Binary matroids that classify forests

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ABSTRACT. Elementary arguments show that a tree or forest is determined (up to isomorphism) by binary matroids defined using the adjacency matrix.

1. Introduction

Let G be a simple graph with $|V(G)| = n$. The adjacency matrix of G is an $n \times n$ matrix $A(G)$, whose entries lie in the two-element field $GF(2)$. We consider two matroids introduced in [10], which are defined using $A(G)$.

Definition 1. The *restricted isotropic matroid* of G is the binary matroid represented by the $n \times 2n$ matrix

$$IA(G) = \begin{pmatrix} I_n & A(G) \end{pmatrix},$$

where I_n is the identity matrix of size n. The matroid is denoted $M[IA(G)]$.

Definition 2. The *isotropic matroid* of G is the binary matroid represented by the $n \times 3n$ matrix

$$IAS(G) = \begin{pmatrix} I_n & A(G) & I_n + A(G) \end{pmatrix},$$

where I_n is the identity matrix of size n. The matroid is denoted $M[IAS(G)]$.

As explained in [6, 10], the names of these matroids reflect their connection with Bouchet's theories of delta-matroids and isotropic systems [1–3, 5]. It is not the purpose of the present paper to give an account of this connection. Rather, our purpose is to present a direct, elementary proof of the following.

Theorem 3. *Isotropic matroids and restricted isotropic matroids are classifying invariants for forests. That is, if F and F' are forests then any one of these three statements implies the other two:*
1) *The forests F and F' are isomorphic.*
2) *The restricted isotropic matroids $M[IA(F)]$ and $M[IA(F')]$ are isomorphic.*
3) *The isotropic matroids $M[IAS(F)]$ and $M[IAS(F')]$ are isomorphic.*

The implications $1 \implies 2$ and $1 \implies 3$ of Theorem 3 are obvious. The converses may be deduced from more general results that have appeared in the literature, as follows. Oum [8, Cor. 3.5] showed that for two bipartite graphs, a condition equivalent to isomorphism of the restricted isotropic matroids implies that the graphs are related through pivoting, and hence also related through local complementation. Bouchet [4] used the split decomposition of Cunningham [7] to show that forests related through local complementation are isomorphic. The implication $2 \implies 1$ follows from these two results. The implication $3 \implies 1$ follows from the same result of Bouchet [4], using the fact that graphs with isomorphic isotropic matroids are related through local complementation [10]. A different way to deduce $3 \implies 1$ from the results of [10] was given in [6].

In the following pages we provide a self-contained exposition of Theorem 3, which requires only elementary ideas of linear algebra and matroid theory. Before proceeding, we should thank Robert Brijder for his long collaboration and friendship, which have provided many instances of inspiration and understanding.

2. Binary matroids

In this section we briefly recall some fundamental elements of the theory of binary matroids. We refer the reader to Oxley [9] for a thorough discussion of matroid theory.

Let B be a matrix with entries in $GF(2)$, the field of two elements. The *binary matroid $M[B]$* represented by B has an element for each column of B. A nonempty subset $D \subseteq M[B]$ is *dependent* if the corresponding columns of B are linearly dependent; a minimal dependent set is a *circuit*

of $M[B]$. An element x of $M[B]$ is a *loop* if $\{x\}$ is dependent; that is, every entry of the corresponding column of B is 0. An element x of $M[B]$ is a *coloop* if no circuit includes x; that is, the corresponding column of B is not included in the linear span of the other columns. Two elements x and y are *parallel* if $\{x, y\}$ is a circuit; that is, the corresponding columns of B are nonzero and identical. A *basis* of $M[B]$ is a subset that corresponds to a basis of the column space of B.

If B' is another matrix with entries in $GF(2)$, then a bijection between the matroids $M[B]$ and $M[B']$ is an *isomorphism* iff it matches dependent sets to dependent sets. In general, there will be many different matrices representing matroids isomorphic to $M[B]$. Here is a simple way to produce some of them.

Lemma 4. *Suppose a particular entry of B, say b_{ij}, is equal to 1. Let κ be any column vector of the same size as the columns of B, whose ith entry is 0. Let B' be the matrix obtained from B by adding κ to each column of B that has a 1 in the ith row. Then $M[B] \cong M[B']$.*

Proof. A set of columns of B sums to 0 if and only if the corresponding set of columns of B' sums to 0. □

If $S \subseteq M[B]$ then S is made into a *submatroid* of $M[B]$ by defining a subset of S to be dependent in S if and only if it is dependent in $M[B]$. The submatroid is also said to be obtained from $M[B]$ by *deleting* the subset $M[B] \setminus S$. The submatroid is denoted $M[B]|S$ or $M[B] \setminus (M[B] \setminus S)$. If B_S is the submatrix of B consisting of columns corresponding to elements of S then it is easy to see that $M[B_S] = M[B]|S$.

Circuits are used to define an equivalence relation $\sim$ on $M[B]$, as follows: every element x has $x \sim x$; if x and y are two elements of one circuit then $x \sim y$; and if $x \sim y \sim z$ then $x \sim z$. The equivalence classes of $\sim$ are the *components* of $M[B]$. Components of cardinality 1 are singletons $\{x\}$, where x is a loop or coloop. As the circuits of a submatroid of $M[B]$ are also circuits of $M[B]$ itself, every component of a submatroid of $M[B]$ is contained in a component of $M[B]$. In general, the components are not the same; a component of $M[B]$ may contain several different components of a given submatroid. In particular, a coloop of a submatroid need not be a coloop of $M[B]$.

If $x \in M[B]$ then the *contraction* $M[B]/x$ is a matroid on the set $M[B] \setminus \{x\}$. A subset $D \subseteq M[B] \setminus \{x\}$ is defined to be dependent in $M[B]/x$ if and only if $D \cup \{x\}$ is dependent in $M[B]$. It is not immediately evident, but it does turn out that $M[B]/x$ is represented by a matrix.

To see why, notice first that we may presume that the column of B representing x has no more than one nonzero entry. If x is a loop, then of course the entries in the x column are all 0. Otherwise, if b_{ij} is a nonzero entry of the column representing x, and κ is the column vector obtained from the jth column of B by changing the ith entry to 0, then according to Lemma 4 we may add κ to every column of B with a 1 in the ith row, without changing the matroid $M[B]$. The effect on the jth column is to change every entry other than b_{ij} to 0.

If x is a loop, then a matrix representing $M[B]/x$ is obtained by removing the x column of B. If x is not a loop, the following proposition applies.

Proposition 5. *Suppose x corresponds to the jth column of B, and b_{ij} is the only nonzero entry of this column. Let B' be the submatrix of B obtained by removing both the ith row and the jth column of B. Then $M[B'] \cong M[B]/x$.*

Proof. If S is a set of columns of B' then the sum of S is 0 if and only if the sum of the corresponding columns of B is equal to either 0 or the jth column of B. Either way, we get a dependent set of columns of B by including the jth column along with the columns corresponding to elements of S. $\square$

Suppose B is an $m \times n$ matrix with entries in $GF(2)$, of rank r. If $r = 0$ or $r = n$ then the *standard representation* of $M[B]$ is $M[0_n]$ or $M[I_n]$, where 0_n and I_n are the $n \times n$ zero and identity matrices. If $r \in \{1, \ldots, n-1\}$, choose r columns of B that constitute a basis of the column space of B. Permuting columns, we may presume these r columns are the first r columns of B. Then the linear relations that hold among the columns of B are precisely the same as the linear relations that hold among the columns of the matrix $\begin{pmatrix} I_r & A \end{pmatrix}$, where I_r is the $r \times r$ identity matrix and A is the matrix whose columns record the coefficients in formulas for the $(r+1)$st, $\ldots, n$th columns of B as linear combinations of the first r columns. As the columns of B and $\begin{pmatrix} I_r & A \end{pmatrix}$ satisfy the same linear relations, it must be that $M[B] \cong M[\begin{pmatrix} I_r & A \end{pmatrix}]$. A matrix of the form $\begin{pmatrix} I_r & A \end{pmatrix}$ is a *standard representation* of the matroid $M[B]$.

The *dual* of $M[B]$ is a matroid on the same ground set as $M[B]$, whose bases are the complements of bases of $M[B]$. The dual of $M[B]$ is denoted $M[B]^*$. The duals of binary matroids are themselves binary matroids:

Proposition 6. *Given an $r \times n$ standard representation $M[(I_r \quad A)]$ of a binary matroid M of rank $r \in \{1, \ldots, n-1\}$, the dual matroid M^* is the binary matroid $M[(A^T \quad I_{n-r})]$, where A^T is the transpose of A.*

Proof. The proposition is equivalent to the claim that for any subset $S \subseteq \{1, \ldots, n\}$ of size r, the columns of $(I_r \quad A)$ with indices from S are linearly independent if and only if the columns of $(A^T \quad I_{n-r})$ with indices not from S are linearly independent.

Suppose S intersects both $\{1, \ldots, r\}$ and $\{r+1, \ldots, n\}$. Then after permuting the rows and columns of $(I_r \quad A)$, we may assume that $S = \{s+1, \ldots, s+r\}$ where $1 \leqslant s \leqslant r-1$. Let the submatrix of $(I_r \quad A)$ with column indices from S be

$$\begin{pmatrix} 0 & A_S \\ I_{r-s} & A' \end{pmatrix}.$$

The columns of this matrix are independent if and only if the $s \times s$ submatrix A_S is nonsingular. If $s + r = n$ then the columns of $(A^T \quad I_{n-r})$ with indices not from S constitute the submatrix A_S^T, which is nonsingular if and only if A_S is nonsingular.

If $s + r < n$, let

$$(I_r \quad A) = \begin{pmatrix} I_s & 0 & A_S & A'' \\ 0 & I_{r-s} & A' & A''' \end{pmatrix}.$$

Then

$$(A^T \quad I_{n-r}) = \begin{pmatrix} A_S^T & (A')^T & I_s & 0 \\ (A'')^T & (A''')^T & 0 & I_{n-r-s} \end{pmatrix},$$

so the columns of $(A^T \quad I_{n-r})$ with indices not from S constitute the submatrix

$$\begin{pmatrix} A_S^T & 0 \\ (A'')^T & I_{n-r-s} \end{pmatrix},$$

which is nonsingular if and only if A_S is nonsingular.

If S does not intersect $\{1, \ldots, r\}$ then a similar argument applies, without A' and A''. If S does not intersect $\{r+1, \ldots, n\}$ then the columns of $(I_r \quad A)$ with indices from S are the columns of I_r, and the columns $(A^T \quad I_{n-r})$ with indices not from S are the columns of I_{n-r}. $\qquad\square$

We emphasize that a matroid and its dual are defined on the same ground set. When two matroids on different ground sets are said to be duals of each other, the relationship between them must involve a specified bijection between their ground sets.

We should also mention that the special cases $r = 0$ and $r = n$ are excluded from Proposition 6; this is done merely for consistency of notation. When $r = 0$, the matroid $M = M[0_n]$ has the dual $M[I_n]$. And when $r = n$, $M = M[I_n]$ has the dual $M[0_n]$.

3. Isotropic matroids of forests

If F is a forest then we use the following notation for the columns of the matrices $IA(F)$ and $IAS(F)$, and for the corresponding elements of the matroids $M[IA(F)]$ and $M[IAS(F)]$. If $v \in V(F)$ the v column of $A(F)$ is denoted $\chi(v)$, or $\chi_F(v)$ if it is necessary to specify F. The column of the identity matrix I_n with a 1 in the row of $IA(F)$ that contains the v row of $A(F)$ is denoted $\phi(v)$, or $\phi_F(v)$. The column of $I_n + A(F)$ corresponding to v is denoted $\psi(v)$, or $\psi_F(v)$.

Some simple observations are given in Proposition 7 below. Recall that a leaf in a forest is a vertex with precisely one neighbor. For clarity, we refer to the connected components of a forest F, and the components of the matroids $M[IA(F)]$ and $M[IAS(F)]$.

Proposition 7. *If F is a forest, $M[IA(F)]$ and $M[IAS(F)]$ have the following properties.*

1) *$M[IA(F)]$ is a submatroid of $M[IAS(F)]$.*
2) *If v is isolated, then $\chi(v)$ is a loop. There are no other loops in $M[IAS(F)]$.*
3) *If v is isolated, then $\phi(v)$ is a coloop in $M[IA(F)]$. Also $\phi(v)$ and $\psi(v)$ are parallel in $M[IAS(F)]$, and $\{\phi(v), \psi(v)\}$ is a component of $M[IAS(F)]$.*
4) *If v is a leaf adjacent to w, then $\chi(v)$ and $\phi(w)$ are parallel.*
5) *If v_1 and v_2 are two leaves adjacent to w, then $\chi(v_1)$ and $\chi(v_2)$ are parallel.*
6) *If a connected component of F has precisely two vertices v and w, then $\psi(v)$ and $\psi(w)$ are parallel in $M[IAS(F)]$.*
7) *All parallels in $M[IAS(F)]$ fall under one of the above descriptions.*

Proof. Only item 7 is not obvious. Suppose x and y are parallel in $M[IAS(F)]$. Then the x and y columns of $IAS(F)$ are nonzero and identical.

If the two columns have more than one nonzero entry, they are not ϕ columns. If they are both χ columns, the two corresponding vertices of F share two neighbors; this is impossible in a forest. If one is a χ column and the other is a ψ column then the corresponding vertices of F

are neighbors, with a shared neighbor; again, this is impossible. If both columns are ψ columns and they have more than two nonzero entries, then again, the two corresponding vertices are neighbors with a shared neighbor, an impossibility. If both columns are ψ columns and they have only two nonzero entries, then they fall under item 6.

If the two columns have only one nonzero entry, then they cannot both be ϕ columns, and it cannot be that one is a χ column and the other is a ψ column. All the other ϕ, χ, ψ combinations are covered in items 3–6. $\square$

Here are two obvious lemmas.

Lemma 8. *If $v \in V(F)$ then $\{\phi(v), \chi(v), \psi(v)\}$ is a dependent set in $M[IAS(F)]$. We call it the* vertex triple *of v. It is a circuit of $M[IAS(F)]$ if and only if v is not an isolated vertex.*

Lemma 9. *If $v \in V(F)$ then $\{\chi(v)\} \cup \{\phi(w) \mid vw \in E(F)\}$ is a circuit in $M[IA(F)]$. We call it the* neighborhood circuit *of v.*

Neighborhood circuits and vertex triples determine the components of $M[IA(F)]$ and $M[IAS(F)]$.

Proposition 10. *For each connected component C of F, the matroid $M[IA(F)]$ has two components. For each $v \in V(F)$, each of the matroid components contains precisely one of $\phi(v), \chi(v)$. If the two matroid components are thought of as matroids on the set $V(C)$, they are duals of each other. On the other hand, $M[IAS(F)]$ has two components for each isolated vertex of F, and one component for each connected component of F with more than one vertex.*

Proof. Let $v \in V(F)$. If v is isolated then items 2 and 3 of Proposition 7 tell us that the vertex triple of v is the union of two components of $M[IAS(F)]$, and $\{\phi(v), \chi(v)\}$ is the union of two components of $M[IA(F)]$. These two components are duals because $\phi(v)$ is a coloop of $M[IA(F)]$, and $\chi(v)$ is a loop.

Suppose v is not isolated. The vertex triple of v and the neighborhood circuit of v are both circuits of $M[IAS(F)]$, so one component of $M[IAS(F)]$ contains the vertex triple of v and also contains $\phi(w)$ for every neighbor w of v. Iterating this observation, we conclude that one component of $M[IAS(F)]$ contains the vertex triple of w for every vertex w in the connected component of F that contains v.

In contrast, vertex triples are not circuits of $M[IA(F)]$. Using the neighborhood circuits, we see that there are two components of $M[IA(F)]$

associated with v. One component includes

$$\{\phi(w) \mid d(v,w) \text{ is finite and even}\} \cup \{\chi(w) \mid d(v,w) \text{ is finite and odd}\},$$

and the other component includes

$$\{\chi(w) \mid d(v,w) \text{ is finite and even}\} \cup \{\phi(w) \mid d(v,w) \text{ is finite and odd}\}.$$

Here $d(v,w)$ is the distance from v to w in F, i.e. the number of edges in a path from v to w, if such a path exists, and ∞ otherwise. (Note that $d(v,v) = 0$ is finite and even.) The two matroid components are naturally represented by matrices of the form $\begin{pmatrix} I_r & A_1 \end{pmatrix}$ and $\begin{pmatrix} A_1^T & I_{|V(C)|-r} \end{pmatrix}$, where

$$\begin{pmatrix} 0 & A_1 \\ A_1^T & 0 \end{pmatrix}$$

is the adjacency matrix of the connected component C containing v. Proposition 6 tells us that the two matroid components are duals of each other, when they are thought of as matroids on $\{1, \ldots, |V(C)|\}$ according to the ordering of the columns in $\begin{pmatrix} I_r & A_1 \end{pmatrix}$ and $\begin{pmatrix} A_1^T & I_{|V(C)|-r} \end{pmatrix}$. $\square$

4.   Examples

In this section we mention some counterexamples to assertions related to Theorem 3.

4.1.   Graphs that are not forests

Let C_3 be the cycle on three vertices, and P_3 the path on three vertices. Then

$$IAS(C_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Let κ be the column vector

$$\kappa = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

If we add κ to each column of $IAS(C_3)$ with a 1 in the second row, we obtain the matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

According to Lemma 4, $M[IAS(C_3)] \cong M[B]$. On the other hand, the columns of B are the same as the columns of

$$IAS(P_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix},$$

in a different order. It follows that $M[IAS(C_3)] \cong M[B] \cong M[IAS(P_3)]$. We see that the implication $3 \implies 1$ of Theorem 3 does not hold for graphs that are not forests.

The restricted isotropic matroid $M[IA(C_3)]$ has only one component, so it illustrates the fact that Proposition 10 does not hold for graphs that are not forests.

Now, let C_4 and P_4 be the cycle and path on four vertices. Then

$$IA(P_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let κ_1 be the first column of $IA(P_4)$, and κ_4 the fourth column. Let B be the matrix obtained from $IA(P_4)$ by adding κ_1 to every column with a 1 in the third row, and κ_4 to every column with a 1 in the second row. Then

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

has the same columns as

$$IA(C_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix},$$

so Lemma 4 tells us that $M[IA(P_4)] \cong M[B] \cong M[IA(C_4)]$. We see that the implication $2 \implies 1$ of Theorem 3 does not hold for graphs that are not forests.

4.2. Strange isomorphisms of isotropic matroids

Theorem 3 asserts that if there is an isomorphism between the (restricted) isotropic matroids of two forests, then there is also an isomorphism between the forests themselves. The theorem does not assert that

a matroid isomorphism gives rise directly to an isomorphism between the forests. Generally speaking, in fact, a matroid isomorphism need not be directly related to any particular graph isomorphism.

One way to appreciate this fact is to observe that the matrix $IA(P_3)$ has three identical columns, and three other columns that constitute a circuit of size three in $M[IA(P_3)]$. In each case, the three corresponding matroid elements are indistinguishable, up to automorphism. It follows that the automorphism group of $M[IA(P_3)]$ is isomorphic to $S_3 \times S_3$, where S_3 is the symmetric group. In contrast, the graph P_3 has only two automorphisms.

Another relevant example is P_4. Here is $IAS(P_4)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Produce a matrix B from $IAS(P_4)$ in three steps: interchange the first and fourth rows, add the second column to each column with a nonzero entry in the fourth row, and add the third column to each column with a nonzero entry in the first row. That is,

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Permuting the rows of a matrix has no effect on the matroid represented by the matrix, and the other two steps in the production of B fall under Lemma 4, so B represents the matroid $M[IAS(P_4)]$, with the columns of B representing matroid elements in the same order as the columns of $IAS(P_4)$. The matroid elements are listed, in order, in the first row of the matrix displayed below. (Here $V(P_4) = \{v_1, v_2, v_3, v_4\}$, with the vertices indexed in order along the path, and ϕ_i, χ_i, ψ_i appear in the matrix as abbreviations for $\phi(v_i), \chi(v_i), \psi(v_i)$.) On the other hand, the columns of B are equal to columns of $IAS(P_4)$, as listed in the second row of the matrix displayed below.

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \chi_3 & \chi_1 & \chi_4 & \chi_2 & \phi_2 & \psi_3 & \psi_2 & \phi_3 & \phi_4 & \psi_4 & \psi_1 & \phi_1 \end{pmatrix}$$

It follows that the 2×12 matrix displayed above represents an automorphism f of $M[IAS(P_4)]$. Theorem 3 is satisfied, of course, as P_4

is isomorphic to itself. But neither automorphism of P_4 appears to be associated in a direct way with f.

In the proof of Theorem 3, the lack of a direct relationship between arbitrary graph and matroid isomorphisms is managed by refining the statement of Theorem 3 to require some agreement between the two types of isomorphisms. See Theorems 13 and 24 below.

5.   Part 1 of the proof

In this section we prove the implication $2 \implies 1$ of Theorem 3.

Lemma 11. *Suppose v is a leaf of a forest F, and w is its only neighbor. Then both of the transpositions $(\phi(v)\chi(w))$ and $(\chi(v)\phi(w))$, in cycle notation, are automorphisms of $M[IA(F)]$.*

Proof. The transposition $(\chi(v)\phi(w))$ is an automorphism of $M[IA(F)]$ because the $\chi(v)$ and $\phi(w)$ columns of $IA(F)$ are equal.

The $\phi(v)$ and $\chi(w)$ columns of $IA(F)$ are the only columns with nonzero entries in the v row. Let κ be the sum of these two columns. Then κ is a column vector with a 0 in the v row, so according to Lemma 4, adding κ to the $\phi(v)$ and $\chi(w)$ columns produces another matrix representing the same matroid. Adding κ to these columns has the same effect as interchanging them, so the transposition $(\phi(v)\chi(w))$ is an automorphism of $M[IA(F)]$. $\square$

Lemma 12. *Let v be a leaf of a forest F, and w its unique neighbor. If M is the component of $M[IA(F)]$ that contains $\chi(v)$, then $M \setminus \chi(v)$ is isomorphic to the component of $M[IA(F \setminus v)]$ that contains $\phi(w)$. The isomorphism is natural, in that the image of each element $\phi(x)$ or $\chi(x)$ from $M \setminus \chi(v)$ is the element $\phi(x)$ or $\chi(x)$ from $M[IA(F \setminus v)]$.*

Proof. The matroid M is represented by the submatrix of $IA(F)$ consisting of the $\phi(x)$ columns such that the distance between w and x is finite and even, and the $\chi(x)$ columns such that the distance between w and x is finite and odd. Every entry of the v row of this submatrix is 0, so the v row may be deleted without affecting the matroid represented by the matrix. If we then remove the $\chi(v)$ column, we obtain the submatrix of $IA(F \setminus v)$ that represents the component of $M[IA(F \setminus v)]$ containing $\phi(w)$. $\square$

Theorem 13. *Suppose T and T' are trees, and there is a matroid isomorphism f between a component of $M[IA(T)]$ and a component of $M[IA(T')]$. Then there is a graph isomorphism $g : T \to T'$, with $g(v) = v'$ whenever $f(\phi(v)) = \phi(v')$ or $f(\chi(v)) = \chi(v')$.*

Proof. Suppose a component of $M[IA(T)]$ is isomorphic to a component of $M[IA(T')]$. According to Proposition 10, it follows that T and T' have the same number n of vertices. If $n = 1$, the theorem follows immediately.

We proceed using induction on $n > 1$. Suppose a component of $M[IA(T)]$ is isomorphic to a component of $M[IA(T')]$. According to Proposition 10, each of the matroids $M[IA(T)], M[IA(T')]$ has two components, and the components are duals of each other. It follows that each component of $M[IA(T)]$ is isomorphic to a component of $M[IA(T')]$.

Let v be a leaf of T, and w its unique neighbor. Let M be the component of $M[IA(T)]$ that contains the parallel pair $\{\chi(v), \phi(w)\}$. Let $f : M \to M'$ be an isomorphism between M and a component M' of $M[IA(T')]$. Of course it follows that $\{f(\chi(v)), f(\phi(w))\}$ is a parallel pair in M'.

According to Proposition 7, the parallel pair $\{f(\chi(v)), f(\phi(w))\}$ includes $\chi(v')$ for some leaf v' of T'. The other element of $\{f(\chi(v)), f(\phi(w))\}$ might be $\chi(v'')$ for some leaf $v'' \neq v'$ which shares the unique neighbor w' of v'. If it is, Lemma 11 tells us that the transposition $(\chi(v'')\phi(w'))$ defines an automorphism of $M[IA(T')]$, and hence of M'. Composing this automorphism with f, if necessary, we may assume that $\{f(\chi(v)), f(\phi(w))\} = \{\chi(v'), \phi(w')\}$. Lemma 11 also tells us that $(\chi(v')\phi(w'))$ defines an automorphism of M'; composing this automorphism with f, if necessary, we may assume that $f(\chi(v)) = \chi(v')$ and $f(\phi(w)) = \phi(w')$.

Then f defines an isomorphism $g : M \setminus \chi(v) \to M' \setminus \chi(v')$ by restriction. The image of $\phi(w)$ under g is $f(\phi(w)) = \phi(w')$. According to Lemma 12, g defines a matroid isomorphism between a component of $M[IA(T \setminus v)]$ and a component of $M[IA(T' \setminus v')]$. The inductive hypothesis tells us that there is a graph isomorphism $h : T \setminus v \to T' \setminus v'$, with $h(w) = w'$. Attaching v to w and v' to w', we see that h extends to an isomorphism between T and T'. □

Now, suppose F and F' are forests such that $M[IA(F)] \cong M[IA(F')]$. Then the two matroids have the same cardinality, so the two forests have the same number n of vertices. If $n = 1$, then of course the two forests are isomorphic.

The argument proceeds by induction on $n > 1$. According to Proposition 10, each $M[IA]$ matroid has twice as many components as the corresponding forest has connected components. If each matroid has only two components, the two forests are trees and Theorem 13 applies. If each matroid has more than two components, choose a component M in $M[IA(F)]$. Under an isomorphism $M[IA(F)] \cong M[IA(F')]$, M corresponds to an isomorphic component of $M[IA(F')]$. Proposition 10 and

Theorem 13 tell us that the corresponding connected components of F and F' are isomorphic trees. After removing these isomorphic trees from F and F', and deleting a component isomorphic to M and a component isomorphic to M^* from each of $M[IA(F)]$ and $M[IA(F')]$, we may apply the inductive hypothesis to the remaining submatroids.

6. Part 2 of the proof

The proof of the implication $3 \implies 1$ of Theorem 3 is a bit longer than the proof of $2 \implies 1$.

Lemma 14. *Let v be a leaf in a forest F, let w be the unique neighbor of v, and let $\beta_{vw} : M[IAS(F)] \to M[IAS(F)]$ be the permutation*

$$\beta_{vw} = (\phi(v)\psi(w))(\chi(v)\phi(w))(\psi(v)\chi(w)),$$

in cycle notation. Then β_{vw} is a matroid automorphism of $M[IAS(F)]$.

Proof. Let κ be the column vector with a 1 in the x row whenever x is either w or a neighbor of w other than v.

The v entry of κ is 0, so according to Lemma 4, a new matrix representing $M[IAS(F)]$ can be obtained by adding κ to every column of $IAS(F)$ whose entry in the v row is 1. The only columns of $IAS(F)$ with nonzero entries in the v row are $\phi(v), \psi(v), \chi(w)$ and $\psi(w)$. For each of the four, the effect of adding κ is the same as the effect of applying the permutation β_{vw}.

The lemma follows, because the columns of $IAS(F)$ corresponding to $\chi(v)$ and $\phi(w)$ are equal. $\square$

Here is a similar observation.

Lemma 15. *Let v be a leaf in a forest F, and let w be the unique neighbor of v. Then the permutation $\gamma_{vw} = (\phi(v)\psi(v))(\chi(w)\psi(w))$ is an automorphism of $M[IAS(F)]$.*

Proof. Let $\kappa = \phi(w)$, and apply Lemma 4 to all the columns of $IAS(F)$ with nonzero entries in the v row. $\square$

Definition 16. A *triangulation* of a matroid is a partition of the matroid into 3-element subsets, each of which is either a circuit or a union of two circuits. Two triangulations are *equivalent* if one is the image of the other under an automorphism of the matroid.

The 3-element subsets included in a triangulation $\mathfrak{T}$ are the *triples* of $\mathfrak{T}$. The *vertex triangulation* of the isotropic matroid of a graph is the triangulation whose triples are vertex triples. In general, an isotropic matroid will have many different triangulations. For instance, if x and y are parallel in $M[IAS(F)]$, then the transposition (xy) is an automorphism of $M[IAS(F)]$, and the image under (xy) of a triangulation $\mathfrak{T}$ may well be different from $\mathfrak{T}$. We call this kind of automorphism a *parallel swap*.

Definition 17. Two triangulations of a matroid are *equivalent through parallel swaps*, or *ps-equivalent*, if one can be obtained from the other through a sequence of parallel swaps.

Lemma 18. *Let T be a tree with four or more vertices, and let τ be a 3-element circuit of $M[IAS(T)]$, which cannot be changed into a vertex triple using parallel swaps. Then T has a degree-2 vertex z, with neighbors x and y, such that parallel swaps change τ into either $\{\phi(x), \phi(y), \chi(z)\}$ or $\{\psi(x), \phi(y), \psi(z)\}$. In the latter case, x is a leaf of T.*

Proof. Suppose $\tau = \{\gamma(x), \delta(y), \epsilon(z)\}$, where $\gamma, \delta, \epsilon \in \{\phi, \chi, \psi\}$ and $x, y, z \in V(T)$. If two of x, y, z are equal, then two elements of τ come from the corresponding vertex triple. It follows that the third element of τ is parallel to the third element of the vertex triple, contradicting the hypothesis that τ cannot be changed into a vertex triple by parallel swaps. Hence $x \neq y \neq z \neq x$.

Every element of τ corresponds to a column of $IAS(T)$. As τ is a circuit in $M[IAS(T)]$, each nonzero entry in one of the three columns is matched by a nonzero entry in precisely one of the other two columns.

Suppose every two of these three columns share a nonzero entry; then each column has at least two nonzero entries, as the three columns do not have a common nonzero entry, so $\gamma, \delta, \epsilon \in \{\chi, \psi\}$. As T is a tree, x, y, z cannot all be neighbors. Suppose that two pairs of x, y, z are neighbors, and the third pair is not; say x and y are not neighbors. As T is a tree, the neighbors x and z cannot share a neighbor; and the same for y and z. If z has a neighbor other than x or y, we contradict the fact that each nonzero entry of $\epsilon(z)$ must be matched in $\gamma(x)$ or $\delta(y)$; hence the degree of z is 2. If x has a neighbor v other than z, the nonzero entry of $\gamma(x)$ in the v row must be matched in $\delta(y)$, as it is not matched in $\epsilon(z)$; but then x, v, y, z, x is a closed walk in T, an impossibility. The same contradiction arises if y has a neighbor other than z, so $V(T) = \{x, y, z\}$, contradicting the hypothesis that $|V(T)| \geqslant 4$. Suppose only one pair of x, y, z are neighbors; say x is adjacent to y, and neither neighbors z. Then each of x, y shares a

neighbor with z; but this is impossible in a tree. We conclude that none of x, y, z are adjacent, and each pair has a shared neighbor. The shared neighbors must be different, as the three columns do not have a common nonzero entry; again, this is impossible in a tree.

Renaming x, y, z if necessary, we may assume that $\gamma(x)$ and $\delta(y)$ do not share a nonzero entry. Then $\epsilon(z)$ has a nonzero entry in every row where $\gamma(x)$ or $\delta(y)$ has a nonzero entry, and nowhere else.

If $\gamma(x)$ has more than two nonzero entries, then x and z share two neighbors, an impossibility in a tree. The same holds for $\delta(y)$, so each of $\gamma(x), \delta(y)$ has one or two nonzero entries.

Suppose $\gamma(x)$ has two nonzero entries. If $\gamma = \chi$ then either x and z share two neighbors, or x and z are neighbors that share a neighbor. Neither situation is possible in a tree, so $\gamma = \psi$. As $\psi(x)$ has only two nonzero entries, x is a leaf. As $\epsilon(z)$ shares the 1 entry of $\psi(x)$ in the x row, x and z are neighbors. Hence the entry of $\psi(x)$ in the z row is 1; $\epsilon(z)$ shares this entry, so $\epsilon = \psi$. If $\delta(y)$ were to have two nonzero entries, then as $\gamma(x)$ and $\delta(y)$ do not share a nonzero entry, neither nonzero entry of $\delta(y)$ would be in the z row. Both nonzero entries of $\delta(y)$ would be matched in $\epsilon(z) = \psi(z)$, so y and z would share two neighbors, an impossibility. Hence $\delta(y)$ has only one nonzero entry, so the $\delta(y)$ column of $IAS(T)$ is equal to the $\phi(y')$ column for some vertex y'. As the nonzero entry of $\phi(y')$ is matched in $\epsilon(z) = \psi(z)$, y' is a neighbor of z. A parallel swap changes τ into $\{\psi(x), \phi(y'), \psi(z)\}$ with x a leaf, z the only neighbor of x, and y' the only other neighbor of z.

If $\delta(y)$ has two nonzero entries, interchange the names of x and y and apply the argument of the preceding paragraph.

The only remaining possibility is that each of $\gamma(x), \delta(y)$ has only one nonzero entry. Using parallel swaps, we can change τ into $\tau' = \{\phi(x'), \phi(y'), \epsilon(z)\}$. If $\epsilon(z) = \psi(z)$ then one of x', y' is z, so τ' contains two elements of the vertex triple of z; a parallel swap then changes τ' into the vertex triple of z, contrary to hypothesis. It follows that $\epsilon(z) = \chi(z)$, and hence x' and y' are the only neighbors of z. $\square$

As mentioned in the introduction, the isotropic matroids of general graphs detect only equivalence under local complementation, not equivalence under isomorphism. A structural reflection of the special properties of isotropic matroids of forests is the fact that triangulations not ps-equivalent to vertex triangulations are not unusual for general graphs, as discussed in [10], but they are quite rare for forests.

Theorem 19. *Let T be a tree whose isotropic matroid has a triangulation that is not ps-equivalent to the vertex triangulation. Then T is isomorphic to the four-vertex path, P_4.*

Proof. It is easy to see that if T is a tree on one or two vertices, then $M[IAS(T)]$ does not have a triangulation that is not ps-equivalent to the vertex triangulation.

Up to isomorphism, there is only one tree on three vertices, the path P_3. If there were a triangulation of $M[IAS(P_3)]$ not ps-equivalent to the vertex triangulation, it would include a triple that meets all three vertex triples. Therefore, the triangulation would not include any vertex triple; every triple would meet all three vertex triples. If v is one of the leaves of P_3 and w is the central vertex, then one triple of the triangulation would include $\phi(v)$ and one of $\chi(w), \psi(w)$, because these are the only elements available to cancel the nonzero entry of $\phi(v)$ in the v row. For the same reason, a triple would include $\psi(v)$ and one of $\chi(w), \psi(w)$. It follows that the triple containing the remaining element of the vertex triple of v, $\chi(v)$, would also contain the remaining element of the vertex triple of w, $\phi(w)$. But this is impossible: the matrix $IAS(P_3)$ does not have a column of zeroes, so there is no way to complete this triple. Hence $M[IAS(P_3)]$ does not have a triangulation that is not ps-equivalent to the vertex triangulation.

Suppose T has four or more vertices and we are given a triangulation $\mathfrak{T}$ of $M[IAS(T)]$, which is not ps-equivalent to the vertex triangulation, and has the smallest possible number of triples that are not vertex triples. Let τ be a non-vertex triple included in $\mathfrak{T}$.

Suppose τ can be changed into a vertex triple using parallel swaps. According to Proposition 7, no ψ element of $M[IAS(T)]$ is parallel to any other element of $M[IAS(T)]$. Hence there is a $v \in V(T)$ with $\psi(v) \in \tau$, and parallel swaps can be used to change τ into the vertex triple of v. The element(s) to be swapped into τ, $\phi(v)$ and/or $\chi(v)$, must appear in one or two other triples of $\mathfrak{T}$. Neither of these other triples could be a vertex triple, as $\psi(v)$ appears in τ. Therefore the parallel swap(s) that change τ into the vertex triple of v also produce a triangulation ps-equivalent to $\mathfrak{T}$, which has a smaller number of non-vertex triples. This contradicts our choice of $\mathfrak{T}$.

According to Lemma 18, we may perform parallel swaps to change τ into $\tau' = \{\phi(x), \phi(y), \chi(z)\}$ or $\widetilde{\tau} = \{\psi(x), \phi(y), \psi(z)\}$, where x, y, z are three distinct vertices of T, and x and y are the only neighbors of z. These parallel swaps transform $\mathfrak{T}$ into a triangulation $\mathfrak{T}'$ or $\widetilde{\mathfrak{T}}$.

If the second of these possibilities occurs, then Lemma 18 tells us that x is a leaf. The triple $\widetilde{\tau}' \in \widetilde{\mathfrak{T}}$ that contains $\phi(x)$ must also contain $\chi(z)$, as these are the only two columns of $IAS(T)$ that are not included in $\widetilde{\tau}$ and have nonzero entries in the x row. In order to complete a dependent set, the third element of $\widetilde{\tau}'$ must have only one nonzero entry, in the y row. Therefore a parallel swap changes $\widetilde{\tau}'$ into $\{\phi(x), \phi(y), \chi(z)\}$.

We conclude that no matter which of the two possibilities mentioned in the paragraph before last occurs, parallel swaps can be used to change $\mathfrak{T}$ into a triangulation $\mathfrak{T}'$ that includes $\tau' = \{\phi(x), \phi(y), \chi(z)\}$, where x and y are the only neighbors of z. The rest of the argument is focused on τ' and $\mathfrak{T}'$; τ and $\widetilde{\tau}$ will not be mentioned again.

Let τ'' be the triple of $\mathfrak{T}'$ that includes $\phi(z)$. One of the other elements of τ'' must correspond to a column of $IAS(T)$ with a nonzero entry in the z row. As x and y are the only neighbors of z, this other element must be one of the following: $\chi(x), \psi(x), \chi(y), \psi(y), \psi(z)$. If $\psi(z) \in \tau''$ then the third element of τ'' is parallel to $\chi(z)$; it cannot equal $\chi(z)$ because $\chi(z) \in \tau'$. This parallel of $\chi(z)$ must correspond to a column of $IAS(T)$ with the same two nonzero entries as $\chi(z)$. But this is impossible, as the corresponding vertex of T would either neighbor both x and y, or equal one of x, y and neighbor the other; either way, the fact that x and y both neighbor z would imply that T contains a circuit. Therefore $\psi(z) \notin \tau''$. Interchanging the names of x and y if necessary, we may presume that τ'' contains $\chi(x)$ or $\psi(x)$.

Case 1. Suppose τ'' contains $\chi(x)$ along with $\phi(z)$. Then $\tau'' = \{\chi(x), \phi(z), \kappa\}$, where κ corresponds to a column of $IAS(T)$ with nonzero entries in all the rows corresponding to neighbors of x other than z, and nowhere else. If κ has more than one nonzero entry, the vertex that contributes this column to $IAS(T)$ is either a neighbor of x that shares a neighbor with x, or a vertex that shares two neighbors with x. Either way, this vertex lies on a closed walk with x, an impossibility as T is a tree. Therefore κ has only one nonzero entry. We conclude that x is of degree two; one neighbor is z and the other, w say, has $\phi(w)$ parallel to κ. In sum: T contains a path w, x, z, y; x and z are of degree 2 in T; and $\mathfrak{T}'$ includes $\tau' = \{\phi(x), \phi(y), \chi(z)\}$ and $\tau'' = \{\phi(z), \chi(x), \kappa\}$, where κ is parallel to $\phi(w)$.

The triangulation $\mathfrak{T}'$ must also include triple(s) that contain $\psi(x)$ and $\psi(z)$. If $\psi(x)$ and $\psi(z)$ were to appear together in a triple, the third element of that triple would have nonzero entries in the w and y rows; but then w and y would share a neighbor, or neighbor each other, and either way the path w, x, z, y would be contained in a closed walk in T.

This is impossible, so $\mathfrak{T}'$ includes a triple τ''' that contains $\psi(x)$ but not $\psi(z)$. Considering that the elements of τ' and τ'' are not available for τ''', the only way for τ''' to match the entry of $\psi(x)$ in the x row is for τ''' to contain $\chi(w)$ or $\psi(w)$. Similarly, to match the entry of $\psi(x)$ in the z row, τ''' must contain $\chi(y)$ or $\psi(y)$. These three elements – $\chi(w)$ or $\psi(w)$, $\psi(x)$, and $\chi(y)$ or $\psi(y)$ – will not provide a dependent triple τ''' if w has any neighbor other than x, or y has any neighbor other than z. We deduce that w, x, y and z are the only vertices in T, and T is isomorphic to P_4.

Case 2. Suppose τ'' contains $\psi(x)$ along with $\phi(z)$. Then $\tau'' = \{\psi(x), \phi(z), \kappa\}$ where κ corresponds to a column of $IAS(T)$ with nonzero entries in the x row, in all the rows corresponding to neighbors of x other than z, and nowhere else. If κ has more than two nonzero entries, the vertex that contributes the κ column to $IAS(T)$ is a neighbor of x that shares a neighbor with x. This is impossible, as T is a tree. If κ is a χ column with two nonzero entries, the same contradiction arises.

Suppose κ has only one nonzero entry. This entry must be in the x row, so the only neighbor of x is z. As $\phi(x) \in \tau'$ and $\kappa \in \tau''$, $\kappa \neq \phi(x)$. Hence $\kappa = \chi(w)$ for some neighbor w of x. But this is impossible; the only neighbor of x is z, and the entry of $\chi(z)$ in the y row is 1, not 0.

The only remaining possibility is that there is a vertex w such that $\kappa = \psi(w)$ has two nonzero entries, in the x and w rows. Then $w \neq x$, because x neighbors z and the z coordinate of $\kappa = \psi(w)$ is 0, and $w \notin \{y, z\}$, because both $\psi(y)$ and $\psi(z)$ have entries equal to 1 in the y and z rows. It follows that w is a leaf, and x is its unique neighbor. Moreover, $\psi(x)$ has nonzero entries only in the w, x and z rows, so w and z are the only neighbors of x.

We conclude that T contains a path w, x, z, y, with w a leaf and x, z of degree two. Two triples of $\mathfrak{T}'$ are $\tau' = \{\phi(x), \phi(y), \chi(z)\}$ and $\tau'' = \{\psi(x), \phi(z), \psi(w)\}$. Some other triple $\tau''' \in \mathfrak{T}'$ must contain $\chi(x)$, and τ''' must contain a second element with a nonzero entry in the w row; τ'' contains $\psi(x)$ and $\psi(w)$, so this second element of τ''' can only be $\phi(w)$. The third element of τ''' must correspond to a column of $IAS(T)$ whose only nonzero entry is in the z row, but it cannot be $\phi(z)$, because $\phi(z) \in \tau''$. Therefore z neighbors a leaf of T. As x is not a leaf, and y is the only other neighbor of z, it follows that y is a leaf, and hence that T is isomorphic to P_4. $\square$

Theorem 20. *If T is a tree, then every triangulation of $M[IAS(T)]$ is equivalent to the vertex triangulation.*

Proof. According to Theorem 19, we need only consider the special case $T = P_4$. Let $V(T) = \{v_1, v_2, v_3, v_4\}$, with the vertices indexed in order along the path, and let $\mathfrak{T}$ be a triangulation of $M[IAS(T)]$.

Let $\tau_1 \in \mathfrak{T}$ be the vertex triple that contains $\phi(v_1)$. Then τ_1 contains one other element corresponding to a column of $IAS(T)$ with a nonzero entry in the v_1 row. This other element must be $\psi(v_1), \chi(v_2)$ or $\psi(v_2)$.

Case 1: $\psi(v_1) \in \tau_1$. As τ_1 is dependent, its third element is parallel to $\chi(v_1)$. Therefore $\mathfrak{T}$ is ps-equivalent to a triangulation $\mathfrak{T}'$, which includes the vertex triple of v_1. The triple $\tau_2 \in \mathfrak{T}'$ that includes $\chi(v_2)$ must also include $\psi(v_2)$, as these are the only remaining columns of $IAS(T)$ with nonzero entries in the v_1 row. The third element of τ_2 is parallel to $\phi(v_2)$; it cannot be $\chi(v_1)$, as $\mathfrak{T}'$ includes the vertex triple of v_1, so it must equal $\phi(v_2)$. Thus $\mathfrak{T}'$ includes the vertex triples of both v_1 and v_2. The two remaining columns of $IAS(T)$ with nonzero entries in the v_2 row are $\chi(v_3)$ and $\psi(v_3)$; they must appear in the same triple of $\mathfrak{T}'$, and the third element of this triple must be parallel to $\phi(v_3)$. A parallel swap (if needed) then transforms $\mathfrak{T}'$ into a triangulation that contains the vertex triples of v_1, v_2 and v_3; this must be the vertex triangulation.

Case 2: $\chi(v_2) \in \tau_1$. The third element of τ_1 must be parallel to $\chi(v_4)$, so $\mathfrak{T}$ is ps-equivalent to a triangulation $\mathfrak{T}'$ with $\tau_1' = \{\phi(v_1), \chi(v_2), \chi(v_4)\} \in \mathfrak{T}'$. Let f be the automorphism of $M[IAS(T)]$ discussed in Section 4.2. Then $f(\mathfrak{T}')$ includes $f(\tau_1')$, which is the vertex triple of v_3. The two remaining columns of $IAS(T)$ with nonzero entries in the v_4 row are $\phi(v_4)$ and $\psi(v_4)$; they must appear together in a triple of $f(\mathfrak{T}')$, and this triple can only be the vertex triple of v_4. The two remaining columns of $IAS(T)$ with nonzero entries in the v_3 row are $\chi(v_2)$ and $\psi(v_2)$; they must be included in a single triple of $f(\mathfrak{T}')$. After a parallel swap (if needed) this triple is the vertex triple of v_2, so $f(\mathfrak{T}')$ is ps-equivalent to a triangulation that includes the vertex triples of v_2, v_3 and v_4. This can only be the vertex triangulation.

Case 3: $\psi(v_2) \in \tau_1$. The third element of τ_1 then corresponds to a column of $IAS(T)$ with precisely two nonzero entries, one in the v_2 row and one in the v_3 row. No such column exists. $\qquad\square$

We should mention that Theorem 20 extends to arbitrary graphs. See [10, Prop. 29] for details.

The next lemma is useful for induction.

Lemma 21. *Let v be a vertex of a forest F, and let $F \setminus v$ be the forest obtained from F by removing v. Then*

$$(M[IAS(F)] \setminus \{\chi(v), \psi(v)\})/\phi(v) \cong M[IAS(F \setminus v)].$$

The isomorphism is natural, in that the image of each element $\phi(x), \chi(x)$ or $\psi(x)$ from $(M[IAS(F)] - \{\chi(v), \psi(v)\})/\phi(v)$ is the element $\phi(x), \chi(x)$ or $\psi(x)$ from $M[IAS(F \setminus v)]$.

Proof. It is obvious that the matrix $IAS(F \setminus v)$ is the same as the submatrix of $IAS(F)$ obtained by deleting the v row and the $\phi(v), \chi(v)$ and $\psi(v)$ columns. As the only nonzero entry of the $\phi(v)$ column occurs in the v row, the lemma follows from Proposition 5. $\qquad \square$

One last lemma will be useful in the proof of the implication $3 \implies 1$ from Theorem 3. Suppose F and F' are forests, and f is an isomorphism between $M[IAS(F)]$ and $M[IAS(F')]$. If w is a vertex of F that is adjacent to a leaf v, then $\phi(w)$ and $\chi(v)$ are parallel in $M[IAS(F)]$, so their images under f must be parallel in $M[IAS(F')]$. It follows from Proposition 7 that either $f(\phi(w)) = \phi(w')$ for some $w' \in V(F')$ that is adjacent to a leaf, or $f(\phi(w)) = \chi(v')$ for some leaf $v' \in V(F')$.

Definition 22. In this situation the *leaf index* $i(f)$ is the number of vertices $w \in V(F)$ such that w is adjacent to a leaf, and $f(\phi(w)) = \chi(v')$ for some leaf $v' \in V(F')$.

Lemma 23. *Suppose $f : M[IAS(F)] \to M[IAS(F')]$ is an isomorphism, which maps each vertex triple of F to a vertex triple of F'. Then there is an isomorphism $f' : M[IAS(F)] \to M[IAS(F')]$ such that (a) f' maps each vertex triple of F to a vertex triple of F'; (b) $i(f') = 0$; and (c) whenever $x \in V(F)$ and $x' \in V(F')$ have $f(\phi(x)) = \phi(x')$, it is also true that $f'(\phi(x)) = \phi(x')$.*

Proof. If $i(f) = 0$, then $f' = f$ satisfies the lemma.

Suppose $i(f) > 0$. Then there is a vertex $w \in V(F)$ such that w neighbors a leaf of F, and $f(\phi(w)) = \chi(v')$ for some leaf v' of F'. Let w' be the unique neighbor of v' in F'. As f is an isomorphism, and $\chi(v')$ is parallel to $\phi(w')$, $f^{-1}(\phi(w'))$ is a parallel of $\phi(w)$. That is, there is a leaf v that neighbors w and has $f(\chi(v)) = \phi(w')$. According to Lemma 14,

$$\beta_{vw} = (\phi(v)\psi(w))(\chi(v)\phi(w))(\psi(v)\chi(w))$$

is an automorphism of $M[IAS(F)]$, so the composition $f' = f \circ \beta_{vw} : M[IAS(F)] \to M[IAS(F')]$ is an isomorphism.

The automorphism β_{vw} is the identity map outside the vertex triples of v and w, and it maps either of these two vertex triples to the other. As f maps vertex triples of F to vertex triples of F', it follows that f' also maps vertex triples of F to vertex triples of F'. As f' agrees with f outside the vertex triples of v and w, and $f'(\phi(w)) = f(\beta_{vw}(\phi(w))) = f(\chi(v)) = \phi(w')$, $i(f') = i(f) - 1$. Notice that $f(\phi(v))$ cannot be a ϕ element of F', as it is an element of the same vertex triple as $f(\chi(v)) = \phi(w')$. Therefore f' has the property that whenever $x \in V(F)$ and $x' \in V(F')$ have $f(\phi(x)) = \phi(x')$, it is also true that $f'(\phi(x)) = \phi(x')$.

Repeating this argument $i(f)$ times, we will find an f' that satisfies the lemma. □

We are now ready to verify the implication $3 \implies 1$ of Theorem 3. The proof uses induction on $n = |V(F)|$. As mentioned at the end of Section 4, we do not prove $3 \implies 1$ as stated in Theorem 3, because that statement does not give us a sufficiently precise inductive hypothesis. Instead we prove a more detailed theorem:

Theorem 24. *Suppose F and F' are forests, and $M[IAS(F)]$ is isomorphic to $M[IAS(F')]$. Then:*

1) *There is an isomorphism $f : M[IAS(F)] \to M[IAS(F')]$, under which the image of the vertex triangulation of $M[IAS(F)]$ is the vertex triangulation of $M[IAS(F')]$.*

2) *For any f that satisfies part 1, there is a graph isomorphism $g : F \to F'$ with $g(x) = x'$ whenever $x \in V(F)$ and $x' \in V(F')$ have $f(\phi(x)) = \phi(x')$.*

Proof. Let F and F' be forests, with an isomorphism $M[IAS(F)] \cong M[IAS(F')]$. Then the matroids have the same cardinality, so F and F' have the same number n of vertices.

If $n = 1$, there are two matroid isomorphisms $f, f' : M[IAS(F)] \to M[IAS(F')]$. One of f, f' matches elements according to their ϕ, χ, ψ designations, and the other of f, f' matches the ϕ element of one matroid to the ψ element of the other. (The χ elements must be matched to each other by any isomorphism, as they are the only loops in the matroids.) Either of the isomorphisms f, f' satisfies the statement, together with the unique graph isomorphism $g : F \to F'$.

The argument proceeds using induction on $n > 1$. Suppose the matroids $M[IAS(F)]$ and $M[IAS(F')]$ have more than one component. If $M[IAS(F)]$ has a component M with more than two elements, then M corresponds to a connected component C of F with $1 < |V(C)| < n$.

The image of M under an isomorphism $M[IAS(F)] \cong M[IAS(F')]$ is a component M' of $M[IAS(F')]$, which corresponds to a connected component C' of F'. The theorem holds for F and F' because the inductive hypothesis applies separately to C and C', on the one hand, and $F \setminus C$ and $F' \setminus C'$, on the other hand. If $M[IAS(F)]$ has no component with more than two elements, then all the vertices of F and F' are isolated. It is easy to see that the theorem holds in this trivial case.

Suppose now that $M[IAS(F)]$ and $M[IAS(F')]$ are isomorphic matroids with only one component. Then F and F' are both connected. The image of the vertex triangulation of $M[IAS(F)]$ under an isomorphism $M[IAS(F)] \cong M[IAS(F')]$ is a triangulation of $M[IAS(F')]$. According to Theorem 20, this triangulation is equivalent to the vertex triangulation of $M[IAS(F')]$. Therefore, we can compose an automorphism of $M[IAS(F')]$ with the original isomorphism $M[IAS(F)] \cong M[IAS(F')]$ to obtain an isomorphism $f : M[IAS(F)] \to M[IAS(F')]$ that satisfies part 1 of the statement.

Now, let $f : M[IAS(F)] \to M[IAS(F')]$ be any isomorphism that satisfies part 1 of the statement. We need to verify that part 2 of the statement is satisfied by some isomorphism $g : F \to F'$. According to Lemma 23, there is an isomorphism $f' : M[IAS(F)] \to M[IAS(F')]$ such that $i(f') = 0$ and any isomorphism $g : F \to F'$ that satisfies part 2 for f' will also satisfy part 2 for f.

Let v be a leaf of F, with unique neighbor w. As $i(f') = 0$, $f'(\phi(w)) = \phi(w')$ for some $w' \in V(F')$ that neighbors a leaf. Then $\chi(v)$ is parallel to $\phi(w)$, so $f'(\chi(v))$ is parallel to $f'(\phi(w)) = \phi(w')$; hence $f'(\chi(v)) = \chi(v')$ for some leaf v' adjacent to w'. As f' maps vertex triples to vertex triples, $f'(\phi(v))$ is either $\phi(v')$ or $\psi(v')$.

If $f'(\phi(v)) = \phi(v')$, let $f'' = f'$. If $f'(\phi(v)) = \psi(v')$, let $f'' = \gamma_{v'w'} \circ f'$, where $\gamma_{v'w'} = (\phi(v')\psi(v'))(\chi(w')\psi(w'))$ is the automorphism of $M[IAS(F')]$ mentioned in Lemma 15. Then $f'' : M[IAS(F)] \to M[IAS(F')]$ is an isomorphism that maps vertex triples to vertex triples and has these properties: $f''(\phi(x)) = \phi(x')$ whenever $f(\phi(x)) = \phi(x')$, $i(f'') = 0$, $f''(\phi(v)) = \phi(v')$, $f''(\chi(v)) = \chi(v')$, and $f''(\phi(w)) = \phi(w')$. The first of these properties implies that any isomorphism $g : F \to F'$ that satisfies part 2 of the statement for f'' will also satisfy part 2 for f.

As $f''(\phi(v)) = \phi(v')$, $f''(\chi(v)) = \chi(v')$, and f'' maps vertex triples to vertex triples, it must be that $f''(\psi(v)) = \psi(v')$. Therefore f'' defines an isomorphism

$$(M[IAS(F)] \setminus \{\chi(v), \psi(v)\})/\phi(v) \to (M[IAS(F')] \setminus \{\chi(v'), \psi(v')\})/\phi(v'),$$

which we denote f'''. According to Lemma 21, it follows that f''' defines an isomorphism $f'''' : M[IAS(F \setminus v)] \to M[IAS(F' \setminus v')]$. The isomorphism f'''' inherits the following properties from f'': f'''' maps vertex triples to vertex triples; $f''''(\phi(x)) = \phi(x')$ whenever $f(\phi(x)) = \phi(x')$; $i(f'''') = 0$; and $f''''(\phi(w)) = \phi(w')$. Applying the inductive hypothesis to f'''', we deduce that there is a graph isomorphism $g_0 : F \setminus v \to F' \setminus v'$ with $g_0(x) = x$ whenever $f''''(\phi(x)) = \phi(x')$. In particular, $g_0(w) = w'$. The inductive step is completed by noting that g_0 can be extended to an isomorphism $g : F \to F'$ by defining $g(v) = v'$. This extended isomorphism satisfies part 2 of the statement for f'', and hence also for f. □

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