On products of 3-paths in finite full transformation semigroups

A. T. Imam and M. J. Ibrahim

Communicated by V. Mazorchuk

Abstract. Let Sing\(_n\) denotes the semigroup of all singular self-maps of a finite set \(X_n = \{1, 2, \ldots, n\}\). A map \(\alpha \in \text{Sing}_n\) is called a 3-path if there are \(i, j, k \in X_n\) such that \(i\alpha = j, j\alpha = k\) and \(x\alpha = x\) for all \(x \in X_n \setminus \{i, j\}\). In this paper, we described a procedure to factorise each \(\alpha \in \text{Sing}_n\) into a product of 3-paths. The length of each factorisation, that is the number of factors in each factorisation, is obtained to be equal to \(\lceil \frac{1}{2}(g(\alpha)+m(\alpha)) \rceil\), where \(g(\alpha)\) is known as the gravity of \(\alpha\) and \(m(\alpha)\) is a parameter introduced in this work and referred to as the measure of \(\alpha\). Moreover, we showed that \(\text{Sing}_n \subseteq P^{[n-1]}\), where \(P\) denotes the set of all 3-paths in \(\text{Sing}_n\) and \(P^{[k]} = P \cup P^2 \cup \cdots \cup P^k\).

1. Introduction

Let \(X_n = \{1, 2, \ldots, n\}\). The full transformation semigroup \(T_n\) on \(X_n\), that is the semigroup of all self-maps of \(X_n\) under composition of mappings, have been much studied. One of the outstanding contribution is given by Howie [5], where it was shown that the subsemigroup \(\text{Sing}_n\), of all singular maps in \(T_n\), is generated by its set \(E_1\) of all idempotents of defect one (that is element \(e \in T_n\) satisfying \(e^2 = e\) and \(|X_n \setminus \text{im}(e)| = 1\)). Later Howie [6] and Iwahori [7] independently computed the minimum number of factors in \(E_1\) required to expressed each \(\alpha \in \text{Sing}_n\) to be \(g(\alpha)\), the

The authors are grateful to an anonymous referee for his/her comments and suggestions.

2020 MSC: 20M20, 20M05.

Key words and phrases: 3-path, length formula, full transformation.
gravity of $\alpha \in \text{Sing}_n$ (see [6] for details). The maximum possible value of this number was also obtained in [6] to be equal to $\lfloor \frac{3}{2}(n-1) \rfloor$, where $[x]$ is the floor of $x$ (that is the unique integer $m$ for which $x - 1 < m \leq x$).

If $E$ denote the set of all idempotents in $\text{Sing}_n$, the minimum number of factors in $E$ required to expressed each $\alpha \in \text{Sing}_n$ was found, by Saito [8], to be equal to $\lceil g(\alpha) \cdot d(\alpha) \rceil$ or $\lceil g(\alpha) \cdot d(\alpha) \rceil + 1$, where $d(\alpha) = |X_n \setminus \text{im}(\alpha)|$ denotes the defect of $\alpha$, and $[x]$ is the ceiling of $x$ (that is the unique integer $m$ for which $x \leq m < x + 1$).

Related lengths problems where addressed, for product of idempotents in semigroups of order-preserving maps in both full and partial cases, by Schein [9], Higgins [4] and Yang [10]. Garba [2] solved similar problems in the semigroup $\mathcal{I}_n$, of all partial transformations of $X_n$. Recently, Garba and Imam [3] also studied similar lengths problems in the symmetric inverse semigroup $\mathcal{I}_n$, of all partial one-to-one maps of $X_n$. Ayik, et al. [1] showed that the semigroup $\text{Sing}_n$ can also be generated by certain primitive elements called path-cycles. Special class of path-cycles called $m$-paths can be regarded as generalisations of idempotents of defect one in the sense that all idempotents of defect one are 2-paths and vice-versa. In general, Ayik et al. [1] proved that the semigroup $\text{Sing}_n$ is generated by its set of $m$-paths for each $m \in \{2, 3, \ldots, n\}$. In this paper, we describe a procedure to factorise each singular map $\alpha$ in $\text{Sing}_n$ into a product of 3-paths. We then obtained the length of each factorisation, that is the number of 3-paths in the factorisation.

2. Preliminaries

Let $X_n = \{1, \ldots, n\}$ and let $\mathcal{T}_n$ be the full transformation semigroup on $X_n$. If $\{x_1, \ldots, x_m\} \subseteq X_n$ and $\alpha \in \mathcal{T}_n$ is defined by

$$x_i\alpha = x_{i+1}, \quad x_m\alpha = x_r \quad (1 \leq r \leq m) \quad \text{and} \quad x\alpha = x \quad (x \in X_n \setminus \{x_1, \ldots, x_m\}),$$

then $\alpha$ is called a path-cycle of length $m$ and period $r$, or simply, an $(m, r)$-path-cycle, and is denoted (in a linear notation) by $\alpha = [x_1, \ldots, x_m|x_r]$. If $r = m$, $\alpha$ is called an $m$-path to $x_m$ or simply an $m$-path; if $m \geq 2$ and $r = 1$, $\alpha$ is called an $m$-cycle; if $m = r = 1$, $\alpha$ is called a loop; if $m = r = 2$, $\alpha$ is an idempotent of defect one; if $m \geq 2$ and $r \neq 1$, $\alpha$ is said to be a proper path-cycle.

Let $\xi = [x_1, x_2, x_3|x_3]$ be an arbitrary 3-path in $\text{Sing}_n$, then $\xi$ maps $x_1$ to $x_2$, $x_2$ to $x_3$ and all other elements of $X_n$ identically. Instead of using the linear notation for $\xi$, we shall throughout this paper extend the array
notation, used for idempotents of defect one (that is 2-paths) used in [6], and write $\xi$ as

$$\xi = \begin{pmatrix} x_2 & x_1 \\ x_3 & x_2 \end{pmatrix}.$$ 

This will enable us a proper adoption of the methods of [6] in proving our results. In the array notation we shall refer to $x_1$ as the upper entry of $\xi$; to $x_2$ as the middle entry of $\xi$; and to $x_3$ as the lower entry of $\xi$.

Let $\alpha$ be in $\text{Sing}_n$. The equivalence relation $\omega$ on $X_n$ defined by

$$\omega = \{(x, y) \in X_n \times X_n : (\exists u, v \geq 0) x\alpha^u = y\alpha^v\},$$

partitioned $X_n$ into orbits $\Omega_1, \ldots, \Omega_k$. These orbits correspond to the connected components of the digraph associated to $\alpha$ with vertex set $X_n$ in which there is a directed edge $(x, y)$ if and only if $x\alpha = y$. Each orbit $\Omega$ has a kernel defined by

$$K(\Omega) = \{x \in \Omega : (\exists r > 0) x\alpha^r = x\}.$$ 

An orbit $\Omega$ is said to be:

- **standard** if and only if $2 \leq |K(\Omega)| < |\Omega|$;
- **acyclic** if and only if $1 = |K(\Omega)| < |\Omega|$;
- **cyclic** if and only if $2 \leq |K(\Omega)| = |\Omega|$;
- **trivial** if and only if $1 = |K(\Omega)| = |\Omega|$.

Every orbit of $\alpha$ falls into exactly one of these four categories and all four cases can arise. Let $c(\alpha)$ be the number of cyclic orbits of $\alpha$ and $f(\alpha)$ be the number of fixed points of $\alpha$, this equals the sum of the number of trivial and the number of acyclic orbits of $\alpha$. The **gravity** of $\alpha$ is defined as

$$g(\alpha) = n + c(\alpha) - f(\alpha).$$

For each standard or acyclic orbit $\Omega$ of $\alpha \in \text{Sing}_n$ and each $x \in \Omega \setminus \text{im}(\alpha)$, the sequence $x, x\alpha, x\alpha^2, \ldots$ eventually arrives in $K(\Omega)$, the kernel of $\Omega$, and remains there for all subsequent iterations. Denote the set of all distinct elements in this sequence by $Z(x)$. Suppose that $\alpha \in \text{Sing}_n$ has $s$ standard orbits $\Omega_1, \Omega_2, \ldots, \Omega_s$. 
We then define \( Z(x_{ij}) \) as:

\[
|Z(x_{ij})| = \begin{cases} 
\max_{1 \leq i \leq k_j} \{|Z(x_{ij})| : \text{if } |Z(x_{ij})| \text{ is even for some } i, \text{ or } \text{if } |Z(x_{ij})| \text{ is even}\} \\
\max_{1 \leq i \leq k_j} \{|Z(x_{ij})| : \text{if } |Z(x_{ij})| \text{ is odd for all } i. \}
\end{cases}
\]

Then there exist \( m_j \geq 1 \) and \( r_j \geq 2 \) (see [6]) such that

\[
K(\Omega_j) = \{x_{1j} \alpha^{m_j}, \ldots, x_{1j} \alpha^{m_j+r_j-1}\},
\]

where \( x_{1j} \alpha^{m_j+r_j} = x_{1j} \alpha^{m_j} \). Note that this definition of \( K(\Omega_j) \) is still valid for every \( x_{ij} \), not only for \( x_{1j} \), and moreover, they are all the same. We then define

\[
Z_1(\Omega_j) = Z(x_{1j}) = \{x_{1j}, x_{1j} \alpha, \ldots, x_{1j} \alpha^{m_j}, \ldots, x_{1j} \alpha^{m_j+r_j-1}\} \quad (1)
\]

and

\[
Z_i(\Omega_j) = \{x_{ij}, x_{ij} \alpha, \ldots, x_{ij} \alpha^{p_{ij}-1}\} \quad (2 \leq i \leq k_j)
\]

where \( x_{ij} \alpha^{p_{ij}} \in (Z_1(\Omega_j) \cup Z_2(\Omega_j) \cup \cdots \cup Z_{i-1}(\Omega_j)) \). Thus, \( \{Z_i(\Omega_j) : 1 \leq i \leq k_j\} \) is a partition of \( \Omega_j \). Also, suppose that \( \alpha \in \text{Sing}_n \) has acyclic orbits; let \( \Phi \) be the union of all its acyclic orbits and denote the set \( \{x \in \Phi : x \alpha = x\} \) by \( \text{Fix}(\Phi) \). Let \( \Phi \setminus \text{im}(\alpha) = \{x_1, x_2, \ldots, x_l\} \) where \( x_1 \) is such that

\[
|Z(x_1)| = \begin{cases} 
\max_{1 \leq u \leq l} \{|Z(x_u)| : \text{if } |Z(x_u)| \text{ is odd for some } u, \text{ or } \text{if } |Z(x_u)| \text{ is odd}\} \\
\max_{1 \leq u \leq l} \{|Z(x_u)| : \text{if } |Z(x_u)| \text{ is even for all } u. \}
\end{cases}
\]

Then, for \( u = 1, 2, \ldots, l \), define

\[
Y_u(\Phi) = \{x_u, x_u \alpha, \ldots, x_u \alpha^{q_u-1}\}, \quad (3)
\]

where \( x_1 \alpha^{q_u} \in \text{Fix}(\Phi) \) and \( x_u \alpha^{q_u} \in (Y_1(\Phi) \cup \cdots \cup Y_{u-1}(\Phi) \cup \text{Fix}(\Phi)) \) \((u = 2, 3, \ldots, l)\). Thus, \( \{Y_u(\Phi) : 1 \leq u \leq l\} \) is a partition of \( \Phi \). We will be interested in the cardinalities of \( Z_i(\Omega_j) \) and \( Y_u(\Phi) \) being odd or even. For this, we define indicator functions \( z_{ij} \) and \( y_u \) by

\[
\begin{align*}
z_{ij} &= \begin{cases} 
0 & \text{if } |Z_i(\Omega_j)| \text{ is even, } \\
1 & \text{if } |Z_i(\Omega_j)| \text{ is odd, }
\end{cases} \\
y_u &= \begin{cases} 
0 & \text{if } |Y_u(\Phi)| \text{ is even, } \\
1 & \text{if } |Y_u(\Phi)| \text{ is odd. }
\end{cases}
\end{align*}
\]
Finally, for each $\alpha \in \text{Sing}_n$ we define the measure of $\alpha$ by

$$m(\alpha) = \begin{cases} 
 l(\alpha) - e(\alpha) & \text{if } l(\alpha) > e(\alpha), \\
 0 & \text{if } l(\alpha) \leq e(\alpha), 
\end{cases}$$

(4)

where $l(\alpha) = \sum_{j=1}^s \sum_{i=1}^{k_j} z_{ij} + \sum_{v=1}^t y_v$ and $e(\alpha)$ denote the number of cyclic orbits of $\alpha$ of even cardinality.

Before closing this section, we illustrate the above definitions and notations in an example.

**Example 1.** Consider the map

$$\alpha = \left(\begin{array}{cccccccccccccccccccc}
\end{array}\right)$$

in $\text{Sing}_{22}$ with orbits

- **standard:** $\Omega_1 = \{1, 2, 3, 4, 5\}, \Omega_2 = \{6, 7, 8, 9\}$;
- **acyclic:** $\Phi_1 = \{10, 11\}, \Phi_2 = \{12, 13, 14\}$;
- **cyclic:** $\Theta_1 = \{15, 16, 17\}, \Theta_2 = \{18, 19\}, \Theta_3 = \{20, 21\}$;
- **trivial:** $\Psi_1 = \{22\}$,

as shown in Figure 1.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (6) at (1,0) {6};
\node (9) at (2,0) {9};
\node (12) at (3,0) {12};
\node (2) at (0,-1) {2};
\node (5) at (1,-1) {5};
\node (7) at (2,-1) {7};
\node (10) at (3,-1) {10};
\node (13) at (4,-1) {13};
\node (15) at (5,-1) {15};
\node (18) at (6,-1) {18};
\node (20) at (7,-1) {20};
\node (3) at (0,-2) {3};
\node (4) at (1,-2) {4};
\node (8) at (2,-2) {8};
\node (11) at (3,-2) {11};
\node (14) at (4,-2) {14};
\node (16) at (5,-2) {16};
\node (17) at (6,-2) {17};
\node (19) at (7,-2) {19};
\node (21) at (8,-2) {21};
\node (22) at (9,-2) {22};
\draw (1) -- (6) -- (9) -- (12);
\draw (2) -- (5) -- (7) -- (10) -- (13) -- (15) -- (18) -- (20);
\draw (3) -- (4) -- (8) -- (11) -- (14) -- (16) -- (17) -- (19) -- (21) -- (22);
\end{tikzpicture}
\caption{Orbits of $\alpha \in \text{Sing}_{22}$.}
\end{figure}

For this $\alpha$, we have $\Phi = \{10, 11, 12, 13, 14\}$ and so, $Z_1(\Omega_1) = \{1, 2, 3, 4\}$, $Z_2(\Omega_1) = \{5\}$ (note that, according to the concerning definitions, it is also possible that $Z_1(\Omega_1) = \{2, 3, 4, 5\}$ and $Z_2(\Omega_1) = \{1\}$), $Z_1(\Omega_2) = \{6, 7, 8\}$, $Z_2(\Omega_2) = \{9\}$, $Y_1(\Phi) = \{12, 13\}$, $Y_2(\Phi) = \{10\}$. Thus, $z_{11} = 0$, $z_{21} = 1$, $z_{12} = 1$, $z_{22} = 1$, $y_1 = 0$, $y_2 = 1$ and so, $l(\alpha) = z_{11} + z_{21} + z_{12} + z_{22} + y_1 + y_2 = 4$, also $e(\alpha) = 2$. Therefore the measure of $\alpha$ is $m(\alpha) = 2$. 
### 3. Products of 3-paths

In [1], it was proved that for a fixed $m$ in $\{2, \ldots, n\}$, every element of $\text{Sing}_n$ is a product of $m$-paths. The result was obtained via decomposing each 2-path in $\text{Sing}_n$ as a product of 2 $m$-paths while each element of $\text{Sing}_n$ is decomposable as a product of 2-paths. In this section, we consider the case when $m = 3$ and obtain a direct decomposition of each $\alpha \in \text{Sing}_n$ as a product of 3-paths.

Let $E$ be the set of all idempotents in $\text{Sing}_n$ and $E_1$ be the set of all idempotents of defect 1 in $E$. First, we note that, in the notation of [6], each idempotent in $E_1$ is of the form $(i \ j)$, with $i, j \in X_n$ and $i \neq j$. Thus, since $n \geq 3$, there is a $k \in X_n$, with $k \neq i$ and $k \neq j$, such that, $(i \ j) = (j \ k)(i \ j)$. (5)

**Theorem 1.** For $n \geq 3$, each $\alpha \in E \setminus E_1$ is expressible as a product of $g(\alpha)$ 3-path in $\text{Sing}_n$.

**Proof.** Let $\alpha \in E \setminus E_1$ and let $A_1, A_2, \ldots, A_r$ be its non-singleton blocks. Then, each of the blocks $A_i$ ($1 \leq i \leq r$) is stationary. If $|A_i| \geq 3$ for some $i$, we can assume without loss of generality that $|A_1| \geq 3$. Let

\[ A_i \setminus \{A_i \alpha\} = \{x_{i1}, x_{i2}, \ldots, x_{ia_i}\} \quad (1 \leq i \leq r) \]

and define products $\xi_i$ ($1 \leq i \leq r$) of 3-paths by

\[ \xi_1 = \begin{pmatrix} x_{12} & x_{11} \\ A_1\alpha & x_{12} \end{pmatrix} \begin{pmatrix} x_{13} & x_{11} \\ A_1\alpha & x_{13} \end{pmatrix} \ldots \begin{pmatrix} x_{ia_1} & x_{11} \\ A_1\alpha & x_{ia_1} \end{pmatrix} \begin{pmatrix} x_{12} & x_{11} \\ A_1\alpha & x_{12} \end{pmatrix} \]

and

\[ \xi_i = \begin{pmatrix} x_{i1} & x_{11} \\ A_i\alpha & x_{i1} \end{pmatrix} \begin{pmatrix} x_{i2} & x_{11} \\ A_i\alpha & x_{i2} \end{pmatrix} \ldots \begin{pmatrix} x_{ia_i} & x_{11} \\ A_i\alpha & x_{ia_i} \end{pmatrix} \quad (2 \leq i \leq r). \]

Then, it is easy to verify that

\[ \alpha = \xi_1 \xi_2 \cdots \xi_r. \]

Also, observe that each point in $A_i \setminus \{A_i \alpha\}$ ($2 \leq i \leq r$) appeared exactly once as a middle entry of a 3-path in $\xi_i$ and each point in $A_1 \setminus \{A_1\alpha, x_{11}, x_{12}\}$ appeared exactly once as a middle entry of a 3-path in $\xi_1$. The point $x_{12}$ appeared exactly twice as a middle entry of
3-paths in $\xi_1$ while the point $x_{11}$ did not appear anywhere as a middle entry. Thus, the number of 3-paths used in the product $\xi_1 \xi_2 \cdots \xi_r$ is

$$\sum_{i=1}^{r} |A_i \setminus \{A_i \alpha\}| = n - f(\alpha) = g(\alpha).$$

If $|A_i| = 2$ for all $i$, let $A_i = \{x_i, x_i \alpha\}$ ($1 \leq i \leq r$). Then,

$$\alpha = \left( \begin{array}{cc} x_1 & x_r \\ x_1 \alpha & x_1 \end{array} \right) \left( \begin{array}{cc} x_2 & x_r \\ x_2 \alpha & x_2 \end{array} \right) \cdots \left( \begin{array}{cc} x_{r-1} & x_r \\ x_{r-1} \alpha & x_{r-1} \end{array} \right) \left( \begin{array}{cc} x_1 & x_r \\ x_r \alpha & x_1 \end{array} \right),$$

and again, the number of 3-paths used is $n - f(\alpha) = g(\alpha)$.

**Example 2.** Consider the idempotent $e = \left( \begin{array}{ccc} \{1, 2, 7, 5\} & \{3, 8, 10, 12\} & \{4, 6, 9, 11\} \\ 2 & 8 & 11 \end{array} \right) = \xi_1 \xi_2 \xi_3$

where

$$\xi_1 = \left( \begin{array}{cc} 5 & 1 \\ 2 & 5 \end{array} \right) \left( \begin{array}{cc} 7 & 1 \\ 2 & 7 \end{array} \right),$$

$$\xi_2 = \left( \begin{array}{cc} 3 & 1 \\ 8 & 3 \end{array} \right) \left( \begin{array}{cc} 10 & 1 \\ 8 & 10 \end{array} \right) \left( \begin{array}{cc} 12 & 1 \\ 8 & 12 \end{array} \right),$$

$$\xi_3 = \left( \begin{array}{cc} 4 & 1 \\ 11 & 4 \end{array} \right) \left( \begin{array}{cc} 6 & 1 \\ 11 & 6 \end{array} \right) \left( \begin{array}{cc} 9 & 1 \\ 11 & 9 \end{array} \right).$$

**Theorem 2.** For $n \geq 3$, each $\alpha \in \text{Sing}_n \setminus E$ is expressible as a product of $\left\lfloor \frac{1}{2}(g(\alpha) + m(\alpha)) \right\rfloor$ 3-paths in $\text{Sing}_n$.

**Proof.** Suppose that $\alpha \in \text{Sing}_n \setminus E$ has orbits as follows:

- **standard:** $\Omega_1, \Omega_2, \ldots, \Omega_s$;
- **acyclic:** $\Phi_1, \Phi_2, \ldots, \Phi_a$;
- **cyclic:** $\Theta_1, \Theta_2, \ldots, \Theta_c$;
- **trivial:** $\Psi_1, \Psi_2, \ldots, \Psi_t$.

For each standard orbit $\Omega_j$ let $\Omega_j \setminus \text{im}(\alpha) = \{x_{1j}, x_{2j}, \ldots, x_{kj}\}$;

$$K(\Omega_j) = \{x_{1j} \alpha^{m_j}, x_{1j} \alpha^{m_j+1}, \ldots, x_{1j} \alpha^{m_j+r_j-1}\};$$

and define $Z_1(\Omega_j)$ and $Z_i(\Omega_j)$ ($i = 2, \ldots, k_j$) as in Equations (1) and (2), respectively. Also, let

$$\Phi = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_a;$$
and define $Y_u(\Phi)$ ($u = 1, 2, \ldots, l$) as in Equation (3). Moreover, let

$$\Theta_v = \{y_v, y_v\alpha, \ldots, y_v\alpha^{p_v - 1}\}$$

(where $y_v\alpha^{p_v} = y_v$). Then we consider six possible cases that may arise.  

**Case 1.** $0 = e(\alpha) = l(\alpha)$. In this case each $Z_i(\Omega_j)$ ($i = 1, 2, \ldots, k_j$) and each $Y_u(\Phi)$ ($u = 1, 2, \ldots, l$) is of even size; also, each $\Theta_v$ is of odd size. Thus, corresponding to each $Z_i(\Omega_j)$, $Z_i(\Omega_j)$ ($i = 2, 3, \ldots, k_j$), $Y_u(\Phi)$ ($u = 1, 2, \ldots, l$) and $\Theta_v$ ($v = 1, 2, \ldots, c$) we define, respectively, products $\xi_{ij}$, $\xi_{ij}$ ($i = 2, 3, \ldots, k_j$), $\tau_u$ ($u = 1, 2, \ldots, l$) and $\eta_v$ ($v = 1, 2, \ldots, c$) of 3-paths by

$$\xi_{ij} = \begin{pmatrix}
(x_{ij}\alpha^{m_j+r_j-1} & x_{ij}\alpha^{m_j+r_j-2} & x_{ij}\alpha^{m_j+r_j-3} & x_{ij}\alpha^{m_j+r_j-4} \\
x_{ij}\alpha^{m_j-1} & x_{ij}\alpha^{m_j-2} & x_{ij}\alpha^{m_j-3} & x_{ij}\alpha^{m_j-4}
\end{pmatrix},$$

$$\xi_{ij} = \begin{pmatrix}
(x_{ij}\alpha^{p_{ij}-1} & x_{ij}\alpha^{p_{ij}-2} & x_{ij}\alpha^{p_{ij}-3} & x_{ij}\alpha^{p_{ij}-4} \\
x_{ij}\alpha^{p_{ij}} & x_{ij}\alpha^{p_{ij}} & x_{ij}\alpha^{p_{ij}} & x_{ij}\alpha^{p_{ij}}
\end{pmatrix},$$

$$\tau_u = \begin{pmatrix}
(x_u\alpha^{q_u-1} & x_u\alpha^{q_u-2} & x_u\alpha^{q_u-3} & x_u\alpha^{q_u-4} \\
x_u\alpha^{q_u} & x_u\alpha^{q_u} & x_u\alpha^{q_u} & x_u\alpha^{q_u}
\end{pmatrix},$$

and

$$\eta_v = \begin{pmatrix}
y_v\alpha^{p_v-1} & y_v\alpha^{p_v-2} & y_v\alpha^{p_v-3} & x_v\alpha \\
z & y_v\alpha^{p_v-1} & y_v\alpha^{p_v-2} & y_v\alpha^{p_v-3}
\end{pmatrix},$$

where $z$ is any point in $X_u \setminus \text{im}(\alpha)$.

For each $j = 1, 2, \ldots, s$, let

$$\beta_j = \xi_{1j}\xi_{2j} \cdots \xi_{kj},$$

then each element $x \in \Omega_j$ appears exactly once either as an upper entry or as a middle entry of a 3-path in the product $\beta_j$. Moreover, with the sole exception of $x = x_{1j}\alpha^{m_j-1}$, an element $x \in \Omega_j$ appearing as a lower entry or a middle entry never subsequently reappears as an upper or middle entry. Hence each $x \neq x_{1j}\alpha^{m_j+r_j-1}$ in $\Omega_j$ is moved by exactly one of the 3-paths appearing in the product $\beta_j$ and moreover, it is moved to $x\alpha$. The exceptional element $x_{1j}\alpha^{m_j+r_j-1}$ is moved to $x_{1j}\alpha^{m_j-1}$ by the first 3-path in the product $\xi_{1j}$ and then is moved, by either

$$\begin{pmatrix}
(x_{1j}\alpha^{m_j-1} & x_{1j}\alpha^{m_j-2} \\
x_{1j}\alpha^{m_j} & x_{1j}\alpha^{m_j-1}
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
x_{1j}\alpha^{m_j} & x_{1j}\alpha^{m_j-1} \\
x_{1j}\alpha^{m_j+1} & x_{1j}\alpha^{m_j}
\end{pmatrix}$$
to \( x_1 j \alpha^{m_j} = x_1 j \alpha^{m_j + \tau_j} \). Thus, \( x \beta_j = x \alpha \) for every \( x \in \Omega_j \), while \( x \beta_j = x \) for every \( x \in X_n \setminus \Omega_j \). Since the orbits \( \Omega_j \) (\( 1 \leq j \leq s \)) are pairwise disjoint, we have a product \( \beta_1 \beta_2 \cdots \beta_s \) of 3-paths such that

\[
x \beta_1 \beta_2 \cdots \beta_s = \begin{cases} x \alpha & \text{if } x \in \bigcup_{j=1}^s \Omega_j, \\
x & \text{if } x \in X_n \setminus \bigcup_{j=1}^s \Omega_j. 
\end{cases}
\]

Similarly, if \( \gamma = \tau_1 \tau_2 \cdots \tau_l \), then each point \( x \in \Phi \) appears either as an upper entry or a middle entry of a 3-path in the product \( \gamma \). Moreover, each \( x \in \Phi \) that appears as a lower entry or a middle entry never subsequently reappears as an upper or middle entry. Hence each \( x \in \Phi \) is moved to \( x \alpha \) by exactly one of the 3-paths appearing in the product \( \gamma \). Thus, \( x \gamma = x \alpha \) for each \( x \in \Phi \) while \( x \gamma = x \) for each \( x \in X_n \setminus \Phi \).

Also, if \( \delta = \eta_1 \eta_2 \cdots \eta_c \), then, again, we can observe that the product \( \delta \) is such that \( x \delta = x \alpha \) for each \( x \in \bigcup_{v=1}^c \Theta_v \) and \( x \delta = x \) for each \( x \in X_n \setminus \bigcup_{v=1}^c \Theta_v \). Hence, it follows that

\[ \alpha = \beta_1 \beta_2 \cdots \beta_s \gamma \delta, \]

a product of 3-paths in \( \text{Sing}_n \).

Let us denote the number of 3-paths in the products \( \xi_{ij}, \tau_u \) and \( \eta_v \) by \( \#(\xi_{ij}), \#(\tau_u) \) and \( \#(\eta_v) \), respectively (we shall also use similar notation in the sequel). Then, counting the number of points appearing at the top of each product \( \xi_{ij}, \tau_i \) and \( \eta_j \), we have \( \#(\xi_{ij}) = \frac{1}{2} |Z_i(\Omega_j)|, \#(\tau_u) = \frac{1}{2} |Y_u(\Phi)| \) and \( \#(\eta_v) = \frac{1}{2} (|\Theta_v| + 1) \). And so,

\[
\#(\beta_j) = \frac{1}{2} \sum_{i=1}^{k_j} |Z_i(\Omega_j)| = \frac{1}{2} |\Omega_j|,
\]

so that,

\[
\#(\beta_1 \beta_2 \cdots \beta_s) = \frac{1}{2} \sum_{j=1}^s |\Omega_j|, \quad \#(\gamma) = \frac{1}{2} \sum_{u=1}^l |Y_u(\Phi)|
\]

and

\[
\#(\delta) = \frac{1}{2} \sum_{v=1}^c (|\Theta_v| + 1) = \frac{1}{2} \left( \sum_{v=1}^c |\Theta_v| + c \right).
\]
Using these, while noting that

$$
\sum_{j=1}^{s} |\Omega_j| + \sum_{v=1}^{c} |\Theta_v| + \sum_{u=1}^{l} |Y_u(\Phi)| = n - (a + t),
$$

we have

$$
\#(\alpha) = \frac{1}{2}(n + c - (a + t)) = \frac{1}{2}(n + c(\alpha) - f(\alpha)) = \frac{g(\alpha)}{2}.
$$

**Case 2.** \(0 = l(\alpha) < e(\alpha)\). As in Case 1, each \(Z_i(\Omega_j)\) \((i = 1, 2, \ldots, k_j)\) and each \(Y_u(\Phi)\) \((u = 1, 2, \ldots, l)\) is of even size. Let \(e(\alpha) = e\) and arrange the cyclic orbits such that

\[
\Theta_1, \Theta_2, \ldots, \Theta_e
\]

are of even sizes and

\[
\Theta_{e+1}, \Theta_{e+2}, \ldots, \Theta_c
\]

are of odd sizes. Then, corresponding to each \(Z_i(\Omega_j)\) \((i = 1, 2, \ldots, k_j)\), \(Y_u(\Phi)\) \((u = 1, 2, \ldots, l)\) and \(\Theta_v\) \((v = e + 1, e + 2, \ldots, c)\), we define, respectively, products \(\xi_{ij}, \tau_u\) and \(\eta_v\) of 3-paths as in Case 1. While if \(e\) is even, then corresponding to the even size cyclic orbits \(\Theta_v\) \((v = 1, 2, \ldots, e)\), we define a product \(\eta_v\eta_{v+1} (v = 1, 2, \ldots, e - 1)\) of 3-paths by

\[
\eta_v\eta_{v+1} = \begin{pmatrix}
  y_v \alpha^{p_v-1} & y_v \alpha^{p_v-2} \\
  z & y_v \alpha^{p_v-1}
\end{pmatrix}
\begin{pmatrix}
  y_v \alpha^{p_v-3} & y_v \alpha^{p_v-4} \\
  y_v \alpha^{p_v-2} & y_v \alpha^{p_v-3}
\end{pmatrix}
\cdots
\begin{pmatrix}
  z & y_{v+1} \alpha^{p_{v+1}-1} \\
  y_v & z
\end{pmatrix}
\begin{pmatrix}
  y_{v+1} \alpha^{2} & y_{v+1} \alpha \\
  y_{v+1} \alpha & y_{v+1} \alpha^2
\end{pmatrix}
\begin{pmatrix}
  y_{v+1} & z \\
  x_{v+1} \alpha & y_{v+1}
\end{pmatrix},
\]

where \(z\) is any point in \(X_n \setminus \text{im}(\alpha)\).

If \(e\) is odd, then for each \(v = 1, 2, \ldots, e - 2\), we define the product \(\eta_v\eta_{v+1}\) of 3-paths as above and for \(v = e\) we define the product \(\eta_e\) of 3-paths by

\[
\eta_e = \begin{pmatrix}
  y_e \alpha^{p_e-1} & y_e \alpha^{p_e-2} \\
  z & y_e \alpha^{p_e-1}
\end{pmatrix}
\begin{pmatrix}
  y_e \alpha^{p_e-3} & y_e \alpha^{p_e-4} \\
  y_e \alpha^{p_e-2} & y_e \alpha^{p_e-3}
\end{pmatrix}
\cdots
\begin{pmatrix}
  y_e & z \\
  y_e \alpha & y_e \alpha^2
\end{pmatrix},
\]

where again \(z\) is any chosen point in \(X_n \setminus \text{im}(\alpha)\). It is then not difficult to observe that

\[
\alpha = \beta_1 \beta_2 \cdots \beta_s \gamma \delta,
\]
where
\[
\beta_j = \xi_{1j} \xi_{2j} \cdots \xi_{kj}, \quad \gamma = \tau_1 \tau_2 \cdots \tau_l, \quad \text{and} \quad \delta = \eta_1 \eta_2 \cdots \eta_c.
\]

Also,
\[
\#(\eta_v \eta_{v+1}) = \frac{1}{2}(|\Theta_v| + |\Theta_{v+1}| + 2)
\]
and
\[
\#(\eta_e) = \frac{1}{2}(|\Theta_e| + 2).
\]

Thus, we have
\[
\#(\delta) = \begin{cases} 
\frac{1}{2} (\sum_{v=1}^c |\Theta_v| + c) & \text{if } e \text{ is even,} \\
\frac{1}{2} (\sum_{v=1}^c |\Theta_v| + c + 1) & \text{if } e \text{ is odd.}
\end{cases}
\]

And therefore
\[
\#(\alpha) = \begin{cases} 
\frac{1}{2} (n + c - (a + t)) & \text{if } e \text{ is even,} \\
\frac{1}{2} (n + c - (a + t) + 1) & \text{if } e \text{ is odd.}
\end{cases}
\]

That is
\[
\#(\alpha) = \left\lceil \frac{1}{2} (n + c(\alpha) - f(\alpha)) \right\rceil = \left\lceil \frac{g(\alpha)}{2} \right\rceil.
\]

**Case 3.** \(0 = e(\alpha) < l(\alpha)\). Here, corresponding to each odd size subset \(Z_i(\Omega_j)\) and \(Y_u(\Phi)\), we define, respectively, products \(\xi_{ij}\) (while noting that \(|Z_1(\Omega_j)| > 1\)) and \(\tau_u\) of 3-paths by

\[
\xi_{ij} = \begin{cases} 
\begin{pmatrix} x_{1j} \alpha^{m_j + r_j - 1} & x_{1j} \alpha^{m_j + r_j - 2} \\
x_{1j} \alpha^{m_j - 1} & x_{1j} \alpha^{m_j + r_j - 1} \\
\end{pmatrix} & \text{if } i = 1, \\
\begin{pmatrix} x_{ij} \alpha^{p_{ij} - 1} & x_{ij} \alpha^{p_{ij} - 2} \\
x_{ij} \alpha^{p_{ij}} & x_{ij} \alpha^{p_{ij} - 1} \\
\end{pmatrix} & \text{if } i \neq 1
\end{cases}
\]

\[
\begin{pmatrix} x_{ij} \alpha^2 & x_{ij} \\
x_{ij} \alpha^3 & x_{ij} \alpha^2 \\
\end{pmatrix} \begin{pmatrix} x_{1j} \alpha & x_{1j} \\
x_{1j} \alpha^2 & x_{1j} \alpha \\
\end{pmatrix}
\]

\[
\begin{pmatrix} x_{ij} \alpha^2 & x_{ij} \\
x_{ij} \alpha^3 & x_{ij} \alpha^2 \\
\end{pmatrix} \begin{pmatrix} x_{ij} & x_{1j} \\
x_{ij} \alpha & x_{ij} \\
\end{pmatrix}
\]

\[
\begin{pmatrix} x_{ij} \alpha^2 & x_{ij} \\
x_{ij} \alpha^3 & x_{ij} \alpha^2 \\
\end{pmatrix} \begin{pmatrix} x_{ij} & x_{1j} \\
x_{ij} \alpha & x_{ij} \\
\end{pmatrix}
\]
and

\[ \tau_u = \begin{pmatrix} x_u \alpha^{q_u-1} & x_u \alpha^{q_u-2} \\ x_u \alpha^{q_u} & x_u \alpha^{q_u-1} \end{pmatrix} \cdot \cdots \cdot \begin{pmatrix} x_u \alpha^2 & x_u \alpha \\ x_u \alpha^3 & x_u \alpha^2 \end{pmatrix} \begin{pmatrix} x_u \alpha & x_u \end{pmatrix} \]

if \(|Y_u(\Phi)| > 1\), otherwise, if \(|Y_u(\Phi)| = 1\), define \(\tau_u\) by

\[
\tau_u = \begin{cases} 
\begin{pmatrix} x_u & x_1 \\ x_u \alpha & x_u \end{pmatrix} & \text{if } 1 < u \leq l, \\
\begin{pmatrix} x_u & z \\ x_u \alpha & x_u \end{pmatrix} & \text{if } 1 = u \leq l,
\end{cases}
\]

where \(z\) is chosen to be any point of \(X_n \setminus \text{im}(\alpha)\) distinct from \(x_i\) which appeared in a standard orbit of \(\alpha\). Note that this choice of \(z\) is possible since \(\alpha \not\in E\).

For the subsets \(Z_i(\Omega_j)\) and \(Y_u(\Phi)\) of even sizes and the cyclic orbits \(\Theta_v\), we define respectively, the products \(\xi_{ij}\), \(\tau_u\) and \(\eta_v\) as in Case 1. Then, here too, we can observe that, if

\[
\beta_j = \xi_{1j} \xi_{2j} \cdots \xi_{kj}, \quad \gamma = \tau_1 \tau_2 \cdots \tau_l, \quad \text{and} \quad \delta = \eta_1 \eta_2 \cdots \eta_c,
\]

then

\[ \alpha = \beta_1 \beta_2 \cdots \beta_s \gamma \delta \]

and that

\[
\#(\beta_j) = \frac{1}{2} \sum_{i=1}^{k_j} (|Z_i(\Omega_j)| + z_{ij}) = \frac{1}{2} (|\Omega_j| + \sum_{i=1}^{k_j} z_{ij}),
\]

\[
\#(\gamma) = \frac{1}{2} \sum_{u=1}^{l} (|Y_u(\Phi)| + y_u)
\]

and

\[
\#(\delta) = \frac{1}{2} \sum_{v=1}^{c} (|\Theta_v| + c).
\]
These give
\[
\#(\alpha) = \frac{1}{2} \left( \sum_{j=1}^{s} |\Omega_j| + \sum_{u=1}^{l} |Y_u(\Phi)| + \sum_{v=1}^{c} |\Theta_v| + \sum_{j=1}^{s} \sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u + c \right)
\]
\[
= \frac{1}{2} \left( n + c - (a + t) + \sum_{j=1}^{s} \sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u \right)
\]
\[
= \frac{1}{2} (n + c(\alpha) - f(\alpha) + l(\alpha))
\]
\[
= \frac{1}{2} (g(\alpha) + m(\alpha)).
\]

**Case 4.** \(0 < l(\alpha) = e(\alpha)\). If \(Z_1(\Omega_j)\) is of odd size, we define a product \(\xi_{1j}\) of 3-paths by
\[
\xi_{1j} = \begin{pmatrix}
  x_{1j}^{m_j+r_j-1} & x_{1j}^{m_j+r_j-2} \\
  x_{1j}^{m_j-1} & x_{1j}^{m_j+r_j-1}
\end{pmatrix}
\begin{pmatrix}
  x_{1j}^{m_j+r_j-3} & x_{1j}^{m_j+r_j-4} \\
  x_{1j}^{m_j+r_j-2} & x_{1j}^{m_j+r_j-3}
\end{pmatrix},
\]
otherwise if \(Z_1(\Omega_j)\) is of even size, we define a product \(\xi_{1j}\) of 3-paths as in Case 1. Corresponding to each \(Z_i(\Omega_j)\) \((i \neq 1)\) and \(Y_u(\Phi)\) of odd sizes, define products \(\xi_{ij}\) and \(\tau_u\) of 3-paths, respectively, by
\[
\xi_{ij} = \begin{pmatrix}
  x_{ij}^{p_{ij}-1} & x_{ij}^{p_{ij}-2} \\
  x_{ij}^{p_{ij}} & x_{ij}^{p_{ij}-1}
\end{pmatrix}
\begin{pmatrix}
  x_{ij}^{p_{ij}-3} & x_{ij}^{p_{ij}-4} \\
  x_{ij}^{p_{ij}-2} & x_{ij}^{p_{ij}-3}
\end{pmatrix}\ldots\begin{pmatrix}
  x_{ij} & y_v \\
  x_{ij}^{\alpha} & x_{ij}
\end{pmatrix},
\]
and
\[
\tau_u = \begin{pmatrix}
  x_u^{q_u-1} & x_u^{q_u-2} \\
  x_u^{q_u} & x_u^{q_u-1}
\end{pmatrix}
\begin{pmatrix}
  x_u^{q_u-3} & x_u^{q_u-4} \\
  x_u^{q_u-2} & x_u^{q_u-3}
\end{pmatrix}\ldots\begin{pmatrix}
  x_u & y_v \\
  x_u^{\alpha} & x_u
\end{pmatrix},
\]
where the points \(y_v\), appearing as upper entries of the last 3-paths in these products, ranges (distinctively) from the even cyclic orbits \(\Theta_v\) \((v = 1, 2, \ldots, e(\alpha))\).

Now corresponding to each \(Z_i(\Omega_j)\) \((i \neq 1)\) and \(Y_u(\Phi)\) of even sizes as well as each cyclic orbit \(\Theta_v\) of odd size, the products \(\xi_{ij}, \tau_u\) and \(\Theta_v\) of 3-paths are respectively defined as in Case 1. For the cyclic orbits \(\Theta_v\) \((v = 1, 2, \ldots, e(\alpha))\) of even sizes, we define products \(\eta_v\) of 3-paths by
\[
\eta_v = \begin{pmatrix}
  y_v^{p_v-1} & y_v^{p_v-2} \\
  y_v & y_v^{p_v-1}
\end{pmatrix}
\begin{pmatrix}
  y_v^{p_v-3} & y_v^{p_v-4} \\
  y_v^{p_v-2} & y_v^{p_v-3}
\end{pmatrix}\ldots\begin{pmatrix}
  y_v^{\alpha} & z \\
  y_v^{\alpha^2} & y_j^{\alpha}
\end{pmatrix},
\]
where \( z \) is the middle entry of the last 3-path, in the (already defined) product corresponding to the odd subset \( Z_i(\Omega_j) \) or \( Y_u(\Phi) \), to which \( y_v \) is an upper entry. As in the earlier cases, it can be observed that

\[
\alpha = \beta_1 \beta_2 \cdots \beta_s \gamma \delta,
\]

where

\[
\beta_j = \xi_{1j} \xi_{2j} \cdots \xi_{kj}, \quad \gamma = \tau_1 \tau_2 \cdots \tau_l \quad \text{and} \quad \delta = \eta_1 \eta_2 \cdots \eta_c.
\]

Also, observing the points appearing at the top of the products \( \xi_{ij}, \tau_i \) and \( \eta_i \), we have

\[
\#(\xi_{ij}) = \begin{cases} 
\frac{1}{2} |Z_i(\Omega_j)| & \text{if } |Z_i(\Omega_j)| \text{ is even,} \\
\frac{1}{2}(|Z_i(\Omega_j)| + 1) & \text{if } |Z_i(\Omega_j)| \text{ is odd,}
\end{cases}
\]

\[
\#(\tau_u) = \begin{cases} 
\frac{1}{2} |Y_u(\Phi)| & \text{if } |Y_u(\Phi)| \text{ is even,} \\
\frac{1}{2}(|Y_u(\Phi)| + 1) & \text{if } |Y_u(\Phi)| \text{ is odd,}
\end{cases}
\]

and

\[
\#(\eta_v) = \begin{cases} 
\frac{1}{2} |\Theta_v| & \text{if } |\Theta_v| \text{ is even,} \\
\frac{1}{2}(|\Theta_v| + 1) & \text{if } |\Theta_v| \text{ is odd.}
\end{cases}
\]

Thus, \( \#(\beta_v) = \frac{1}{2} \sum_{i=1}^{k_j} (|Z_i(\Omega_j)| + z_{ij}) = \frac{1}{2} (|\Omega_j| + \sum_{i=1}^{k_j} z_{ij}) \), \( \#(\gamma) = \frac{1}{2} \sum_{u=1}^{l} (|Y_u(\Phi)| + y_u) \) and \( \#(\delta) = \frac{1}{2} (\sum_{v=1}^{c} |\Theta_v| + c - e) \). Hence,

\[
\#(\alpha) = \frac{1}{2} \left( \sum_{j=1}^{s} |\Omega_j| + \sum_{u=1}^{l} |Y_u(\Phi)| + \sum_{v=1}^{c} |\Theta_v| + \sum_{j=1}^{s} \sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u + c - e \right)
\]

\[= \frac{1}{2} \left( n - (a + t) + \sum_{j=1}^{s} \sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u + c - e \right)\]

\[= \frac{1}{2} (n - f(\alpha) + l(\alpha) + c(\alpha) - e(\alpha))\]

\[= \frac{1}{2} (n + c(\alpha) - f(\alpha)) \quad \text{(since } l(\alpha) = e(\alpha))\]

\[= g(\alpha) - \frac{e(\alpha)}{2}.
\]

**Case 5.** \( 0 < l(\alpha) < e(\alpha) \). Here, corresponding to each \( Z_i(\Omega_j) \) and each \( Y_u(\Phi) \) of odd sizes and exactly \( l(\alpha) \) cyclic orbits \( \Theta_v \) of even sizes, we define,
respectively, products $\xi_{ij}$, $\tau_u$ and $\eta_v$ as described in Case 4. Corresponding to the even sizes subsets $Z_i(\Omega_j)$ and $Y_u(\Phi)$, as well as the odd sizes cyclic orbits $\Theta_v$, we define, respectively, the products $\xi_{ij}$, $\tau_u$ and $\eta_v$ as described in Case 1. For the remaining $e(\alpha) - l(\alpha)$ cyclic orbits $\Theta_v$ of even sizes, we define the products $\eta_v$ as described in Case 2. It is then easily seen that, if

$$\beta_j = \xi_1 \xi_2 \cdots \xi_{kj}, \quad \gamma = \tau_1 \tau_2 \cdots \tau_l, \quad \text{and} \quad \delta = \eta_1 \eta_2 \cdots \eta_c,$$

then

$$\alpha = \beta_1 \beta_2 \cdots \beta_s \gamma \delta$$

and that

$$\#(\beta_j) = \frac{1}{2} \sum_{i=1}^{k_j} (|Z_i(\Omega_j)| + z_{ij}) = \frac{1}{2} (|\Omega_j| + \sum_{i=1}^{k_j} z_{ij}),$$

$$\#(\gamma) = \frac{1}{2} \sum_{u=1}^{l} (|Y_u(\Phi)| + y_u)$$

and

$$\#(\delta) = \begin{cases} \\
\frac{1}{2} \left( \sum_{v=1}^{c} |\Theta_v| + c \right) & \text{if } e(\alpha) - l(\alpha) \text{ is even}, \\
- \left( \sum_{j=1}^{s} \sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u \right) & \\
\frac{1}{2} \left( \sum_{v=1}^{c} |\Theta_v| + c \right) & \text{if } e(\alpha) - l(\alpha) \text{ is odd}, \\
- \left( \sum_{j=1}^{s} \sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u \right) + 1 & 
\end{cases}$$

Thus, in this case we have

$$\#(\alpha) = \begin{cases} \\
\frac{1}{2} (n + c - (a + t)) & \text{if } e(\alpha) - l(\alpha) \text{ is even}, \\
\frac{1}{2} (n + c - (a + t) + 1) & \text{if } e(\alpha) - l(\alpha) \text{ is odd}. 
\end{cases}$$

That is $\#(\alpha) = \lfloor \frac{1}{2} (n + c(\alpha) - f(\alpha)) \rfloor = \lfloor \frac{g(\alpha)}{2} \rfloor$.

**Case 6.** $0 < e(\alpha) < l(\alpha)$. Here, corresponding to each cyclic orbits $\Theta_v$ of even size and exactly $e(\alpha)$ subsets $Z_i(\Omega_j)$ and $Y_u(\Phi)$ of odd sizes, we define,
respectively, products $\eta_v$, $\xi_{ij}$ and $\tau_u$ as described in Case 4. Corresponding to the even sizes subsets $Z_i(\Omega_j)$ and $Y_u(\Phi)$, as well as the odd sizes cyclic orbits $\Theta_v$, we define, respectively, the products $\xi_{ij}$, $\tau_u$ and $\eta_v$ as described in Case 1. For the remaining $l(\alpha) - e(\alpha)$ subsets $Z_i(\Omega_j)$ and $Y_u(\Phi)$ of odd sizes, we define, respectively, the products $\xi_{ij}$ and $\tau_u$ as described in Case 3. Then, it is easily seen that, if

$$\beta_j = \xi_{1j}\xi_{2j} \cdots \xi_{k_j}, \quad \gamma = \tau_1\tau_2 \cdots \tau_l \quad \text{and} \quad \delta = \eta_1\eta_2 \cdots \eta_c,$$

then

$$\alpha = \beta_1\beta_2 \cdots \beta_s\gamma\delta$$

and that

$$\#(\beta_j) = \frac{1}{2} \sum_{i=1}^{k_j} (|Z_i(\Omega_j)| + z_{ij}) = \frac{1}{2} (|\Omega_j| + \sum_{i=1}^{k_j} z_{ij}),$$

$$\#(\gamma) = \frac{1}{2} \sum_{u=1}^l (|Y_u(\Phi)| + y_i) \quad \text{and} \quad \#(\delta) = \frac{1}{2} \sum_{v=1}^c (|\Theta_v| + c - e).$$

Thus here,

$$\#(\alpha) = \frac{1}{2} \left( n - (a + t) + \sum_{j=1}^s \sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^l y_u + c - e \right)$$

$$= \frac{1}{2} \left( n + c(\alpha) - f(\alpha) + l(\alpha) - e(\alpha) \right) = \frac{1}{2} \left( g(\alpha) + m(\alpha) \right).$$

Hence in all cases we have expressed $\alpha \in \text{Sing}_n$ as a product of $k(\alpha) = \lceil \frac{1}{2}(22+2) \rceil = 12$ 3-paths and so the proof of the theorem is now complete. $\square$

**Example 3.** Let $\alpha \in \text{Sing}_{22}$ be the map given in Example 1. Then $g(\alpha) = 22 + 3 - 3 = 22$ and $m(\alpha) = 2$, so that $\alpha$ can be expressed as a product of $k(\alpha) = \lceil \frac{1}{2}(22+2) \rceil = 12$ 3-paths in $\text{Sing}_{22}$. The processes of decomposition described in the proof of Theorem 2 give $\alpha = \xi_{11}\xi_{12}\xi_{21}\xi_{22}\tau_1\tau_2\eta_1\eta_2\eta_3$ where $\xi_{11} = (\frac{4}{1}, \frac{3}{4}, \frac{2}{3})$, $\xi_{12} = (\frac{5}{4}, \frac{18}{5})$, $\xi_{21} = (\frac{8}{6}, \frac{7}{6})$, $\xi_{22} = (\frac{9}{7}, \frac{20}{9})$, $\tau_1 = (\frac{13}{14}, \frac{12}{13})$, $\tau_2 = (\frac{10}{11}, \frac{12}{11})$, $\eta_1 = (\frac{15}{14}, \frac{17}{15})$, $\eta_2 = (\frac{18}{19}, \frac{19}{18})$ and $\eta_3 = (\frac{21}{20}, \frac{9}{21})$.

In the next lemma we obtain the maximum value of $g(\alpha) + m(\alpha)$ in $\text{Sing}_n$. 
Lemma 1. Let \( n \geq 3 \) and \( \text{Sing}_n \) be the semigroup of all singular self-maps of \( X_n \). Then \( \max\{g(\alpha) + m(\alpha) : \alpha \in \text{Sing}_n\} = 2(n-1) \).

Proof. From [6, Lemma 2.5], we have

\[
\max\{g(\alpha) : \alpha \in \text{Sing}_n\} = \left\lfloor \frac{3}{2}(n-1) \right\rfloor
\]

and from (4), the maximum value of \( m(\alpha) \) is attained by making \( l(\alpha) \) as large as possible while keeping \( e(\alpha) \) as small as possible. It is clear that any map \( \alpha \in \text{Sing}_n \) of height one has \( l(\alpha) = n-1 \) and \( e(\alpha) = 0 \), which are the maximum and least possible values of \( l(\alpha) \) and \( e(\alpha) \) respectively. Thus,

\[
\max\{m(\alpha) : \alpha \in \text{Sing}_n\} = n-1.
\]

Now, for a map \( \alpha \in \text{Sing}_n \) of height one, \( g(\alpha) = n-1 \) and so,

\[
\max\{g(\alpha) + m(\alpha) : \alpha \in \text{Sing}_n\} \geq 2(n-1).
\]

Next, we show the opposite inequality, that is,

\[
\max\{g(\alpha) + m(\alpha) : \alpha \in \text{Sing}_n\} \leq 2(n-1).
\]

Suppose for some \( \beta \in \text{Sing}_n \), \( g(\beta) + m(\alpha) > 2(n-1) \). Then, since \( m(\beta) \leq n-1 \), we must have \( g(\beta) > n-1 \). Also, \( g(\beta) \leq \left\lfloor \frac{3}{2}(n-1) \right\rfloor \), for if \( g(\beta) = \left\lfloor \frac{3}{2}(n-1) \right\rfloor \), then \( m(\beta) = 0 \) and \( g(\beta) + m(\beta) = \left\lfloor \frac{3}{2}(n-1) \right\rfloor \leq 2(n-1) \) for all \( n \geq 3 \), which is a contradiction to the choice of \( \beta \in \text{Sing}_n \). It then follows that \( g(\alpha) + m(\alpha) \leq 2(n-1) \) for all \( \alpha \in \text{Sing}_n \). \( \square \)

Let \( P \) be the set of all 3-paths in \( \text{Sing}_n \), and for each positive integer \( k \) write \( P^{[k]} \) for the set of all product of elements in \( P \) of length \( k \) or less. That is \( P^{[k]} = P \cup P^2 \cup \cdots \cup P^k \). Then, from Theorem 2 and Lemma 1 we deduce the following.

Corollary 1. For each \( n \geq 3 \), we have \( \text{Sing}_n \subseteq P^{[n-1]} \).

Remark 1. At the moment, we do not know whether the formula obtained in Theorem 2 is best possible, that is whether there is a number smaller then \( \left\lfloor \frac{3}{2}(g(\alpha) + m(\alpha)) \right\rfloor \) expressing \( \alpha \in \text{Sing}_n \) as a product of 3-paths.
References


Contact Information

A. T. Imam  
Department of Mathematics,  
Ahmadu Bello University Zaria, Nigeria  
E-Mail(s): atimam@abu.edu.ng

M. J. Ibrahim  
Department of Mathematics and Computer  
Science, Sule Lamido University Jigawa, Nigeria  
E-Mail(s): mjibrahim@slu.edu.ng

Received by the editors: 17.02.2021  
and in final form 12.05.2021.