\{g(\alpha) : \alpha \in \mathrm{Sing}_n\} = \lfloor \frac{3}{2}(n-1) \rfloor$$

and from (4), the maximum value of $m(\alpha)$ is attained by making $l(\alpha)$ as large as possible while keeping $e(\alpha)$ as small as possible. It is clear that any map $\alpha \in \mathrm{Sing}_n$ of height one has $l(\alpha) = n-1$ and $e(\alpha) = 0$, which are the maximum and least possible values of $l(\alpha)$ and $e(\alpha)$ respectively. Thus,

$$\max\{m(\alpha) : \alpha \in \mathrm{Sing}_n\} = n-1.$$

Now, for a map $\alpha \in \mathrm{Sing}_n$ of height one, $g(\alpha) = n-1$ and so,

$$\max\{g(\alpha) + m(\alpha) : \alpha \in \mathrm{Sing}_n\} \geqslant 2(n-1).$$

Next, we show the opposite inequality, that is,

$$\max\{g(\alpha) + m(\alpha) : \alpha \in \mathrm{Sing}_n\} \leqslant 2(n-1).$$

Suppose for some $\beta \in \mathrm{Sing}_n$, $g(\beta) + m(\alpha) > 2(n-1)$. Then, since $m(\beta) \leqslant n-1$, we must have $g(\beta) > n-1$. Also, $g(\beta) < \lceil \frac{3}{2}(n-1) \rceil$, for if $g(\beta) = \lceil \frac{3}{2}(n-1) \rceil$, then $m(\beta) = 0$ and $g(\beta) + m(\beta) = \lceil \frac{3}{2}(n-1) \rceil \leqslant 2(n-1)$ for all $n \geqslant 3$, which is a contradiction to the choice of $\beta \in \mathrm{Sing}_n$. It then follows that $g(\alpha) + m(\alpha) \leqslant 2(n-1)$ for all $\alpha \in \mathrm{Sing}_n$. □

Let P be the set of all 3-paths in Sing_n, and for each positive integer k write $P^{[k]}$ for the set of all product of elements in P of length k or less. That is $P^{[k]} = P \cup P^2 \cup \cdots \cup P^k$. Then, from Theorem 2 and Lemma 1 we deduce the following.

Corollary 1. *For each $n \geqslant 3$, we have* $\mathrm{Sing}_n \subseteq P^{[n-1]}$.

Remark 1. At the moment, we do not know whether the formula obtained in Theorem 2 is best possible, that is whether there is a number smaller then $\lceil \frac{1}{2}(g(\alpha) + m(\alpha)) \rceil$ expressing $\alpha \in \mathrm{Sing}_n$ as a product of 3-paths.

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