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# Quasi semiprime multiplication modules over a pullback of a pair of Dedekind domains

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Abstract. The main purpose of this article is to classify all indecomposable quasi semiprime multiplication modules over pullback rings of two Dedekind domains and establish a connection between the quasi semiprime multiplication modules and the pure-injective modules over such rings. First, we introduce and study the notion of quasi semiprime multiplication modules and classify quasi semiprime multiplication modules over local Dedekind domains. Second, we get all indecomposable separated quasi semiprime multiplication modules and then, using this list of separated quasi-semiprime multiplication modules, we show that non-separated indecomposable quasi semiprime multiplication $R$-modules with finite-dimensional top are factor modules of finite direct sums of separated indecomposable quasi semiprime multiplication modules.

## Introduction

The idea of investigating a mathematical structure via its representations in simpler structures is commonly used and often successful. One of the aims of the modern representation theory is to solve classification problems for subcategories of modules over a unitary ring $R$. The reader is referred to [1], [34, Chapters 1 and 14], [37] and [2] for a detailed discussion problems, their representation types (finite, tame, or wild), and useful computational reduction procedures, see [23] and [36].

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Let $v_1 : R_1 \longrightarrow \bar{R}$ and $v_2 : R_2 \longrightarrow \bar{R}$ be a homomorphism of two local Dedekind domains $R_i$, $i = 1, 2$, onto a common field $\bar{R}$. Denote the pullback

$$R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$$

by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$, where $\bar{R} = R_1/J(R_1) = R_2/J(R_2)$. Then $R$ is a ring under coordinate-wise multiplication. Denote the kernel of $v_i$, $i = 1, 2$, by $P_i$. Then

$$\mathrm{Ker}(R \longrightarrow \bar{R}) = P = P_1 \times P_2, \qquad R/P \cong \bar{R} \cong R_2/P_2,$$
$$\text{and} \quad P_1P_2 = P_2P_1 = 0$$

(so $R$ is not a domain). Furthermore, for $i \neq j$, $0 \longrightarrow P_i \longrightarrow R \longrightarrow R_j \longrightarrow 0$ is a exact sequence of $R$-modules (see [24]).

A commutative ring $R$ is local if it has a unique maximal ideal. If $R$ is commutative and $S$ is a multiplicative closed subset of $R$, then we denote by $S^{-1}R$ the localization of $R$ with respect to $S$. If $P$ is a prime ideal of $R$ and $S = R \setminus P$ we write $S^{-1}R$ as $R_P$.

We know that every module is an elementary substructure of a pure-injective module. In fact, there is a minimal pure-injective elementary extension of each module $M$, denoted by $h(M)$, called the pure-injective hull of $M$ and it is unique up to isomorphism fixing $M$. The class of pure-injectives is closed under direct summands and finite direct sums, but an infinite direct sum of pure-injectives need not be pure-injective. Observe that any finite module is pure-injective. In a sense, then, pure-injective modules are model theoretically typical: for example, classification of the complete theories of $R$-modules reduces to classifying the (complete theories of) pure-injectives. Also, for some rings, “small" (finite-dimensional, finitely generated, ...) modules are classified and in many cases this classification can be extended to give a classification of (indecomposable) pure-injective modules. Indeed, there is sometimes a strong connection between infinitely generated pure-injective modules and families of finitely generated modules. Therefore, pure-injective modules are very important (see [22] and [32]). One point of this paper is to introduce a subclass of pure-injective modules.

Modules over pullback rings have been studied by several authors (see for example, [3,5,7,9–16,19,29] and [39]). The important work of Levy [25] provides a classification of all finitely generated indecomposable modules over Dedekind-like rings. L. Klingler [20] extended this classification to lattices over certain non-commutative Dedekind-like rings, and Klingler and J. Haefner ([17], [18]) classified lattices over certain non-commutative pullback rings, which they called special quasi triads. Common to all these classifications is the reduction to a "matrix problem" over a division ring (see [31], [34] and [35] for background on matrix problems and their applications).

In the present article, we introduce a new class of $R$-modules, called quasi-semiprime multiplication modules (see Definition 3), and we study it in detail from the classification problem point of view. We are mainly interested in case either $R$ is a Dedekind domain or $R$ is a pullback of two local Dedekind domains. First, we give a complete description of the quasi-semiprime multiplication modules over a local Dedekind domain. Let $R$ be a pullback of two local Dedekind domains over a common factor field. Next, the main purpose of this paper is to give a complete description of the indecomposable quasi-semiprime multiplication $R$-modules with finite-dimensional top over $R/\operatorname{Rad}(R)$ (for any module $M$ we define its top as $M/\operatorname{Rad}(R)M$).

The classification is divided into two stages: the description of all indecomposable separated quasi semiprime multiplication $R$-modules and then, using this list of separated quasi semiprime multiplication modules, we show that non-separated indecomposable quasi semiprime multiplication $R$-modules with finite-dimensional top are factor modules of finite direct sums of separated indecomposable quasi semiprime multiplication $R$-modules. Then we use the classification of separated indecomposable quasi semiprime multiplication modules from Section 3, together with results of Levy [26] on the possibilities for amalgamating finitely generated separated modules, to classify the non-separated indecomposable quasi semiprime multiplication modules $M$ with finite-dimensional top. We will see that the non-separated modules may be represented by certain amalgamation chains of separated indecomposable quasi semiprime multiplication modules and where adjacency corresponds to amalgamation in the socles of these separated quasi semiprime multiplication modules.

For the sake of completeness, we state some definitions and notations used throughout. We have identified all indecomposable quasi-semiprime multiplication modules over a local Dedekind domain and so in this article all rings are commutative with identity and all modules unitary. Let $R$ be the pullback ring as mentioned in the beginning of introduction. An $R$-module $S$ is defined to be separated if there exist $R_i$-modules $S_i$, $i = 1, 2$, such that $S$ is a submodule of $S_1 \oplus S_2$ (the latter is made into an $R$-module by setting $(r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2)$). Equivalently, $S$ is separated if it is a pullback of an $R_1$-module and an $R_2$-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \longrightarrow S/PS \longleftarrow S/P_1S)$ [24, Corollary 3.3] and $S \subseteq (S/P_2S) \oplus (S/P_1S)$. Also, $S$ is separated if and only if $P_1S \cap P_2S = 0$ [24, Lemma 2.9].

If $R$ is a pullback ring, then every $R$-module is a epimorphic image of a separated $R$-module, indeed every $R$-module has a "minimal" such representation: a separated representation of an $R$-module $M$ is an epimorphism $\varphi : S \longrightarrow M$ of $R$-modules where $S$ is separated and, if $\varphi$ admits a factorization $\varphi = (S \xrightarrow{f} S' \longrightarrow M)$ with $S'$ separated, then $f$ is one-to-one. The module $K = \mathrm{Ker}(\varphi)$ is then an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [24, Proposition 2.3]. An exact sequence $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ of $R$-modules with $S$ separated and $K$ an $\bar{R}$-module is a separated representation of $M$ if and only if $P_iS \cap K = 0$ for each $i$ and $K \subseteq PS$ [24, Proposition 2.8]. Every module $M$ has a separated representation, which is unique up to isomorphism [24, Theorem 2.8]. Moreover, $R$-homomorphisms lift to a separated representation, preserving epimorphisms and monomorphisms [24, Theorem 2.6].

**Definition 1.** (a) If $R$ is a ring and $N$ is a submodule of an $R$-module $M$, the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of $M$.

(b) (i) A proper submodule $N$ of a module $M$ over a commutative ring $R$ is said to primary submodule (resp., prime submodule) if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r^n \in (N : M)$ for some $n$ (resp., $m \in N$ or $r \in (N : M)$), so $\mathrm{Rad}(N : M) = P$ (resp., $(N : M) = P'$) is a prime ideal of $R$, and $N$ is said to be $P$-primary (resp., $P'$-prime) submodule. The set of all primary submodules (resp., prime submodules) in an $R$-module $M$ is denoted pSpec$(M)$ (resp., Spec$(M)$).

(ii) A proper submodule $N$ of a module $M$ over commutative ring $R$ is said to be 2-absorbing, if $abm \in N$ for some $a, b \in R$ and $m \in M$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$ (see [30]).

(c) A proper ideal $I$ of a commutative ring $R$ is called semiprime, if $a^k \in I$ for some $a \in R$ and a positive integer $k$, then $a \in I$.

(d) A proper submodule $N$ of an $R$-module $M$ is called to be semiprime, if for $a \in R$ and $m \in M$, $a^k m \in N$ for some positive integer $k$ implies that $am \in N$. The set of all semiprime submodules in an $R$-module $M$ is denoted by $seSpec(M)$.

(e) An $R$-module $M$ is defined to be a multiplication module if for each submodule $N$ of $M$, $N = IM$, for some ideal $I$ of $R$. In this case, we can take $I = (N : M)$.

(f) An $R$-module $M$ is defined to be a weak multiplication module if $\mathrm{Spec}(M) = \varnothing$ or for every prime submodule $N$ of $M$, $N = IM$, for some ideal $I$ of $R$ (see [10]).

(g) An $R$-module $M$ is defined to be a primary multiplication module if $\mathrm{pSpec}(M) = \varnothing$ or for every primary submodule $N$ of $M$, $N = IM$, for some ideal $I$ of $R$ (see [12]).

(h) An $R$-module $M$ is defined to be a semiprime multiplication module if for every semiprime submodule $N$ of $M$, $N = IM$, for some ideal $I$ of $R$ (see [13]).

(i) A submodule $N$ of an $R$-module $M$ is called a pure submodule, if any finite system of equations over $N$ which is solvable in $M$ is also solvable in $N$. A submodule $N$ of an $R$-module $M$ is called relatively divisible (or an $RD$-submodule) in $M$ if $rN = N \cap rM$ for all $r \in R$ (see [22, 23, 38] and [32]).

(j) A module $M$ is pure-injective if it has the injective property relative to all pure exact sequences (see [32, 38]).

**Remark 1.**

(i) An $R$-module is pure-injective if and only if it is algebraically compact (see [22] and [39]).

(ii) Let $R$ be a Dedekind domain, $M$ an $R$-module and $N$ a submodule of $M$. Then $N$ is pure in $M$ if and only if $IN = N \cap IM$ for each ideal $I$ of $R$. Moreover, $N$ is pure in $M$ if and only if $N$ is an $RD$-submodule of $M$ [38].

## 1. Quasi semiprime multiplication modules over a local Dedekind domain

The aim of this section is to classify quasi semiprime multiplication modules over a local Dedekind domain. First, we collect basic properties concerning quasi semiprime multiplication modules.

**Definition 2.** Let $R$ be a commutative ring and $M$ be an $R$-module. A proper submodule $N$ of $M$ is said to be quasi semiprime, if $(N : M)$ is a semiprime ideal of $R$. The set of all quasi semiprime submodules in an $R$-module $M$ is denoted by $\mathrm{qsSpec}(M)$.

An $R$-module $M$ is called to be quasi semiprime, if its zero submodule is a quasi-semiprime submodule of $M$.

prime submodules $\Rightarrow$ semiprime submodules $\Rightarrow$ quasi-semiprime submodules

**Example 1.** **(quasi semiprime submodule that is not semiprime)** Consider $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$ and the submodule $N = \langle (4, 0) \rangle$. Then $(N : M) = 0$ is a semiprime ideal of $\mathbb{Z}$. Thus $N$ is a quasi semiprime submodule, but $N$ is not a semiprime submodule of $M$. Because $2^2(1, 0) \in N$, but $2(1, 0) \notin N$.

**Example 2.** **(semiprime submodule that is not prime)** Consider $\mathbb{Z}$-module $M = \mathbb{Z}_{30}$. Let $N = \langle 6 \rangle$ is a submodule of $M$. Then $N$ is a semiprime submodule of $M$, but it is not a prime submodule of $M$.

**Proposition 1.** *Let $M$ be an $R$-module. Then*

(i) *Let $K \subset N$ be submodules of $M$. Then $N$ is a quasi semiprime submodule of $M$ if and only if $N/K$ is a quasi semiprime submodule of $M/K$.*

(ii) *If $N$ is a quasi semiprime submodule of $M$, then $M/N$ is a quasi semiprime $R$-module.*

*Proof.* The proof is straightforward. □

**Lemma 1.** *Let $M$ be an $R$-module, $N$ a quasi semiprime submodule of $M$ and $I$ an ideal of $R$ with $I \subset (0 : M)$. Then $N$ is a quasi semiprime submodule of $M$ as an $R/I$-module.*

*Proof.* By Proposition 1, $M/N$ is a quasi semiprime $R$-module. Let $(a + I)^k \in (N :_{R/I} M)$ for some $a + I \in R/I$ and $k \in \mathbb{N}$, so $a^k \in (N :_R M) = (0 :_R M/N)$. Hence $a \in (0 :_R M/N) = (N :_R M)$ since $M/N$ is a quasi semiprime module, so $a + I \in (N :_{R/I} M)$, as needed. □

**Definition 3.** Let $R$ be a commutative ring. An $R$-module $M$ is said to be a quasi semiprime multiplication module, if $\mathrm{qsSpec}(M) = \varnothing$ or for every quasi semiprime submodule $N$ of $M$, $N = IM$, for some ideal $I$ of $R$.

**Lemma 2.** *Let $M$ be a quasi semiprime multiplication module over a commutative ring $R$. Then the following hold:*

(i) *If $I$ is an ideal of $R$ and $N$ a non-zero $R$-submodule of $M$ with $I \subseteq (N : M)$, then $M/N$ is a quasi semiprime multiplication $R/I$-module.*

(ii) *If $N$ is a submodule of $M$, then $M/N$ is a quasi semiprime multiplication $R$-module.*

(iii) *Every direct summand of $M$ is a quasi semiprime multiplication $R$-module.*

*Proof.* (i) Let $K/N$ be a quasi semiprime submodule of $M/N$. Then by Proposition 1, $K$ is a quasi semiprime submodule of $M$, then $K = (K : M)M$. An inspection will show that $K/N = (K/N :_{R/I} M/N)M/N$.

(ii) Take $I = 0$ in (i).

(iii) Apply (ii). □

**Lemma 3.** *Let $R$ and $R'$ be any commutative rings, $f : R \to R'$ a surjective homomorphism and $M$ an $R'$-module. Then the following hold:*

(i) *If $M$ is a quasi semiprime $R$-module, then $M$ is a quasi semiprime $R'$-module.*

(ii) *If $N$ is a quasi semiprime $R$-submodule of $M$, then $N$ is a quasi semiprime $R'$-submodule of $M$.*

(iii) *If $M$ is a quasi semiprime multiplication $R'$-module, then $M$ is a quasi semiprime multiplication $R$-module.*

*Proof.* (i) It is obvious.

(ii) Clearly, $M/N$ is a quasi semiprime $R$-module, so $M/N$ is a quasi semiprime $R'$-module by (i), hence $N$ is a quasi semiprime $R'$-submodule of $M$.

(iii) Let $N$ be a quasi semiprime $R$-submodule of $M$. Then $N$ is a quasi semiprime $R'$-submodule of $M$ by (ii), so $N = I'M$ for some ideal $I'$ of $R'$. Set $I = f^{-1}(I')$. Then $I$ is an ideal of $R$ and $f(I) = f(f^{-1}(I')) = I'$; hence $IM = f(I)M = I'M = N$. Therefore, $M$ is a quasi semiprime multiplication $R$-module. □

**Remark 2.** (i) Assume that $M$ is a divisible quasi semiprime multiplication module over an integral domain $R$ and let $N$ be a proper submodule of $M$. Then $M$ divisible module gives $(N :_R M) = 0$; so $N = 0$. Thus every divisible quasi-semiprime multiplication module over $R$ is simple.

(ii) We know that if $N$ is a semiprime submodule of an $R$-module $M$, then $(N :_R M)$ is a semiprime ideal of $R$; so every semiprime submodule is aquasi-semiprime submodule. Let $R$ be a local Dedekind domain with unique maximal ideal $P = Rp$. By [6, Lemma 2.6], every non-zero proper submodule $L$ of $E = E(R/P)$, the injective hull of $R/P$ is of the form $L = A_n = (0 :_E P^n)$ $(n \geqslant 1)$, $L = A_n = Ra_n$ and $PA_{n+1} = A_n$. Hence no $A_n$ is a semiprime submodule of $E$, for if $n$ is a positive integer then $P^3 A_{n+3} = A_n$, but $PA_{n+3} = A_{n+2} \nsubseteq A_n$. So it is a semiprime multiplication module. Since for any $n$, $(A_n :_R E) = 0$, hence for any $n$, $A_n$ is a quasi semiprime submodule of $E$. If $E$ is a quasi semiprime multiplication module, then $A_n = (A_n : E)E = 0$, a contradiction. Hence a semiprime multiplication module need not be a quasi semiprime multiplication module. For $R$-module $Q(R)$, we know that for any non-zero submodule $L$ of $Q(R)$, we have $(L : Q(R)) = 0$ is a semiprime (prime) ideal of $R$. Similarly, $Q(R)$ is not a quasi-semiprime multiplication module.

(iii) Let $R$ be a local Dedekind domain with unique maximal ideal $P = Rp$. Consider the divisible modules $E = E(R/P)$, the injective hull of $R/P$, and $Q(R)$, the field of fractions of $R$. By [6, Lemma 2.6], $E$ has non-zero proper submodules and $Q(R)$ has the non-zero proper submodule $R$. Therefore these modules are not quasi semiprime multiplication modules by (ii). Moreover, the cyclic modules $R$ and $R/P^n$ are quasi semiprime multiplication modules since they are multiplication modules.

multiplication modules $\Rightarrow$ quasi-semiprime multiplication modules $\Rightarrow$
semiprime multiplication modules $\Rightarrow$ weak multiplication modules

Thus, the class of quasi semiprime multiplication modules contains the class of semiprime multiplication modules, and the class semiprime multiplication modules contains the class weak multiplication modules.

**Proposition 2.** *Let $M$ be a quasi semiprime multiplication module over an integral domain $R$ (with is not a field). Then $M$ is either torsion or torsion-free.*

*Proof.* Assume that $T(M)$ is the torsion submodule of $M$ and $T(M) \neq M$. Then $T(M)$ is a prime submodule (so a quasi semiprime submodule) of $M$ with $(T(M) : M) = 0$ by [28, Lemma 3.8]. It follows that $T(M) = (T(M) : M)M = 0$. Thus $M$ is a torsion-free module and this complete the proof. $\square$

If $R$ is a local Dedekind domain with unique maximal ideal $P$, then if $p \in P \setminus P^2$ then the ideal generated by $p$ is $P$ and hence for each $n$, we have $P^n = p^n R$. Moreover, every non-zero element of $R$ has the form $up^m$ where $u$ is a unit in $R$. Also, if $u$ is a unit in $R$, then $P^n = uP^n$ for all positive integer $n$ and the set of all proper ideals of $R$ is $\{0, P, P^2, \cdots\}$.

**Lemma 4.** *Let $R$ be a local Dedekind domain with a unique maximal ideal $P = Rp$. Then the following hold:*

(i) *If $I$ is a semiprime ideal of $R$, then $I = 0$ or $I = P$.*

(ii) *If $M$ is a quasi semiprime multiplication module, then* $\mathrm{qsSpec}(M) \neq \varnothing$.

*Proof.* (i) The ideals 0 and $P$ are semiprime (prime). If $I = P^n$ with $n \geqslant 2$, then $p^n \in I$, but $p \notin I$. Then $P^n$ is not a semiprime ideal for all $n \geqslant 2$.

(ii) Let $\mathrm{qsSpec}(M) = \varnothing$. Since $\mathrm{Spec}(M) \subseteq \mathrm{qsSpec}(M) = \varnothing$, it follows from [27, Lemma 1.3 and Proposition 1.4] that $M$ is a torsion divisible $R$-module with $PM = M$ and $M$ is not finitely generated. By an argument like that in [6, Proposition 2.7], $M \cong E(R/P)$, which is a contradiction by Remark 2. □

**Theorem 1.** *Let $R$ be a local Dedekind domain with a unique maximal ideal $P = Rp$. Then the following is a complete list, up to isomorphism, of the indecomposable quasi semiprime multiplication modules:*

(i) *$R$;*

(ii) *$R/P^n$ $(n \geqslant 1)$ the indecomposable torsion module.*

*Proof.* First, we note that each of the preceding modules is indecomposable (by [5, Proposition 1.3]) and quasi semiprime multiplication module by Remark 2.

Now let $M$ be an indecomposable quasi semiprime multiplication module, and choose any non-zero element $a \in M$. Let $h(a) = sup\{n \mid a \in P^n M\}$ (so $h(a)$ is a non-negative integer or $\infty$). Also, $(0 : a) = \{r \in R \mid ra = 0\}$, thus $(0 : a)$ is an ideal of the form $P^n$ or 0. Because $(0 : a) = P^{m+1}$ implies that $P^m a \neq 0$ and $P(P^m a) = 0$, we can choose $a$ so that $(0 : a) = P$ or 0. Now we consider the various possibilities for $h(a)$ and $(0 : a)$.

If $h(a) = n$, $(0 : a) = 0$, (resp. $h(a) = n$, $(0 : a) = P$), then by a similar argument like that in [8, Theorem 2.12, Case 2] (resp. [8, Theorem 2.12, Case 3]), we get $M \cong R$ (resp. $M \cong R/P^{n+1}$). So we may assume that $h(a) = \infty$. If $(0 : a) = P$ (resp. $(0 : a) = 0$), then by an argument like that in [6, Proposition 2.7, Case 2] (resp. [8, Theorem 2.12, Case 3]), we get $M \cong E(R/P)$ (resp. $M \cong Q(R)$) that is a contradiction by Remark 2. □

**Corollary 1.** *Let $M$ be a quasi semiprime multiplication module over a local Dedekind domain with maximal ideal $P = Rp$.*

(i) *$M \neq R$ is a direct sum of copies of $R/P^n$ $(n \geqslant 1)$;*

(ii) *Every quasi semiprime multiplication $R$-module not isomorphic to $R$ is pure-injective.*

*Proof.* (i) Let $N$ denote the indecomposable summand of $M$. Then by Lemma 2, $N$ is an indecomposable quasi semiprime multiplication module. Now the assertion follows from Theorem 1.

(ii) Apply [5, Proposition 1.3]. □

## 2. The separated quasi semiprime multiplication modules

Throughout this section, we shall assume unless otherwise stated, that

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \tag{1}$$

is the pullback of two local Dedekind domains $R_1, R_2$ with maximal ideals $P_1, P_2$ generated, respectively, by $p_1, p_2$, $P$ denotes $P_1 \oplus P_2$ and $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ is a field. In particular, $R$ is a commutative Noetherian local ring with unique maximal ideal $P$. The other prime ideals of $R$ are easily seen to be $P_1 \oplus 0$ and $0 \oplus P_2$. Let $r = (a, b) \in R$ with $a \neq 0$ and $b \neq 0$. Then we can write $a = (p_1^n, p_2^m)$ for some positive integers $m, n$, so $ann(a) = 0$; hence $Ra \cong R$. If $a = (0, p_2^m)$ for some positive integer $m$, then $ann(a) = P_1 \oplus 0$ and so $R(0, p_2^m) \cong R/(P_1 \oplus 0) \cong R_2$. Similarly, $R(p_1^n, 0) \cong R/(0 \oplus P_2) \cong R_1$. The other ideals $I$ of $R$ are of the form $I = P_1^n \oplus P_2^m = (\langle p_1^n \rangle, \langle p_2^m \rangle)$ for some positive integers $m, n$.

**Remark 3** ([11, Remark 3.1]). Let $R$ be the pullback ring as in (1) and $T$ be an $R$-module of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$, with projection maps $\pi_i : S \to S_i$. Set $T_1 = \{t_1 \in S_1 : (t_1, t_2) \in T$ for some $t_2 \in S_2\}$ and $T_2 = \{t_2 \in S_2 : (t_1, t_2) \in T$ for some $t_1 \in S_1\}$.

Then for each $i$, $i = 1, 2$, $T_i$ is an $R_i$-submodule of $S_i$ and $T \leqslant T_1 \oplus T_2$. Moreover, we can define a mapping $\acute{\pi}_1 = \pi_1 | T : T \to T_1$ by sending $(t_1, t_2)$ to $t_1$; hence $T_1 \cong T/(0 \oplus \mathrm{Ker}(f_2) \cap T) \cong T/(T \cap P_2 S) \cong (T + P_2 S)/P_2 S \subseteq S/P_2 S$. So we may assume that $T_1$ is a submodule of $S_1$. Similarly, we may assume that $T_2$ is a submodule of $S_2$ (note that $\mathrm{Ker}(f_1) = P_1 S_1$ and $\mathrm{Ker}(f_2) = P_2 S_2$).

**Lemma 5.** *Let $R$ be a pullback ring as in* (1)*. Then the ideals* $0$*,* $P_1 \oplus 0$*,* $0 \oplus P_2$ *and* $P_1 \oplus P_2$ *are semiprime.*

*Proof.* Let $(a, b)^n \in 0$ for some $(a, b) \in R$ and $n \in \mathbb{N}$, then $a^n = 0$ and $b^n = 0$. Thus $a = 0$ and $b = 0$ since $0$ is semiprime ideal of $R_i$, $i = 1, 2$, so

0 is semiprime ideal of $R$. Since $P_1 \oplus 0$, $0 \oplus P_2$ and $P_1 \oplus P_2$ are prime, so they are semiprime. $\square$

**Proposition 3.** *Let $R$ be a pullback ring as in* (1)*. Then the following hold:*

(i) *If $T$ is a quasi semiprime submodule of a non-zero separated $R$-module $S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S/P_1S = S_2)$, then $T_1$ is a quasi semiprime submodule of $S_1$ and $T_2$ is a quasi semiprime submodule of $S_2$.*

(ii) *If $T$ is a quasi semiprime submodule of a non-zero separated $R$-module $S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S/P_1S = S_2)$, then $(T :_R S) = 0$ or $P_1 \oplus 0$ or $0 \oplus P_2$ or $P_1 \oplus P_2$.*

(iii) *Let $T$ be a quasi semiprime submodule of a non-zero separated $R$-module $S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S/P_1S = S_2)$ with $S_i$ is a quasi semiprime multiplication $R_i$-module, for each $i = 1, 2$. If $(T :_R S) = 0$, then $T = 0$ and $S_i \neq 0$, for each $i = 1, 2$.*

*Proof.* (i) Let $a_1^n \in (T_1 :_{R_1} S_1)$ where $a_1 \in R_1$ and $n \in \mathbb{N}$. Since $(T_1 :_{R_1} S_1) \subseteq P_1$, then $(a_1^n, 0) \in R$ because $v_1(a_1^n) = 0 = v_2(0)$. Let $(s_1, s_2) \in S$. Since $a_1^n s_1 \in P_1S_1 \cap T_1$ and $0 \in P_2S_2 \cap T_2$ and $f_1(a_1^n s_1) = 0 = f_2(0)$, so $(a_1^n, 0)(s_1, s_2) \in T$. Hence $(a_1, 0)^n \in (T :_R S)$, and so $(a_1, 0) \in (T :_R S)$ since $(T :_R S)$ is a semiprime ideal of $R$. Therefore we get $a_1 \in (T_1 :_{R_1} S_1)$. Thus $T_1$ is a quasi semiprime submodule of $S_1$. Similarly, $T_2$ is a quasi semiprime submodule of $S_2$.

(ii) Since $T$ is a quasi semiprime submodule of $S$, so $(T :_R S)$ is a semiprime ideal of $R$. Hence the assertion follows from Lemma 5.

(iii) First we show that $(T_1 :_{R_1} S_1) = 0$ and $(T_2 :_{R_2} S_2) = 0$. By assumption, either $S_1 \neq 0$ or $S_2 \neq 0$. Suppose that $S_1 \neq 0$ . Assume to the contrary, $0 \neq r_1 \in (T_1 :_{R_1} S_1) \subseteq P_1$ (so $v_1(r_1) = 0 = v_2(0)$; hence $(r_1, 0) \in R$). Therefore $r_1S_1 \subseteq T_1$. Let $(s_1, s_2) \in S$. Then $(r_1, 0)(s_1, s_2) = (r_1s_1, 0) \in T$ since $f_1(r_1s_1) = 0 = f_2(0)$ and $T$ is separated; hence $0 \neq (r_1, 0)S \subseteq T$ that is a contradiction. So $(T_1 :_{R_1} S_1) = 0$. If $S_2 = 0$, then $S = (P_1 \oplus 0)S$, and so $(0 \oplus P_2)S = 0$. Therefore $0 \neq (0 :_R S) \subseteq (T :_R S)$ which is a contradiction. So $S_2 \neq 0$. Similarly, $S_2 \neq 0$ gives $(T_2 :_{R_2} S_2) = 0$. Now $S_1$ is a quasi semiprime multiplication $R_1$-module gives $T_1 = 0$. Similarly, $T_2 = 0$. Thus $T = 0$. $\square$

**Proposition 4.** *Let $S$ is any non-zero separated quasi-semiprime multiplication module over a pullback ring as in* (1)*. Then* $\mathrm{qsSpec}(S) \neq \varnothing$*.*

*Proof.* Let $\pi$ be the projection map of $R$ onto $R_1$. By Lemma 4 (ii), $\mathrm{qsSpec}(S_1) \neq \varnothing$, so there is a quasi-semiprime submodule $T_1$ of $S_1$. Then by Lemma 3, $T_1 = T/(0 \oplus P_2)S$ is a quasi semiprime $R$-submodule of $S_1 \cong S/(0 \oplus P_2)S$; so $T$ is a quasi semiprime $R$-submodule of $S$ by Proposition 1 (i) and hence $\mathrm{qsSpec}(S) \neq \varnothing$. □

**Theorem 2.** *Let $S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$ be any separated module over the pullback ring as (1). Then $S$ is a quasi semiprime multiplication $R$-module if and only if $S_i$ is a quasi semiprime multiplication $R_i$-module, $i = 1, 2$.*

*Proof.* Let $S$ be a separated quasi semiprime multiplication $R$-module. First suppose that $\bar{S} = 0$. Then by [5, Lemma 2.7(i)], $S = S_1 \oplus S_2$; hence $S_i$ is a quasi semiprime multiplication $R_i$-module by Lemma 2 (iii), for each $i = 1, 2$. So we may assume that $\bar{S} \neq 0$. Since $(0 \oplus P_2) \subseteq ((0 \oplus P_2)S : S)$, Lemma 2 gives $S_1 \cong S/(0 \oplus P_2)S$ is a quasi semiprime multiplication $R/(0 \oplus P_2) \cong R_1$-module. Similarly, $S_2$ is a quasi semiprime multiplication $R_2$-module.

Conversely, assume that each $S_i$ is a quasi semiprime multiplication $R_i$-module and let $T$ be a quasi semiprime submodule of $S$. We split the proof into three cases for $(T :_R S)$ (see Lemma 5).

*Case 1.* $(T :_R S) = 0$. By Proposition 3 (iii), $T = 0 = (T :_R S)S$, as required.

*Case 2.* $(T :_R S) = P_1 \oplus 0$. Then $T_1 = P_1S_1$ and $T_2 = 0$ since $S_1$ and $S_2$ are quasi semiprime multiplication. We show that $T = (P_1 \oplus 0)S$. Since $(T :_R S) = P_1 \oplus 0$, then $(T :_R S)S = (P_1 \oplus 0)S \subseteq T$. Let $t = (t_1, 0) \in T$. Then $t_1 = p_1s_1$ for some $s_1 \in S_1$. Hence there exist $s_2 \in S_2$ such that $(s_1, s_2) \in S$; so $t = (p_1s_1, 0) = (p_1, 0)(s_1, s_2) \in (P_1 \oplus 0)S$ and so $T = (P_1 \oplus 0)S$. Similarly, if $(T :_R S) = 0 \oplus P_2$, we get $T = (0 \oplus P_2)S$.

*Case 3.* $(T :_R S) = P_1 \oplus P_2$. Then $(T_1 :_{R_1} S_1) = P_1$, and $(T_2 :_{R_2} S_2) = P_2$. Now by Proposition 3(i), $T_1 = P_1S_1$ since $S_1$ is quasi-semiprime multiplication. Similarly, $T_2 = P_2S_2$. We have $(T :_R S)S = (P_1 \oplus P_2)S \subseteq T$, we show that $T \subseteq (P_1 \oplus P_2)S$. Let $(t_1, t_2) \in T$. Then $t_1 = p_1s_1$ for some $p_1 \in P_1$, $s_1 \in S_1$, and $t_2 = p_2s_2$ for some $p_2 \in P_2$, $s_2 \in S_2$. Thus there exists $v_2 \in S_2$ such that $(s_1, v_2) \in S$ and there exists $v_1 \in S_1$ such that $(v_1, s_2) \in S$. We have $(t_1, t_2) = (p_1s_1, p_2s_2) = (p_1, 0)(s_1, v_2) + (0, p_2)(v_1, s_2) \in (P_1 \oplus P_2)S$. So $T = (P_1 \oplus P_2)S$ and hence $S$ is a quasi semiprime multiplication $R$-module. □

**Lemma 6.** *Let $R$ be the pullback ring as in* (1). *Then, up to isomorphism, the following separated $R$-modules are indecomposable quasi semiprime multiplication modules:*

(1) $S = (R_1 \longrightarrow \bar{R} \longleftarrow R_2)$;
*and for all positive integers $m, n$,*
(2) $S = (R_1/P_1^n \longrightarrow \bar{R} \longleftarrow R_2/P_2^m)$;
(3) $S = (R_1 \longrightarrow \bar{R} \longleftarrow R_2/P_2^m)$;
(4) $S = (R_1/P_1^n \longrightarrow \bar{R} \longleftarrow R_2)$.

*Proof.* By [5], these modules are indecomposable, quasi semiprime multiplicativity follows from Theorem 1 and Theorem 2. □

**Theorem 3.** *Let $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ be an indecomposable separated quasi semiprime multiplication module over the pullback ring as* (1). *Then $S$ is isomorphic to one of the modules listed in Lemma 6.*

*Proof.* Let $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ be a non-zero indecomposable separated quasi semiprime multiplication $R$-module. If $S = R$, we are done. So we may assume that $S \neq R$. First note that $\text{qsSpec}(S) \neq \varnothing$ by Proposition 4. Next we show that $\bar{S} \neq 0$. If $T$ is a quasi semiprime submodule of $S$, then $T_i$ is a quasi semiprime submodule of $S_i$ by Proposition 3(i); so $T_i \neq S_i$ for each $i$, because if $T_1 = S_1$, then $(T_1 :_{R_1} S_1) = R_1$ is a quasi semiprime ideal of $R_1$ that is a contradiction by Lemma 4 (i)). Now, we consider the various possibilities for $(T :_R S)$.

*Case* 1. $(T :_R S) = 0$. If $\bar{S} = 0$, then $S = S_1 \oplus S_2$ by [5, Lemma 2.7(i)]; hence either $S_1 = 0$ or $S_2 = 0$ (since $S$ is indecomposable) that is a contradiction by Proposition 3(iii). Thus $\bar{S} \neq 0$.

*Case* 2. $(T :_R S) = P = P_1 \oplus P_2$. It follows that $PS \subseteq T \neq S$, so $PS \neq S$.

*Case* 3. $(T :_R S) = P_1 \oplus 0$. Then since $S$ is a quasi semiprime multiplication, $T = (P_1 \oplus 0)S$ which implies that $T_1 = P_1 S_1 = PS/P_2 S \neq S_1 = S/P_2 S$; hence $PS \neq S$. Similarly, when $(T :_R S) = 0 \oplus P_2$, we get $S \neq PS$.

By Theorem 2, $S_i$ is a quasi semiprime multiplication $R_i$-module, for each $i = 1, 2$. Choose $t \in S_1 \cup S_2$ with $\bar{t} \neq 0$ and let $o(t)$ denote the least positive integer $k$ such that $P^k t = 0$ if there is such $k$ and if no such $k$, $o(t) = \infty$ and $o(t)$ is minimal among such $t$. Assume $t \in S_2$, and so write $t = t_2$ and $m = k = o(t_2)$. Now pick $t_1 \in S_1$ with $\bar{t_1} = \bar{t_2} = \bar{t}$ and $o(t_1) = n$ minimal. If $o(t_2) = m$ (resp. $o(t_1) = n$), then $R_2 t_2 \cong R_2/P_2^m$ (resp. $R_1 t_1 \cong R_1/P_1^n$) is pure in $S_2$ (resp. is pure in $S_1$). If $o(t_1) = \infty$

(resp. $o(t_2)=\infty$), then $R_1t_1 \cong R_1$ (resp. $R_2t_2 \cong R_2$) is pure in $S_1$ (resp. is pure in $S_2$), see [5, Theorem 2.9]. Let $\bar{M}$ be the $\bar{R}$-subspace of $\bar{S}$ generated by $\bar{t}$. Then $\bar{M} \cong \bar{R}$. Let $M=(R_1t_1 = M_1 \longrightarrow \bar{M} \longleftarrow M_2 = R_2t_2)$. Then $M$ is an $R$-submodule of $S$ which is a direct summand of $S$; so $S=M$ since $S$ is indecomposable [5, Theorem 2.9]. Therefore $S$ is one of the modules listed (2)–(4) in the Lemma 6, as required. □

**Corollary 2.** *Let $R$ be the pullback ring as in* (1)*. Then every separated quasi semiprime multiplication $R$-module not isomorphic to $R$ is pure-injective.*

*Proof.* Apply Theorem 2 and [5, Lemma 2.9]. □

## 3. Non-separated quasi semiprime multiplication modules

We continue to use the notion already established, so $R$ is the pullback ring as in (1). In this section, we find the indecomposable non-separated quasi semiprime multiplication modules with finite-dimensional top. It turns out that each can be obtained by amalgamating finitely many separated indecomposable quasi semiprime multiplication modules.

We need the following lemma proved in [7, Lemma 3.1].

**Lemma 7.** *Let $R$ be the pullback ring as in* (1) *and let $M$ be any $R$-module. Let $0 \longrightarrow K \xrightarrow{i} S \xrightarrow{\varphi} M \longrightarrow 0$ be a separated representation of $M$. Then*

(i) *If $N$ is a non-zero submodule of $M$, then $0 \longrightarrow K \longrightarrow \varphi^{-1}(N) = T \longrightarrow N \longrightarrow 0$ is a separated representation of $N$.*

(ii) *If $M$ is non-separated, then $P^nM \neq 0$ and $K \subseteq P^nS$ for all positive integers $n$.*

**Proposition 5.** *Let $R$ be the pullback ring as in* (1) *and let $M$ be any $R$-module. Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of $M$. Then*

(i) *If $S$ has a submodule $T$ with $(T :_R S) = 0$, then $M$ is separated.*

(ii) *If $S$ has a submodule $T$ with $(T :_R S) = P_1 \oplus 0$, then $M$ is separated.*

(iii) *If $S$ has a submodule $T$ with $(T :_R S) = 0 \oplus P_2$, then $M$ is separated.*

*Proof.* (i) $(0 :_R S) = 0$ since $(0 :_R S) \subseteq (T :_R S)$. Let $r=(r_1,r_2) \in (0 :_R M)$. Then $rM = rS/K = 0$, so$rS \subseteq K$ which implies that $rPS \subseteq PK = 0$. It follows that $r_1p_1 = 0$ and $r_2p_2 = 0$; hence $r=0$ (since $R_i$ is a domain, for each $i=1,2$). Thus $(0 :_R M) = 0$. It follows that

$(P_1 \oplus 0)M \neq 0$ and $(0 \oplus P_2)M \neq 0$. Let $m \in (P_1 \oplus 0)M \cap (0 \oplus P_2)M$. If $m = 0$, we are done. So suppose that $m \neq 0$. Then there exists $x \in S$ such that $m = \varphi(x)$. Since $\varphi^{-1}(Rm) = \varphi^{-1}(\varphi(Rx)) = Rx$ and $Rm \neq 0$, $0 \longrightarrow K \longrightarrow Rx \longrightarrow Rm \longrightarrow 0$ is a separated representation of $Rm$ with $K \subseteq P(Rx)$ by Lemma 7. By hypothesis, there exist $m_1, m_2 \in M$ such that $m = (p_1^s, 0)m_1 = (0, p_2^t)m_2$ for some integers $s, t$. Then $(p_1, 0)m = 0 = (0, p_2)m$ gives $Pm = 0$ and so $\varphi(Px) = 0$; hence $\varphi(P_1 x) = \varphi(P_2 x) = 0$. Since $\varphi$ is one-to-one on $P_i S$ for each $i$, we get $Px = 0$; so $K \subseteq P(Rx) = 0$. Thus $M$ is a separated $R$-module.

(ii) Since $(0 :_R S) \subseteq (T :_R S) = P_1 \oplus 0$, either $(0 :_R S) = 0$ or $(0 :_R S) = P_1^m \oplus 0$ for some positive integer $m$. If $(0 :_R S) = 0$, then we are done by (i). So suppose that $(0 :_R S) = P_1^m \oplus 0$. As $P_1^m \oplus 0 \subseteq ((0 \oplus P_2)S :_R S)$ and $0 \oplus P_2 \subseteq ((0 \oplus P_2)S :_R S)$, we get $P_1^m \oplus P_2 \subseteq ((0 \oplus P_2)S :_R S))$, and so $(P_1^m \oplus P_2)S \subseteq ((0 \oplus P_2)S :_R S)S \subseteq (0 \oplus P_2)S$. It then follows from Lemma 7 that $K \subseteq P^m S \subseteq (P_1^m \oplus P_2)S \subseteq (0 \oplus P_2)S$; hence $K = 0$ since $K \cap (0 \oplus P_2)S = 0$. Thus $M$ is separated.

(iii) It is similar to (i). □

**Theorem 4.** *Let $R$ be the pullback ring as in* (1) *and $M$ be any non-separated $R$-module. Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of $M$. Then $S$ is quasi semiprime multiplication if and only if $M$ is quasi semiprime multiplication.*

*Proof.* Assume that $S$ is a quasi-semiprime multiplication $R$-module. Then $S \cong M/K$ is quasisemiprime multiplication by Lemma 2(ii). Conversely, suppose that $M$ is a quasi semiprime multiplication $R$-module, and let $T$ be a quasi semiprime submodule of $S$. Since $M$ is non-separated, $(T :_R S) = P$ by Lemma 5(ii) and Proposition 5. Hence we have $K \subseteq PS \subseteq T$ by Lemma 7. By Proposition 1, $T/K$ is a quasi semiprime submodule of $S/K$; so $T/K = P(S/K) = (PS + K)/K) = PS/K$ since $M$ is quasi semiprime multiplication, and hence $T = PS$. Thus $S$ is a quasi semiprime multiplication $R$-module. □

**Proposition 6.** *Let $R$ be the pullback ring as in* (1)*. Then*

(i) *The $R$-module $E(R/P)$, the injective hull of $R/P$, is not a quasi semiprime multiplication module.*

(ii) *Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of a quasi semiprime multiplication non-separated $R$-module $M$ with $M/PM$ finite dimensional over $\bar{R}$. Then $\mathrm{qsSpec}_R(M) \neq \varnothing$.*

*Proof.* (i) Let, for a contradiction, that this is not the case. Assume that $L$ is a non-zero submodule of $E(R/P)$ described in [10, Proposition 3.1], say $L = E_1 + A_n$. Then $(L :_R E(R/P)) = 0$; so $L$ is a quasi semiprime submodule of $E(R/P)$ since 0 is a semiprime ideal of $R$ by Lemma 5, which implies that $L = 0$ that is a contradiction, and this completes the proof.

(ii) By Proposition 4, $\mathrm{qsSpec}_R(S) \neq \varnothing$ and so $S$ has a quasi semiprime submodule $T$. Since $M$ is non-separated, $(T :_R S) = P$ by Lemma 5 and Proposition 5. By an argument like that in Theorem 4, we get $K \subseteq T$. Now by Proposition 1(i), $T/K$ is a quasi-semiprime submodule $S/K \cong M$; hence $\mathrm{qsSpec}_R(M) \neq \varnothing$. □

**Proposition 7.** *Let $R$ be the pullback ring as in* (1) *and $M$ be an indecomposable quasi semiprime multiplication non-separated $R$-module with finite dimensional top over $\bar{R}$. Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of $M$. Then the following hold:*

(i) *$R$ do not occur among the direct summands of $S$.*
(ii) *$S$ has finite-dimensional top and is pure-injective.*
(iii) *$S$ is a direct sum of finitely many indecomposable quasi semiprime multiplication modules.*

*Proof.* (i) If $S = R \oplus T$ for some submodule $T$ of S, then $K \subseteq T$ since $Soc(R) = 0$; so $M \cong T/K \oplus R$ that is a contradiction since $M$ is indecomposable and non-separated.

(ii) By [5, Proposition 2.6(i)], $S/PS \cong M/PM$; so $S$ is finite-dimensional top. By Theorem 4, $S$ is quasi semiprime multiplication. Now the assertion follows from Corollary 2 and (i).

(iii) If $\dim_{\bar{R}} M/PM = n$ with $n \geqslant 0$, then by (ii), $\dim_{\bar{R}} S/PS = n$. By Theorem 3 and (i), we can write $S = \bigoplus_{i=1}^{n} S^i$ where for each $i$, $S$ is a separated $R$-module as described in (2)–(4) of Lemma 6 (see [5, p. 4055]). □

Let $R$ be a pullback ring as in (1). Let $M$ be any $R$-module and let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of $M$. We have shown already that if $M$ is indecomposable quasi-semiprime multiplication with $M$ finite-dimensional top then $S$ is a direct sum of just finitely many indecomposable separated quasi semiprime multiplication modules and these are known by Theorem 3. In any separated representation $0 \longrightarrow K \xrightarrow{i} S \xrightarrow{\varphi} M \longrightarrow 0$ the kernel of the map $\varphi$ to $M$ is annihilated by $P$, hence is contained in the socle of the separated module $S$. Thus $M$

is obtained by amalgamation in the socles of the various direct summands of $S$. This explains Proposition 7 (i): the module $R$ has zero socle and so cannot be amalgamated with any other direct summands of $S$ and hence cannot occur in a separated (hence "minimal") representation. So the questions are: does this provide any further condition on the possible direct summands of $S$? How can these summands be amalgamated in order to form $M$? For the case of finitely generated R-modules $M$ these questions are answered by Levy's description [25], see also [26, § 11].

Levy shows that the indecomposable finitely generated $R$-modules are of two non-overlapping types which he calls deleted cycle and block cycle types. It is the modules of deleted cycle type which are most relevant to us. Such a module is obtained from a direct summand, $S$, of indecomposable separated modules by amalgamating the direct summands of $S$ in pairs to form a chain but leaving the two ends unamalgamated. Reflecting the fact that the dimension over $\bar{R}$ of the socle of any finitely generated indecomposable separated module is $\leqslant 2$ each indecomposable summand of $S$ may be amalgamated with at most two other indecomposable summands. Consider the indecomposable separated $R$-modules $S(n,m) = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$ with $n, m \geqslant 2$ (it is generated over $R$ by $((1+P_1^n, 1+P_2^m)$. Actually, separated indecomposable $R$-modules also include $R_1/P_1^n$ for $n \geqslant 2$ which can be regarded up to isomorphism as $S(n,1) = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2)$. Similarly, for $m \geqslant 2$, $S(1,m) = (R_1/P_1 \to \bar{R} \leftarrow R_2/P_2^m)$ is a separated indecomposable R-module. Moreover, $R_1$, $R_2$ and $R$ themselves can be viewed as separated indecomposable $R$-modules, corresponding to the cases $n = \infty$ and $m = 1$, $n = 1$ and $m = \infty$, $n = m = \infty$. Deleted cycle indecomposable $R$-modules are introduced as follows.

Let $S$ be a direct sum of finitely many modules $S(i) = S(n_{i,1}, n_{i,2})$ (with $i < s$ a non-negative integer). Here $n_{i,j} \geqslant 2$ for every $j < s$ and $j = 1, 2$, with two possible exceptions $i = 0$, $j = 1$ and $i = s-1$ and $j = 2$, where the values $n_{i,j} = 1$ or $\infty$ are allowed. Then amalgamate the direct summands in $S$ by identifying the $P_2$-part of the socle of $S(i)$ and the $P_1$-part of the socle $S(i+1)$ for every $i < s-1$. For instance, given the separated modules $S_1 = (R_1 \to \bar{R} \leftarrow R_2/P_2^3) = Ra$ with $P_2^3 a = 0$ and $S_1 = (R_1/P_1^7 \to \bar{R} \leftarrow R_2/P_2^2) = Ra$ with $P_1^7 a' = 0 = P_2^2 a'$. Then one can form the non-separated module $(S_1 \oplus S_2)/(R(p_2^2 a - p_1^6 a') = Rc + Rc'$ where $c = a + R(p_2^2 a - p_1^6 a')$, $c' = a' + R(p_2^2 a - p_1^6 a')$, $P_2^3 c = 0 = P_1^7 c' = P_2^2 c$ and $P_2^2 c = P_1^6 c'$ which is obtained by identifying the $P_2$-part of the socle of $S_1$ with the $P_1$-part of the socle of $S_2$. We will use that same description, but with quasi semiprime multiplication separated modules in place of the finitely generated ones, gives us the non-zero indecomposable quasi semiprime multiplication non-separated R-modules. As a consequence, any non-zero indecomposable quasi semiprime multiplication separated module with 1-dimensional socle may occur only at one of the ends of the amalgamation chain (see [5, Proposition 3.4]). It remains to show that the modules obtained by these amalgamations are, indeed, indecomposable quasi semiprime multiplication. We do that now and thus complete the classification of the indecomposable quasi semiprime multiplication non-separated modules with finite-dimensional top.

**Theorem 5.** *Let $R = (R_1 \longrightarrow \bar{R} \longleftarrow R_2)$ be the pullback ring of two local Dedekind domains $R_1$, $R_2$ with common factor field $\bar{R}$. Then the class of indecomposable non-separated quasi-semiprime multiplication modules with finite-dimensional top up to isomorphism, are the following:*

(i) *The indecomposable modules of finite length (apart from $R/P$ which is separated), that is, $M = \sum_{i=1}^{s} Ra_i$ with $p_1^{n_s} a_s = 0 = p_2^{m_1} a_1$, $p_1^{n_i - 1} a_i = p_2^{m_{i+1}-1} a_{i+1}$ $(1 \leqslant i \leqslant s-1)$, $m_i, n_i \geqslant 2$ except for $m_1 \geqslant 1$, $n_s \geqslant 1$.*

(ii) *$M = R/P_2^n + \sum_{i=1}^{s-1} Ra_i$ with $p_2^{m_{s-1}} a_{s-1}$, $p_2^{n-1} b_0$, $p_1 a_0 = 0 = p_2 b_0$ and $p_2^{m_i - 1} a_i = p_1^{i+1-1} a_{i+1}$ for all $1 \leqslant i \leqslant s-2$, $R/P_2^n \cong Rb_0$, where $n, m_i, n_i \geqslant 2$ except for $m_{s-1} \geqslant 1$.*

(iii) *$M = R/P_1^n + \sum_{i=1}^{s-1} Ra_i$ with $p_1^{n_{s-1}} a_{s-1} = 0$, $p_1^{n-1} a_0 = P_2^{m_1 - 1} a_1$, and $p_1^{n_i - 1} a_i = p_2^{m_{i+1}-1} a_{i+1}$ for all $1 \leqslant i \leqslant s-2$, $R/P_1^n \cong Ra_0$, where $n, m_i, n_i \geqslant 2$ except for $n_{s-1} \geqslant 1$.*

(iv) *$M = R/P_2^n + \sum_{i=1}^{s-2} Ra_i + R/P_1^r$ with $p_2^{n_1} b_0 = p_1^{n_1 - 1} a_1$, $p_1^{r_1} a_0 = p_2^{m_{s-2}-1} a_{s-2}$, and $p_2^{m_i - 1} a_i = p_1^{n_{i+1}-1} a_{i+1}$ for all $1 \leqslant i \leqslant s-3$, where $R/P_2^n \cong Rb_0$, $R/P_1^r \cong Ra_0$, and $n, m_i, n_i, r \geqslant 2$.*

*Proof.* Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of $M$. By Proposition 7(iii), $S$ is a direct sum of finitely many indecomposable quasi semiprime multiplication separated modules. We know already that every indecomposable quasi semiprime multiplication non-separated module has one of these forms so it remains to show that the modules obtained by these amalgamation are, indeed, indecomposable quasi semiprime multiplication modules. Since a quotient of any quasi semiprime multiplication $R$-module is quasi semiprime multiplication by Lemma 2(ii), they are quasi semiprime multiplication. The indecomposability follows from [26, § 1.9]. □

**Corollary 3.** *Let $R = (R_1 \longrightarrow \bar{R} \longleftarrow R_2)$ be the pullback ring of two local Dedekind domains $R_1$, $R_2$ with common factor field $\bar{R}$. Then*

(i) *Every indecomposable quasi semiprime multiplication module with finite-dimensional top is pure-injective.*

(ii) *This article comprises the classification of indecomposable quasi semiprime multiplication modules with finite-dimensional top over $k$-algebra $k[x, y : xy = 0]_{(x,y)}$.*

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## References

[1] I. Assem, D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras,* Vol. 1, Techniques of Representation Theory, London Math. Soc. Student Texs 65, Cambridge-New Yor:Cambridge Univ. Press, 2007.

[2] D. Arnold, *Abelian Groups and Representations of Finite Partially Ordered sets,* Springer, 2000.

[3] D. Arnold, R. Laubenbacher, *Finitely generated modules over pullback rings,* J. Algebra, **184**, 1996, pp.304-332.

[4] M. C. R. Butler, C. M. Ringel, *Auslander-Reiten sequences with few middle terms, with applications to string algebra,* Comm. Algebras, **15**, 1987, pp.145-179.

[5] S. Ebrahimi Atani, *On pure-injective modules over pullback rings,* Comm. Algebra, **28**, 2000, pp.4037-4069.

[6] S. Ebrahimi Atani, *On secondary modules over Dedekind domains,* Southeast Asian Bull. Math., **25(25)**, 2001, pp.1-6.

[7] S. Ebrahimi Atani, *On secondary modules over puuback rings,* Comm. Algebra, **30**, 2002, pp.2675-2685.

[8] S. Ebrahimi Atani, *Indecomposable weak multiplication modules over Dedekind domains,* Demonstratio Math., **41**, 2008, pp.33-43.

[9] S. Ebrahimi Atani, S. Dolati Pish Hesari, M. Khoramdel, M. Sedghi Shanbeh Bazari, *Pseudo-absorbing multiplication modules over a pullback of ring,* J. Pure and Applied Algebra, **222(10)**, 2018, pp.3124-3136.

[10] S. Ebrahimi Atani, F. Farzalipour, *Weak multiplication modules over a pullback of Dedekind domains,* Colloquium Math., **114**, 2009, pp.99-112.

[11] S. Ebrahimi Atani, F. Esmaeili Khalil Saraei, *Indecomposable primary comultiplication modules over a pullback of two Dedekind domains,* Colloquium Math., **120**, 2010, pp.23-42.

[12] R. Ebrahimi Atani, S. Ebrahimi Atani, *On primary multiplication modules over pullback rings,* Algebra and Discrete Mathematics, **11(2)**, 2011, pp.1-17.

[13] R. Ebrahimi Atani, S. Ebrahimi Atani, *On semiprime multiplication modules over pullback rings,* Comm. Algebra, **41**, 2013, pp.776-791.

[14] F. Esmaeili Khalil Saraei, *Pseudo-prime multiplication modules over pullback rings,* Colloquium Math., **143**, 2016, pp.63-78.

[15] A. Facchini, P. Vamos, *Injective modules over pullback rings,* J. London Math. Soc., **31**, 1985, pp.125-138.

[16] F. Farzalipour, *On 2-absorbing multiplication modules over pullback rings,* Indian J. Pure Appl. Math., **50(40)**, 2019, pp.1021-1038.

[17] J. Haefner, L. Klingler, *Special quasi-triads and integral group rings of finite representation type I,* J. Algebra, **158**, 1993, pp.279-322.

[18] J. Haefner, L. Klingler, *Special quasi-triads and integral group rings of finite representation type II,* J. Algebra, **158**, 1993, pp.323-374.

[19] V. V. Kirichenko, *Classification of the pairs of mutually annihilating operators in a graded space and representations of a dyad of generalized uniserial algebra,* In: Rings an Linear Group, Zap, Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **75**, 1978, pp.91-109 and pp.196-197 (in Russian).

[20] L. Klingler, *Integral representation of groups of square-free order,* J. Algebra, **129**, 1990, pp.26-74.

[21] I. Kaplansky, *Modules over Dedekind rings and valuation rings,* Trans. Amer. Math. Soc., 1952, pp.327-340.

[22] R. Kielpiński, *On* $\Gamma$*-pure-injective modules,* Bull. Acad. Sci. Math., **15**, 1967, pp.127-131.

[23] R. Kielpiński, D. Simson, *On pure homological dimension,* Bull. Acad. Sci. Math., **23**, 1975, pp.1-6.

[24] L. S. Levy, *Modules over pullbacks and subdirect sums,* J. Algebra, **71**, 1981, pp.50-61.

[25] L. S. Levy, *Modules over Dedekind-like rings,* J. Algebra, **93**, 1985, pp.1-116.

[26] L. S. Levy, *Mixed modules over ZG, G, cyclic of prime order, and over related Dedekind pullbacks,* J. Algebra, **71**, 1981, pp.62-114.

[27] R. L. MacCasland, M. E. Moore, P. F. Smith, *On the spectrum of a module over a commutative ring,* Comm. Algebra, **25**, 1997, 79-103.

[28] M. Moore, S. J. Smith, *Prime and radical submodules of modules over commutative rings,* Comm. Algebra, **10**, 2002, pp.5037-5064.

[29] L. A. Nazarova, A. V. Roiter, *Finitely generated modules over a dyad of local Dedekind rings and finite groups having an abelian normal subgroup of index,* p. Izv. Acad. Nauk. SSSR, **33**, 1969, pp.65-69.

[30] Sh. Payrovi, S. Babaei, *On 2-absorbing submodules,* Algebra Colloq., **19(1)**, 2012, pp.913-920.

[31] J. A. dela Pena, D. Simson, *Projective modules, reflection functors, quadraic forms and Auslander-Reiten sequences,* Trans. Amer. Math. Soc., **329**, 1992, pp.733-753.

[32] M. Prest, *Model Theory and Modules,* Cambridge: London Mathematical Society, Cambridge University Press, 1988.

[33] M. Prest, *Ziegler spectrum of tame hereditary algebras,* J. Algebra, **207**, 1998, pp.146-164.

[34] D. Simson, *Linear representations of partially ordered sets and vector space categories,* Algebra, Logic and Applications, Vol. **4**. Switzerland-Australia: Gordon and Breach Science Publisher, 1992.

[35] D. Simson, *Prinjective modules, propartite modules, representations of bocses and lattices over orders,* J. Math. Soc. Japan, **49**, 1997, pp.31-68.

[36] D. Simson, *On Corner type Endo-Wild algebras,* J. Pure and Applied Algebra, **202**, 2005, pp.118-132.

[37] D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras,* Vol. **3**. Representation-Infinite Tilted Algebras, London Math. Soc. Student Texts 72. Cambridge-New York: Cambridge Univ. Press, 2007.

[38] R. B. Warfield, *Purity and algebraic compactness for modules,* Pacific J. Math., **28**, 1969, pp.699-719.

[39] A. N. Wiseman, *Projective modules over pullback rings,* Math. Proc. Cambridge Philos. Soc., **97**, 1985, pp.399-406.

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